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On a pointwise convergence of trigonometric interpolations with shifted nodes

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Abstract

We consider trigonometric interpolations with shifted equidistant nodes and investigate their accuracies depending on the shift parameter. Two different types of interpolations are in the focus of our attention: the Krylov-Lanczos and the rational-trigonometric-polynomial interpolations. In both cases, we find optimal shifts that provide with the best accuracy in different frameworks.

Key Words: Trigonometric Interpolation, Krylov-Lanczos Interpolation, Rational Corrections, Shifted Nodes
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Introduction

In this paper, we consider interpolations with shifted equidistant nodes

$$x_k^\sigma = \frac{2k+\sigma}{2N+1}, \ |k| \le N, \ |\sigma| \le 1, \tag{1}$$

where σ is the shift parameter, and investigate their convergence depending on σ.

Different authors deal with interpolations with shifted nodes and state optimization problems for finding optimal shifts that will improve interpolation properties (accuracy, stability, etc.). Paper [7] explores trigonometric interpolations on the union of two or three shifted each other equidistant grids and by optimal selection of the shift it obtains minimized uniform and L_2-errors of the corresponding interpolations. Plonka, in a series of papers (see [4]-[6] with references therein), examines periodic interpolations with shifted nodes. Paper

[4] discusses periodic Hermite spline interpolation, paper [5] considers periodic spline interpolation and in paper [6] the optimal choice of the shift parameter is found that minimizes the norm of the related interpolation operator.

Essentially different idea is suggested in paper [3] to improve the interpolation properties by shifting the Chebyshev nodes by the change of variable (preserving the spectral convergence), in such a way that the shifted nodes are closer to equidistant. Berrut, et al., in a series of papers (see [1] with references therein), applies this idea for improving the properties of rational (polynomial) interpolations.

Here, we consider trigonometric (2-periodic) interpolations for grid (1) and find optimal σ in the framework of the pointwise convergence. We deal with two different types of interpolations: the Krylov-Lanczos trigonometric-polynomial (see [2] and [10] with references therein) and the rational-trigonometric-polynomial (see [13] with references therein) interpolations with shifted nodes.

The Krylov-Lanczos (KL) interpolation performs convergence acceleration of the classical trigonometric interpolation by polynomial corrections. Paper [10] investigates the KL interpolation for grid (1), when $\sigma = 0$, in the framework of the pointwise convergence. In the current paper, we perform similar investigations in the presence of the shift parameter σ and show that, in some cases, $\sigma = 0$ is not an optimal choice.

Comparison with the results of [10] shows that the optimal choice improves the accuracy by factor $O(N)$. The KL interpolation with shifted nodes is considered also in [8] and [9], in the framework of the L_2 convergence (see also [10] in case of $\sigma = 0$), showing the optimal σ. At the end of the current paper, we represent those results for comparison, to show that L_2-framework is less sensitive to the shifts than the pointwise framework.

The rational-trigonometric-polynomial (RTP) interpolation performs additional acceleration of convergence of the KL interpolation by rational periodic corrections which contain some extra parameters θ_k. Paper [12] investigates convergence of the RTP interpolation with θ_k defined by $\theta_k = 1 - \tau_k/N$ where τ_k are some new parameters, independent of N. Paper [13] continues these investigations deriving exact constants of the asymptotic errors in the framework of the pointwise convergence in the regions away from the endpoints.

Determination of parameters τ_k is a crucial problem for realization of the RTP interpolations. Papers [12] and [13] assume that τ_k are the roots of the Laguerre polynomial $L_p^q(x)$. Paper [11] suggests other choices for τ_k. One approach that is important for our investigations leads to the pointwise minimal RTP interpolations.

In the current paper, we perform similar to [13] investigations and derive exact constants of the asymptotic errors for the RTP interpolations with shifted nodes. We show that, similar to the KL interpolation, the optimal selection of parameter σ improves the pointwise accuracy of rational interpolations by factor $O(N)$. Moreover, based on the estimates, we find τ_k that provide with additional accuracy. By these investigations we improve the realization of the pointwise minimal RTP interpolation.

1 Krylov-Lanczos Interpolation

Let us introduce the KL interpolation with shifted nodes. The KL interpolation realizes convergence acceleration of the classical trigonometric interpolation

$$I_N(f,x,\sigma) = \sum_{n=-N}^{N} \check{f}_n^{\sigma} e^{i\pi n x}, \tag{2}$$

where

$$\check{f}_n^{\sigma} = \frac{1}{2N+1} \sum_{k=-N}^{N} f(x_k^{\sigma}) e^{-i\pi n x_k^{\sigma}}, \tag{3}$$

by application of polynomial corrections. Polynomial corrections represent the discontinuities in the function f and some of its first q derivatives (jumps)

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1),\ k = 0, \dots, q-1. \tag{4}$$

The corresponding representation of the interpolated function is known as the Lanczos representation (see [2])

$$f(x) = F(x) + \sum_{m=0}^{q-1} A_m(f) B_m(x), \tag{5}$$

where $B_m(x)$ are 2-periodic Bernoulli polynomials defined by the recurrence relations

$$B_0(x) = \frac{x}{2},\ B_k(x) = \int B_{k-1}(x)dx,\ x \in [-1,1],\ \int_{-1}^{1} B_k(x)dx = 0 \tag{6}$$

with the Fourier coefficients

$$B_n(m) = \frac{(-1)^{n+1}}{2(i\pi n)^{m+1}},\ n \neq 0,\ B_0(m) = 0. \tag{7}$$

Here, F is a 2-periodic and relatively smooth function on the real line $F \in C^{q-1}(\mathbb{R})$ if $f \in C^{q-1}[-1,1]$ and the Fourier coefficients are defined as follows

$$f_n = \frac{1}{2} \int_{-1}^{1} f(x) e^{-i\pi n x} dx. \tag{8}$$

Interpolation of F by (2) leads to the KL interpolation with shifted nodes

$$I_{N,q}(f,x,\sigma) = \sum_{n=-N}^{N} \check{F}_n^{\sigma} e^{i\pi n x} + \sum_{m=0}^{q-1} A_m(f) B_m(x) \tag{9}$$

with the error

$$r_{N,q}(f,x,\sigma) = f(x) - I_{N,q}(f,x,\sigma), \tag{10}$$

where

$$\check{F}_n^{\sigma} = \check{f}_n^{\sigma} - \sum_{m=0}^{q-1} A_m(f) \check{B}_n^{\sigma}(m). \tag{11}$$

Paper [10] studies the KL interpolation for $\sigma = 0$, showing that, in the regions away from the endpoints ($|x| < 1$), the rate of convergence is $O(N^{-q-2})$ for odd q (see Theorem 9) and $O(N^{-q-1})$ for even q (see Theorem 8). We show that such asymmetric behavior of the KL interpolation is due to the shift parameter σ and its optimal choice overcomes the asymmetry by making the rates of convergence equal to $O(N^{-q-2})$ for both even and odd q (see Theorem 7).

2 Convergence Acceleration by Rational Corrections

Now, let us introduce the rational-trigonometric-polynomial interpolation with shifted nodes.

Consider a vector of complex numbers $\theta = \{\theta_1, \dots, \theta_p\}$. We introduce the following generalized finite differences $\delta_n^k(\theta, y_n)$ determined recurrently:

$$\delta_n^0(\theta, y_n) = y_n, \tag{12}$$

and

$$\delta_n^k(\theta, y_n) = \delta_n^{k-1}(\theta, y_n) + \theta_k \delta_{n-1}^{k-1}(\theta, y_n) + \theta_k \left(\delta_{n+1}^{k-1}(\theta, y_n) + \theta_k \delta_n^{k-1}(\theta, y_n)\right) \tag{13}$$

for some sequence y_n. By $\delta_n^s(y_n)$, we denote the sequence that corresponds to $\theta_k = 1$, $k = 1, \dots, s$. By $\Delta_n^k(y_n)$, we denote the differences defined by the relations

$$\begin{aligned} \Delta_n^0(y_n) &= y_n, & (14)\\ \Delta_n^k(y_n) &= \Delta_n^{k-1}(y_n) + \Delta_{n-1}^{k-1}(y_n). & (15) \end{aligned}$$

Now, it is easy to verify that

$$\delta_n^k(y_n) = \Delta_{n+k}^{2k}(y_n), \tag{16}$$

where

$$\Delta_n^w(y_n) = \sum_{\ell=0}^{w} \binom{w}{\ell} y_{n-\ell}. \tag{17}$$

We have from (5), (9) and (10)

$$r_{N,q}(f, x, \sigma) = \sum_{n=-N}^{N} (F_n - \check{F}_n^\sigma) e^{i\pi n x} + \sum_{|n|>N} F_n e^{i\pi n x}. \tag{18}$$

The following transformation is easy to verify (details see in [13])

$$\begin{aligned} r_{N,q}(f, x, \sigma) = &- \theta_1 \check{F}_N^\sigma \frac{e^{i\pi(N+1)x}}{(1+\theta_1 e^{i\pi x})(1+\theta_1 e^{-i\pi x})} + \theta_1 \check{F}_{-N-1}^\sigma \frac{e^{-i\pi N x}}{(1+\theta_1 e^{i\pi x})(1+\theta_1 e^{-i\pi x})} \\ &- \theta_1 \check{F}_{-N}^\sigma \frac{e^{-i\pi(N+1)x}}{(1+\theta_1 e^{i\pi x})(1+\theta_1 e^{-i\pi x})} + \theta_1 \check{F}_{N+1}^\sigma \frac{e^{i\pi N x}}{(1+\theta_1 e^{i\pi x})(1+\theta_1 e^{-i\pi x})} \\ &+ \frac{1}{(1+\theta_1 e^{i\pi x})(1+\theta_1 e^{-i\pi x})} \sum_{|n|=N+1}^{\infty} \delta_n^1(\theta, F_n) e^{i\pi n x} \\ &+ \frac{1}{(1+\theta_1 e^{i\pi x})(1+\theta_1 e^{-i\pi x})} \sum_{n=-N}^{N} \delta_n^1(\theta, F_n - \check{F}_n^\sigma) e^{i\pi n x}. \end{aligned} \tag{19}$$

Definition of the discrete Fourier coefficients (3) provides with the following periodicity relation

$$\check{F}^{\sigma}_{n+k(2N+1)} = e^{-i\pi k\sigma}\check{F}^{\sigma}_{n}. \tag{20}$$

Therefore,

$$\check{F}^{\sigma}_{-N-1} = \check{F}^{\sigma}_{N-(2N+1)} = e^{i\pi\sigma}\check{F}^{\sigma}_{N}, \tag{21}$$

and

$$\check{F}^{\sigma}_{N+1} = \check{F}^{\sigma}_{-N+(2N+1)} = e^{-i\pi\sigma}\check{F}^{\sigma}_{-N}. \tag{22}$$

Substituting these into (19), we get the following transformation of the error

$$\begin{aligned} r_{N,q}(f,x,\sigma) =& \theta_1 \check{F}^{\sigma}_{N}\frac{e^{-i\pi Nx}e^{i\pi\sigma} - e^{i\pi(N+1)x}}{(1+\theta_1 e^{i\pi x})(1+\theta_1 e^{-i\pi x})} + \theta_1 \check{F}^{\sigma}_{-N}\frac{e^{i\pi Nx}e^{-i\pi\sigma} - e^{-i\pi(N+1)x}}{(1+\theta_1 e^{i\pi x})(1+\theta_1 e^{-i\pi x})} \\ &+ \frac{1}{(1+\theta_1 e^{i\pi x})(1+\theta_1 e^{-i\pi x})}\sum_{|n|=N+1}^{\infty}\delta_n^1(\theta,F_n)e^{i\pi nx} \\ &+ \frac{1}{(1+\theta_1 e^{i\pi x})(1+\theta_1 e^{-i\pi x})}\sum_{n=-N}^{N}\delta_n^1(\theta,F_n-\check{F}^{\sigma}_n)e^{i\pi nx}. \end{aligned} \tag{23}$$

Reiteration of this transformation up to p times leads to the final expansion of the error

$$\begin{aligned} r_{N,q}(f,x) =& (e^{-i\pi Nx}e^{i\pi\sigma} - e^{i\pi(N+1)x})\sum_{k=1}^{p}\frac{\theta_k\delta_N^{k-1}(\theta,\check{F}^{\sigma}_n)}{\prod_{s=1}^{k}(1+\theta_s e^{i\pi x})(1+\theta_s e^{-i\pi x})} \\ &+ (e^{i\pi Nx}e^{-i\pi\sigma} - e^{-i\pi(N+1)x})\sum_{k=1}^{p}\frac{\theta_k\delta_{-N}^{k-1}(\theta,\check{F}^{\sigma}_n)}{\prod_{s=1}^{k}(1+\theta_s e^{i\pi x})(1+\theta_s e^{-i\pi x})} \\ &+ \frac{1}{\prod_{s=1}^{p}(1+\theta_s e^{i\pi x})(1+\theta_s e^{-i\pi x})}\sum_{|n|=N+1}^{\infty}\delta_n^p(\theta,F_n)e^{i\pi nx} \\ &+ \frac{1}{\prod_{s=1}^{p}(1+\theta_s e^{i\pi x})(1+\theta_s e^{-i\pi x})}\sum_{n=-N}^{N}\delta_n^p(\theta,F_n-\check{F}^{\sigma}_n)e^{i\pi nx}. \end{aligned} \tag{24}$$

This implies the following RTP interpolation with shifted nodes

$$\begin{aligned} I_{N,q}^{p}(f,x,\sigma) =& \sum_{n=-N}^{N}\check{F}^{\sigma}_n e^{i\pi nx} + \sum_{m=0}^{q-1}A_m(f)B_m(x) \\ &+ (e^{-i\pi Nx}e^{i\pi\sigma} - e^{i\pi(N+1)x})\sum_{k=1}^{p}\frac{\theta_k\delta_N^{k-1}(\theta,\check{F}^{\sigma}_n)}{\prod_{s=1}^{k}(1+\theta_s e^{i\pi x})(1+\theta_s e^{-i\pi x})} \\ &+ (e^{i\pi Nx}e^{-i\pi\sigma} - e^{-i\pi(N+1)x})\sum_{k=1}^{p}\frac{\theta_k\delta_{-N}^{k-1}(\theta,\check{F}^{\sigma}_n)}{\prod_{s=1}^{k}(1+\theta_s e^{i\pi x})(1+\theta_s e^{-i\pi x})} \end{aligned} \tag{25}$$

with the error

$$
\begin{aligned}
r^p_{N,q}(f,x,\sigma) &= f(x) - I^p_{N,q}(f,x,\sigma)\\
&= \frac{1}{\prod_{s=1}^p (1+\theta_s e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{|n|=N+1}^{\infty} \delta_n^p(\theta, F_n) e^{i\pi n x}\\
&+ \frac{1}{\prod_{s=1}^p (1+\theta_s e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{n=-N}^{N} \delta_n^p(\theta, F_n - \check{F}_n^{\sigma}) e^{i\pi n x}.
\end{aligned}
\tag{26}
$$

RTP interpolation (25) is undefined untill parameters θ_k are unknown. In this paper, we follow [11]-[13] and assume that

$$
\theta_k = 1 - \frac{\tau_k}{N}, \ \ k = 1, \cdots, p, \tag{27}
$$

where new parameters τ_k are independent of N. Paper [13] studies RTP interpolations with (27) for $\sigma = 0$ showing that in the regions away from the endpoints the convergence rate is $O(N^{-2p-q-1})$ for even q and $O(N^{-2p-q-2})$ for odd q. Again, we show that such asymmetric behavior can be corrected by appropriate choice of the shift parameter σ.

As we mentioned above, different approaches are known for determination of τ_k (see [11]-[13] with references therein). Based on the asymptotic estimates derived here, we suggest approach for determination of τ_k that provides with additional accuracy. This approach supplements investigations performed in [11]. In the numerical experiments, we compare this approach with the one (see [12] and [13]) where τ_k are the roots of the Laguerre polynomial $L_p^q(x)$.

3 Preliminaries

In this section, we prove some results concerning the behavior of the generalized finite differences. Let θ_k be defined as in (27) and let $\gamma_k(\tau)$ be the coefficients of the polynomial

$$
\prod_{s=1}^p (1+\tau_s x) = \sum_{s=0}^p \gamma_s(\tau) x^s, \tag{28}
$$

where $\tau = \{\tau_1, \ldots, \tau_p\}$.

Let

$$
\phi_m^{\pm}(\sigma) = \sum_{s=-\infty}^{\infty} e^{i\pi s\sigma} \frac{(-1)^s}{(2s \pm 1)^m} \tag{29}
$$

and

$$
\psi_{m,p}^{\pm}(\sigma,\tau) = \sum_{s=0}^p (-1)^s \gamma_s(\tau) \sum_{k=0}^p \gamma_k(\tau)(2p-k-s+m)! \phi^{\pm}_{2p-k-s+m+1}(\sigma). \tag{30}
$$

In view of definitions of $\delta_n^k(\theta, y_n)$, $\delta_n^k(y_n)$ and their connection with $\Delta_n^k(y_n)$, we get (see also [13] for details)

$$\begin{aligned}\delta_{\pm N}^w(\delta_n^p(\theta, \check{B}_n^\sigma(m))) = \sum_{s=0}^{p}(-1)^s \frac{\gamma_s(\tau)}{N^s} \sum_{k=0}^{p}(-1)^k \frac{\gamma_k(\tau)}{N^k} \\ \times \sum_{\ell=0}^{2w+2p-k-s} \binom{2w+2p-k-s}{\ell} \check{B}_{\pm N+p-s+w-\ell}^\sigma(m).\end{aligned} \tag{31}$$

Taking into account the relation

$$\check{B}_n^\sigma(m) = \sum_{s=-\infty}^{\infty} e^{i\pi s\sigma} B_{n+s(2N+1)}(m) \tag{32}$$

and proceeding as in [13], we obtain

$$\begin{aligned}\delta_{\pm N}^w(\delta_n^p(\theta,\check{B}_n^\sigma(m))) &= \frac{(-1)^{N+p+w+1}}{2(i\pi N)^{m+1}N^{2p+2w}m!}\psi_{2w+m}^\pm(\sigma,\tau) \\ &- \frac{(-1)^{N+p+w+1}}{4(i\pi N)^{m+1}N^{2p+2w+1}m!}\sum_{s=0}^{p}(-1)^s\gamma_s(\tau) \\ &\times \sum_{k=0}^{p}\gamma_k(\tau)(2p+2w-k-s+m+1)!\phi_{2p+2w-k-s+m+1}^\pm(\sigma) \\ &+ \frac{(-1)^{N+p+w+1}}{2(i\pi N)^{m+1}N^{2p+2w+1}m!}\sum_{s=0}^{p}(-1)^s\gamma_s(\tau) \\ &\times \sum_{k=0}^{p}\gamma_k(\tau)(2p+2w-k-s+m+1)!\frac{s-k}{2}\phi_{2p+2w-k-s+m+2}^\pm(\sigma) \\ &\pm \frac{(-1)^{N+p+w+1}}{4(i\pi N)^{m+1}N^{2p+2w+1}m!}\psi_{2w+m+1}^\pm(\sigma,\tau) + O(N^{-m-2p-2w-3}).\end{aligned} \tag{33}$$

This is the main estimate, showing how the shift parameter can be chosen in order to vanish the first term in the right-hand side of (33). Thus, when m is odd and $\sigma = 0$ or m is even and $\sigma = \pm 1$, then the three first terms in the right-hand side of (33) vanish and, we get the best possible rate of convergence for $\delta_{\pm N}^w(\delta_n^p(\theta, \check{B}_n^\sigma(m)))$ for θ_k defined by (27). These observations immediately lead to the next four lemmas.

Lemma 1 *Let $m \geq 1$ be odd and $\sigma = 0$. Let*

$$\theta_k = 1 - \frac{\tau_k}{N}, \; k = 1, \cdots, p. \tag{34}$$

Then, the following estimate holds for $p \geq 1$ and $w \geq 0$ as $N \to \infty$

$$\delta_{\pm N}^w(\delta_n^p(\theta, \check{B}_n^\sigma(m))) = \frac{(-1)^{N+p+w+1}}{4(i\pi N)^{m+1}N^{2w+2p+1}m!}\psi_{2w+m+1,p}^+(0,\tau) + O(N^{-2w-2p-m-3}). \tag{35}$$

Lemma 2 *Let $m \geq 0$ be even and $\sigma = \pm 1$. Let*

$$\theta_k = 1 - \frac{\tau_k}{N}, \ k = 1, \cdots, p. \tag{36}$$

Then, the following estimate holds for $p \geq 1$ and $w \geq 0$ as $N \to \infty$

$$\delta_{\pm N}^{w}(\delta_n^p(\theta, \check{B}_n^{\sigma}(m))) = \pm \frac{(-1)^{N+p+w+1}}{4(i\pi N)^{m+1} N^{2w+2p+1} m!} \psi_{2w+m+1,p}^{+}(\pm 1, \tau) + O(N^{-2w-2p-m-3}). \tag{37}$$

Lemma 3 *Let $m \geq 1$ be odd and $\sigma = \pm 1$. Let*

$$\theta_k = 1 - \frac{\tau_k}{N}, \ k = 1, \cdots, p. \tag{38}$$

Then, the following estimate holds for $p \geq 1$ and $w \geq 0$ as $N \to \infty$

$$\delta_{\pm N}^{w}(\delta_n^p(\theta, \check{B}_n^{\sigma}(m))) = \frac{(-1)^{N+p+w+1}}{2(i\pi N)^{m+1} N^{2w+2p} m!} \psi_{2w+m,p}^{+}(\pm 1, \tau) + O(N^{-2w-2p-m-2}). \tag{39}$$

Lemma 4 *Let $m \geq 0$ be even and $\sigma = 0$. Let*

$$\theta_k = 1 - \frac{\tau_k}{N}, \ k = 1, \cdots, p. \tag{40}$$

Then, the following estimate holds for $p \geq 1$ and $w \geq 0$ as $N \to \infty$

$$\delta_{\pm N}^{w}(\delta_n^p(\theta, \check{B}_n^{\sigma}(m))) = \pm \frac{(-1)^{N+p+w+1}}{2(i\pi N)^{m+1} N^{2w+2p} m!} \psi_{2w+m,p}^{+}(0, \tau) + O(N^{-2w-2p-m-2}). \tag{41}$$

We omit the proves of the next two lemmas as similar ones are proved in [13].

Lemma 5 *Let*

$$\theta_k = 1 - \frac{\tau_k}{N}, \ k = 1, \cdots, p. \tag{42}$$

Then, the following estimate holds for $p \geq 1$ and $w, m \geq 0$ as $N \to \infty$ and $|n| \geq N + 1$,

$$\delta_n^{w}(\delta_n^p(\theta, B_n(m))) \ = \ \frac{1}{N^{2p}} O(n^{-2w-m-1}). \tag{43}$$

Lemma 6 *Let*

$$\theta_k = 1 - \frac{\tau_k}{N}, \ k = 1, \cdots, p. \tag{44}$$

Then, the following estimate holds for $p \geq 1$ and $w, m \geq 0$ as $N \to \infty$ and $|n| \leq N$

$$\delta_n^{w}(\delta_n^p(\theta, \check{B}_n^{\sigma}(m) - B_n(m))) = O(N^{-2w-2p-m-1}). \tag{45}$$

Actually, Lemmas 1-6 are valid also for $p = 0$, with corresponding modifications, which we use while estimating the error of the KL interpolation.

4 Accuracy of the RTP interpolation

Here, we study the convergence of the RTP interpolations in the regions away from the endpoints. First, we present a result from [13] concerning the convergence of the RTP interpolations for even q and $\sigma = 0$ which is non-optimal choice as we see from comparison of Lemmas 2 and 4.

Theorem 1 *[13] Let q be even and $f^{(q+2p+1)} \in AC[-1,1]$ for some $p \geq 1$ and $q \geq 0$. Let*

$$\theta_k = 1 - \frac{\tau_k}{N}, \ k = 1, \cdots, p. \tag{46}$$

Then, the following estimate holds for $|x| < 1$

$$\begin{aligned} r_{N,q}^p(f,x,0) = {} & A_q(f)\frac{(-1)^{N+p+\frac{q}{2}}}{2^{2p+1}\pi^{q+1}N^{2p+q+1}q!}\frac{\sin\frac{\pi x}{2}(2N+1)}{\cos^{2p+1}\frac{\pi x}{2}}\psi_{q,p}^{+}(0,\tau) \\ & + o(N^{-2p-q-1}), \ N \to \infty. \end{aligned} \tag{47}$$

Now, we show that by optimal choice $\sigma = \pm 1$, we get improved accuracy and improvement is by factor $O(1/N)$ if interpolated function has additional smoothness.

Theorem 2 *Let q be even and $f^{(q+2p+2)} \in AC[-1,1]$ for some $p \geq 1$ and $q \geq 0$. Let*

$$\theta_k = 1 - \frac{\tau_k}{N}, \ k = 1, \cdots, p. \tag{48}$$

Then, the following estimate holds for $|x| < 1$ as $N \to \infty$

$$\begin{aligned} r_{N,q}^p(f,x,\pm 1) = {} & A_q(f)\frac{(-1)^{N+p+\frac{q}{2}}}{2^{2p+2}\pi^{q+1}N^{2p+q+2}q!}\frac{\sin\frac{\pi x}{2}\cos\frac{\pi x}{2}(2N+1)}{\cos^{2p+2}\frac{\pi x}{2}}\psi_{q+1,p}^{+}(\pm 1,\tau) \\ & + A_{q+1}(f)\frac{(-1)^{N+p+\frac{q}{2}+1}}{2^{2p+1}\pi^{q+2}N^{2p+q+2}(q+1)!}\frac{\cos\frac{\pi x}{2}(2N+1)}{\cos^{2p+1}\frac{\pi x}{2}}\psi_{q+1,p}^{+}(\pm 1,\tau) \\ & + o(N^{-2p-q-2}). \end{aligned} \tag{49}$$

Proof. We omit some details as similar theorems are proved in paper [13], in more details. Let $\sigma = \pm 1$. the application of transformation (23) to (26) twice with $\theta_{p+1} = \theta_{p+2} = 1$ implies

$$\begin{aligned} r_{N,q}^p(f,x,\pm 1) = {} & -\frac{\delta_N^p(\theta,\check{F}_n^\sigma)}{c(x)}\left(e^{-i\pi Nx} + e^{i\pi(N+1)x}\right) - \frac{\delta_{-N}^p(\theta,\check{F}_n^\sigma)}{c(x)}\left(e^{i\pi Nx} + e^{-i\pi(N+1)x}\right) \\ & - \frac{\delta_N^1(\delta_n^p(\theta,\check{F}_n^\sigma))}{d(x)}\left(e^{-i\pi Nx} + e^{i\pi(N+1)x}\right) - \frac{\delta_{-N}^1(\delta_n^p(\theta,\check{F}_n^\sigma))}{d(x)}\left(e^{i\pi Nx} + e^{-i\pi(N+1)x}\right) \\ & + \frac{1}{d(x)}\sum_{n=-N}^{N}\delta_n^2(\delta_n^p(\theta,F_n - \check{F}_n^\sigma))e^{i\pi nx} + \frac{1}{d(x)}\sum_{|n|=N+1}^{\infty}\delta_n^2(\delta_n^p(\theta,F_n))e^{i\pi nx}, \end{aligned} \tag{50}$$

where

$$c(x) = 4\cos^2\frac{\pi x}{2}\prod_{s=1}^{p}(1 + 2\theta_s\cos\pi x + \theta_s^2) \tag{51}$$

and

$$
\begin{aligned}
d(x) &= (1+e^{i\pi x})^2(1+e^{-i\pi x})^2\prod_{s=1}^{p}(1+\theta_s e^{i\pi x})(1+\theta_s e^{-i\pi x}) \\
&= 16\cos^4\frac{\pi x}{2}\prod_{s=1}^{p}(1+2\theta_s\cos\pi x+\theta_s^2). \qquad (52)
\end{aligned}
$$

According to Lemmas 2, 3, 5 and 6 we derive

$$
\begin{aligned}
r_{N,q}^{p}(f,x,\pm 1) &= -\frac{\delta_N^p(\theta,\check{F}_n^\sigma)}{c(x)}\left(e^{-i\pi Nx}+e^{i\pi(N+1)x}\right) \\
&- \frac{\delta_{-N}^p(\theta,\check{F}_n^\sigma)}{c(x)}\left(e^{i\pi Nx}+e^{-i\pi(N+1)x}\right)+o(N^{-q-2p-2}). \qquad (53)
\end{aligned}
$$

Then, in view of the smoothness of f and the Lanczos representation (5), we get

$$
\delta_{\pm N}^p(\theta,\check{F}_n^\sigma) = \sum_{m=q}^{q+2p+2}A_m(f)\delta_{\pm N}^p(\theta,\check{B}_n^\sigma(m))+o(N^{-2p-q-3}). \qquad (54)
$$

Now, application of Lemmas 2 and 3, with $w=0$, completes the proof. $\square$

Analog of this theorem for odd values of q and optimal $\sigma=0$ is proved in [13].

Theorem 3 *[13] Let q be odd and $f^{(q+2p+2)}\in AC[-1,1]$ for some $p,q\geq 1$. Let*

$$
\theta_k = 1-\frac{\tau_k}{N},\ k=1,\cdots,p. \qquad (55)
$$

Then, the following estimate holds for $|x|<1$ as $N\to\infty$

$$
\begin{aligned}
r_{N,q}^{p}(f,x,0) &= A_q(f)\frac{(-1)^{N+p+\frac{q+1}{2}+1}}{2^{2p+2}\pi^{q+1}N^{2p+q+2}q!}\frac{\sin\frac{\pi x}{2}\sin\frac{\pi x}{2}(2N+1)}{\cos^{2p+2}\frac{\pi x}{2}}\psi_{q+1,p}^{+}(0,\tau) \\
&+ A_{q+1}(f)\frac{(-1)^{N+p+\frac{q+1}{2}}}{2^{2p+1}\pi^{q+2}N^{2p+q+2}(q+1)!}\frac{\sin\frac{\pi x}{2}(2N+1)}{\cos^{2p+1}\frac{\pi x}{2}}\psi_{q+1,p}^{+}(0,\tau) \qquad (56) \\
&+ o(N^{-2p-q-2}).
\end{aligned}
$$

For comparison, in the next theorem, we present the behavior of $r_{N,q}^{p}(f,x,\sigma)$ for odd q and non-optimal $\sigma=\pm 1$. We omit the proof as it mimics the one of Theorem 2 and is based on Lemmas 1, 4-6.

Theorem 4 *Let q be odd and $f^{(q+2p+1)}\in AC[-1,1]$ for some $p,q\geq 1$. Let*

$$
\theta_k = 1-\frac{\tau_k}{N},\ k=1,\cdots,p. \qquad (57)
$$

Then, the following estimate holds for $|x|<1$ as $N\to\infty$

$$
r_{N,q}^{p}(f,x,\pm 1) = A_q(f)\frac{(-1)^{N+p+\frac{q+1}{2}}}{2^{2p+1}\pi^{q+1}N^{2p+q+1}q!}\frac{\cos\frac{\pi x}{2}(2N+1)}{\cos^{2p+1}\frac{\pi x}{2}}\psi_{q,p}^{+}(\pm 1,\tau)+o(N^{-2p-q-1}). \qquad (58)
$$

Again, we see that optimal choice of σ improves accuracy by factor $O(1/N)$.

Estimates of Theorems 2 and 3 allow optimal determination of parameters τ_k for realization of the RTP interpolations. We see that the best choice is selection of such τ_k that will vanish $\psi^+_{q+1,p}(\pm 1)$ for even q and $\psi^+_{q+1,p}(0)$ for odd q. This approach was realized in [11] for odd q due to Theorem 3, while for even q, it becomes possible only based on estimate of Theorem 2. Following [11], we determine parameters τ_k from the following systems of equations

$$\psi^+_{q+1+w,p}(\pm 1, \tau) = 0, \ w = 0, \dots, p-1 \tag{59}$$

for even q, and

$$\psi^+_{q+1+w,p}(0, \tau) = 0, \ w = 0, \dots, p-1 \tag{60}$$

for odd q.

Table 1 shows the values of parameters τ_k derived from systems (59) and (60).

	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$
$p=1$	$\tau_1 = 3.51241$	$\tau_1 = 4.44288$	$\tau_1 = 5.48656$	$\tau_1 = 6.47656$	$\tau_1 = 7.48479$
$p=2$	$\tau_1 = 2.86992$	$\tau_1 = 3.61729$	$\tau_1 = 4.49905$	$\tau_1 = 5.32537$	$\tau_1 = 6.18287$
	$\tau_2 = 8.11083$	$\tau_2 = 9.25264$	$\tau_2 = 10.4512$	$\tau_2 = 11.60618$	$\tau_2 = 12.7675$
$p=3$	$\tau_1 = 2.56805$	$\tau_1 = 3.21937$	$\tau_1 = 4.01446$	$\tau_1 = 4.74893$	$\tau_1 = 5.52092$
	$\tau_2 = 6.72467$	$\tau_2 = 7.69530$	$\tau_2 = 8.73304$	$\tau_2 = 9.73173$	$\tau_2 = 10.74534$
	$\tau_3 = 13.14220$	$\tau_3 = 14.37939$	$\tau_3 = 15.6563$	$\tau_3 = 16.8964$	$\tau_3 = 18.13979$
$p=4$	$\tau_1 = 2.38334$	$\tau_1 = 2.973187$	$\tau_1 = 3.712370$	$\tau_1 = 4.38580$	$\tau_1 = 5.10074$
	$\tau_2 = 6.01259$	$\tau_2 = 6.881871$	$\tau_2 = 7.822761$	$\tau_2 = 8.72540$	$\tau_2 = 9.64729$
	$\tau_3 = 11.08278$	$\tau_3 = 12.16542$	$\tau_3 = 13.29462$	$\tau_3 = 14.39122$	$\tau_3 = 15.49740$
	$\tau_4 = 18.40354$	$\tau_4 = 19.69718$	$\tau_4 = 21.02201$	$\tau_4 = 22.31507$	$\tau_4 = 23.61049$

Table 1: The values of τ_k derived from systems (59) and (60).

From Theorems 2, 3 and systems (59), (60), we immediately get.

Theorem 5 *Let q be even and $f^{(q+2p+2)} \in AC[-1,1]$ for some $p \geq 1$ and $q \geq 0$. Let*

$$\theta_k = 1 - \frac{\tau_k}{N}, \ k = 1, \cdots, p, \tag{61}$$

where τ_k are determined from system (59). Then, the following estimate holds for $|x| < 1$ as $N \to \infty$

$$r^p_{N,q}(f, x, \pm 1) = o(N^{-2p-q-2}). \tag{62}$$

Theorem 6 *[11] Let q be odd and $f^{(q+2p+2)} \in AC[-1,1]$ for some $p, q \geq 1$. Let*

$$\theta_k = 1 - \frac{\tau_k}{N}, \ k = 1, \cdots, p, \tag{63}$$

where τ_k are determined from system (60)*. Then, the following estimate holds for $|x| < 1$ as $N \to \infty$*

$$r^p_{N,q}(f, x, 0) = o(N^{-2p-q-2}). \tag{64}$$

5 Accuracy of the KL interpolation

In this section, we investigate the convergence of the KL interpolation with shifted nodes in the regions away from the endpoints $x = \pm 1$ and show the optimal values of parameter σ which provide with the best rate of convergence. Results are similar to the ones presented in the previous section. We omit the proofs as theorems for the KL interpolation can be derived from the corresponding theorems of the RTP interpolations by taking $p = 0$. At the end of the section we consider optimizations in the framework of the L_2-convergence.

The next theorem investigates even q and shows the behavior of the KL interpolation in the regions away from the endpoints for optimal $\sigma = \pm 1$.

Theorem 7 *Let $q \geq 0$ be even and $f^{(q+2)} \in AC[-1, 1]$. Then, the following estimate holds for $|x| < 1$ as $N \to \infty$*

$$\begin{aligned} r_{N,q}(f, x, \pm 1) = A_q(f) \frac{(-1)^{N+\frac{q}{2}}(q+1)}{4\pi^{q+1}N^{q+2}} \frac{\sin\frac{\pi x}{2}\cos\frac{\pi x}{2}(2N+1)}{\cos^2\frac{\pi x}{2}} \sum_{s=-\infty}^{\infty} \frac{1}{(2s+1)^{q+2}} \\ + A_{q+1}(f) \frac{(-1)^{N+\frac{q}{2}+1}}{2\pi^{q+2}N^{q+2}} \frac{\cos\frac{\pi x}{2}(2N+1)}{\cos\frac{\pi x}{2}} \sum_{s=-\infty}^{\infty} \frac{1}{(2s+1)^{q+2}} + o(N^{-q-2}). \end{aligned} \tag{65}$$

Paper [10] proved the analog of Theorem 7 while investigating the pointwise convergence of the KL interpolation for even values of q but considering non-optimal $\sigma = 0$.

Theorem 8 *[10] Let $q \geq 0$ be even and $f^{(q+1)} \in AC[-1, 1]$. Then, the following estimate holds for $|x| < 1$ as $N \to \infty$*

$$r_{N,q}(f, x, 0) = A_q(f) \frac{(-1)^{N+\frac{q}{2}}}{2\pi^{q+1}N^{q+1}} \frac{\sin\frac{\pi x}{2}(2N+1)}{\cos\frac{\pi x}{2}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{q+1}} + o(N^{-q-1}). \tag{66}$$

Comparison of Theorems 7 and 8 shows that the optimal choice of parameter $\sigma = \pm 1$ improves the convergence rate and improvement is by factor $O(1/N)$ compared to $\sigma = 0$.

The next theorem investigates odd q and shows the behavior of the KL interpolation in the regions away from the endpoints for the optimal $\sigma = 0$.

Theorem 9 *[10] Let $q \geq 1$ be odd and $f^{(q+2)} \in AC[-1, 1]$. Then, the following estimate holds for $|x| < 1$ as $N \to \infty$*

$$\begin{aligned} r_{N,q}(f, x, 0) = A_q(f) \frac{(-1)^{N+\frac{q+1}{2}+1}(q+1)}{4\pi^{q+1}N^{q+2}} \frac{\sin\frac{\pi x}{2}\sin\frac{\pi x}{2}(2N+1)}{\cos^2\frac{\pi x}{2}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{q+2}} \\ + A_{q+1}(f) \frac{(-1)^{N+\frac{q+1}{2}}}{2\pi^{q+2}N^{q+2}} \frac{\sin\frac{\pi x}{2}(2N+1)}{\cos\frac{\pi x}{2}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{q+2}} + o(N^{-q-2}). \end{aligned} \tag{67}$$

The next theorem shows the analog of Theorem 9 for non-optimal value $\sigma = \pm 1$.

Theorem 10 *Let $q \geq 1$ be odd and $f^{(q+1)} \in AC[-1,1]$. Then, the following estimate holds for $|x| < 1$ as $N \to \infty$*

$$r_{N,q}(f,x,\pm 1) = A_q(f)\frac{(-1)^{N+\frac{q+1}{2}}}{2\pi^{q+1}N^{q+1}}\frac{\cos\frac{\pi x}{2}(2N+1)}{\cos\frac{\pi x}{2}}\sum_{s=-\infty}^{\infty}\frac{1}{(2s+1)^{q+1}} + o(N^{-q-1}). \tag{68}$$

We again see that the optimal choice provides with improved accuracy and improvement is by factor $O(1/N)$.

Now, we study the convergence of the KL interpolation with shifted nodes on the entire interval $[-1,1]$ in the framework of the L_2 norm.

Theorem 11 *[8],[9] Let $f^{(q)} \in AC[-1,1]$ for some $q \geq 1$. Then*

$$\lim_{N\to\infty} N^{q+\frac{1}{2}}\|r_{N,q}(f,x,\sigma)\|_{L_2(-1,1)} = |A_q(f)|a(q,\sigma), \tag{69}$$

where

$$a(q,\sigma) = \frac{1}{\pi^{q+1}}\left(\frac{1}{2q+1} + \frac{1}{2}\int_{-1}^{1}\left|\sum_{s\neq 0}\frac{(-1)^s e^{i\pi s\sigma}}{(x+2s)^{q+1}}\right|^2 dx\right)^{1/2}. \tag{70}$$

Proof. We have from (18)

$$\|r_{N,q}(f,x,\sigma)\|^2_{L_2(-1,1)} = 2\sum_{n=-N}^{N}|F_n - \check{F}_n^\sigma|^2 + 2\sum_{|n|>N}|F_n|^2. \tag{71}$$

Then, taking into account the smoothness of f and the Lanczos representation (5), we get

$$F_n = A_q(f)\frac{(-1)^{n+1}}{2(i\pi n)^{q+1}} + o(n^{-q-1}),\ n \to \infty. \tag{72}$$

From here and the relation

$$\check{F}_n^\sigma = \sum_{s=-\infty}^{\infty} e^{i\pi s\sigma}F_{n+s(2N+1)} \tag{73}$$

we derive

$$F_n - \check{F}_n^\sigma = -A_q(f)\frac{(-1)^{n+1}}{2(i\pi)^{q+1}}\sum_{s=-\infty}^{\infty}e^{i\pi s\sigma}\frac{(-1)^s}{(n+s(2N+1))^{q+1}} + o(N^{-q-1}),\ N \to \infty. \tag{74}$$

We complete the proof, by substituting all these into (71) and tending N to infinity by replacing the sums by the corresponding integrals. $\square$

Theorem 11, for $\sigma = 0$, can be found also in [10]. Let us show that by appropriate choice of σ, we can minimize the value of $a(q, \sigma)$.

Figure 1 shows the behavior of constants $a(1, \sigma)$ and $a(2, \sigma)$. Experiments show that the behavior of $a(q, \sigma)$, for odd q, mimics the behavior of $a(1, \sigma)$, and for even q, the behavior of $a(2, \sigma)$. We see from Figure 1 that for even q the optimal is $\sigma_{opt} = 0$, and for odd q, we have two equal optimal values which we find numerically and show them in Table 2 with the corresponding values of $a(q, \sigma_{opt})$. For comparison, in Table 3 we calculated also the values of $a(q, \sigma, 0)$ and $a(q, \sigma, \pm 1)$.

We see from comparison of Tables 2 and 3 that only for $q = 1$ and $q = 3$ we have slight improvement in accuracy. For larger values of q there are no practically significant differences. Also, we see from Figure 1 that for $q = 1$ and $q = 2$ the differences in accuracies are small while changing the value of σ on interval $[-1, 1]$.

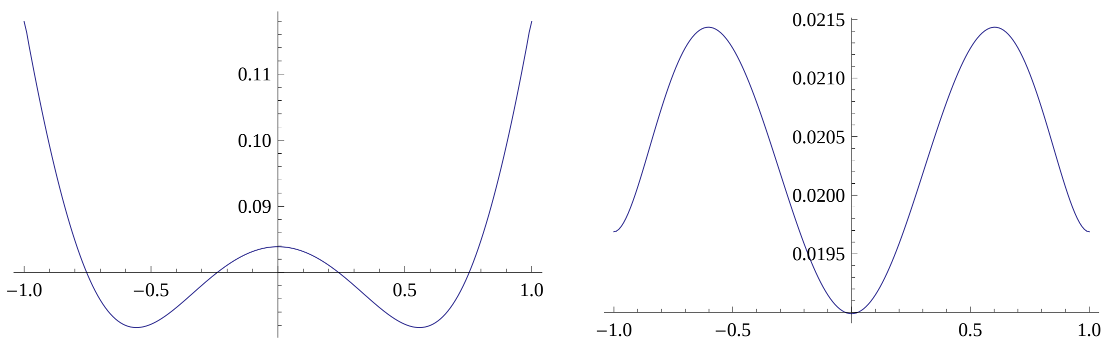

Figure 1: The graphs of $a(1, \sigma)$ (left) and $a(2, \sigma)$ (right).

	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$	$q=6$	$q=7$	$q=8$
$a(q, \sigma_{opt})$	0.0717	0.0190	0.00537	0.00152	0.000442	0.000130	0.0000385	0.0000115
σ_{opt}	±0.5576	0	±0.4950	0	±0.4782	0	±0.4728	0

Table 2: The optimal values of σ that minimize $a(q, \sigma)$ with the corresponding values $a(q, \sigma_{opt})$.

	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$	$q=6$	$q=7$	$q=8$
$a(q, \pm 1)$	0.118	0.0197	0.00568	0.00153	0.000445	0.000130	0.0000385	0.0000115
$a(q, 0)$	0.0839	0.0190	0.00554	0.00152	0.000444	0.000130	0.0000385	0.0000115

Table 3: Numerical values of $a(q, \pm 1)$ and $a(q, 0)$.

Similar investigations can be carried out for the RTP interpolations. Note that Theorem 11 is valid also for $q = 0$ which corresponds to the classic trigonometric interpolation for functions with $A_0(f) \neq 0$.

6 Numerical results

We present some results of numerical experiments showing the impact of parameter σ on accuracy while interpolating function $\sin(x-1)$ for moderate values of N.

First, we introduce some results for the KL interpolations for different values of q and $\sigma = 0, \pm 1$. Figure 2 shows the graphs of $|r_{N,q}(f,x,\sigma)|$ where the top figures correspond to $\sigma = 0$ and the bottom figures to $\sigma = \pm 1$. As it was expected from the above analysis, $\sigma = 0$ (top line) and even q (the second and the fourth graphs), $\sigma = \pm 1$ (bottom line) and odd q (the first and the third graphs), provide with the best accuracies. The same situation we have in Tables 4 and 5, where we presented $L_2(-0.7, 0.7)$ errors for moderate values of N.

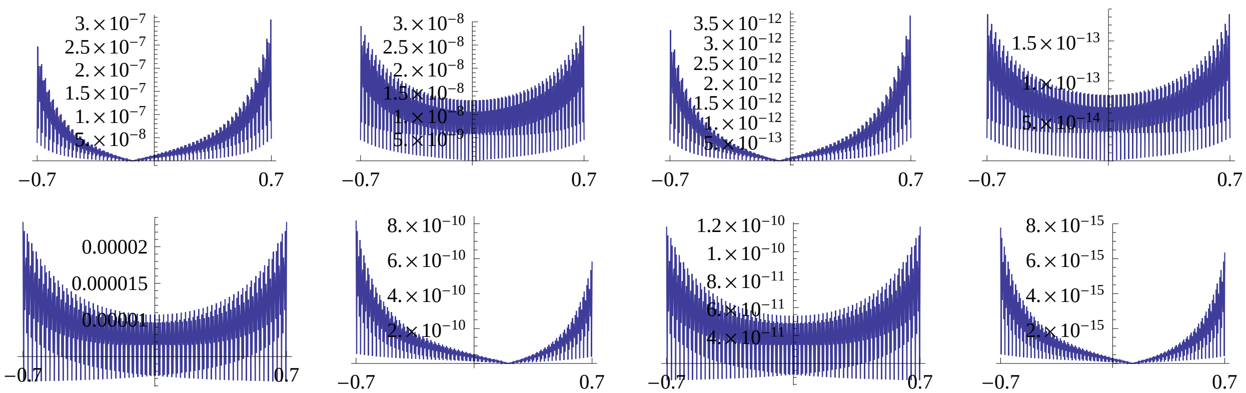

Figure 2: The graphs of $|r_{N,q}(f,x,\sigma)|$ for $q = 1,2,3,4$ (from left to right) on interval $[-0.7, 0.7]$ for $\sigma = 0$ (top line) and $\sigma = \pm 1$ (bottom line) when $N = 128$ and $f(x) = sin(x-1)$.

	N=32	N=64	N=128	N=256	N=512
$\|r_{N,1}(f,x,0)\|_{L_2(-0.7,0.7)}$	$5.3 \cdot 10^{-6}$	$6.6 \cdot 10^{-7}$	$8.4 \cdot 10^{-8}$	$1.1 \cdot 10^{-8}$	$1.3 \cdot 10^{-9}$
$\|r_{N,2}(f,x,0)\|_{L_2(-0.7,0.7)}$	$9.3 \cdot 10^{-7}$	$1.2 \cdot 10^{-7}$	$1.5 \cdot 10^{-8}$	$1.9 \cdot 10^{-9}$	$2.4 \cdot 10^{-10}$
$\|r_{N,3}(f,x,0)\|_{L_2(-0.7,0.7)}$	$1.0 \cdot 10^{-9}$	$3.2 \cdot 10^{-11}$	$1.1 \cdot 10^{-12}$	$3.3 \cdot 10^{-14}$	$1.0 \cdot 10^{-15}$
$\|r_{N,4}(f,x,0)\|_{L_2(-0.7,0.7)}$	$9.1 \cdot 10^{-11}$	$2.9 \cdot 10^{-12}$	$9.4 \cdot 10^{-14}$	$3.0 \cdot 10^{-15}$	$9.4 \cdot 10^{-17}$

Table 4: $L_2(-0.7, 0.7)$ errors while approximating $sin(x-1)$ by $I_{N,q}(f,x,\sigma)$ with $\sigma = 0$.

	N=32	N=64	N=128	N=256	N=512
$\|r_{N,1}(f,x,\pm 1)\|_{L_2(-0.7,0.7)}$	$1.9 \cdot 10^{-4}$	$4.8 \cdot 10^{-5}$	$1.2 \cdot 10^{-5}$	$3.0 \cdot 10^{-6}$	$7.5 \cdot 10^{-7}$
$\|r_{N,2}(f,x,\pm 1)\|_{L_2(-0.7,0.7)}$	$5.1 \cdot 10^{-8}$	$3.4 \cdot 10^{-9}$	$2.2 \cdot 10^{-10}$	$1.4 \cdot 10^{-11}$	$8.5 \cdot 10^{-13}$
$\|r_{N,3}(f,x,\pm 1)\|_{L_2(-0.7,0.7)}$	$1.5 \cdot 10^{-8}$	$9.5 \cdot 10^{-10}$	$6.0 \cdot 10^{-11}$	$3.8 \cdot 10^{-12}$	$2.4 \cdot 10^{-13}$
$\|r_{N,4}(f,x,\pm 1)\|_{L_2(-0.7,0.7)}$	$7.7 \cdot 10^{-12}$	$1.3 \cdot 10^{-13}$	$2.1 \cdot 10^{-15}$	$3.4 \cdot 10^{-17}$	$5.3 \cdot 10^{-19}$

Table 5: $L_2(-0.7, 0.7)$ errors while approximating $sin(x-1)$ by $I_{N,q}(f,x,\sigma)$ with $\sigma = \pm 1$.

Similar results we have for the RTP interpolations. Now, let us show some results showing the impact of τ_k selection on the accuracies. Figures 3 and 4 compare behaviors of

$|r^p_{128,q}(f,x,\sigma)|$ where parameters τ_k are the roots of the Laguerre polynomials (Figure 3) and are derived from systems (59), (60) (Figure 4). We see that even for moderate values of N the impact of τ_k selection is significant and in Figure 4 all interpolations are more accurate than in Figure 3.

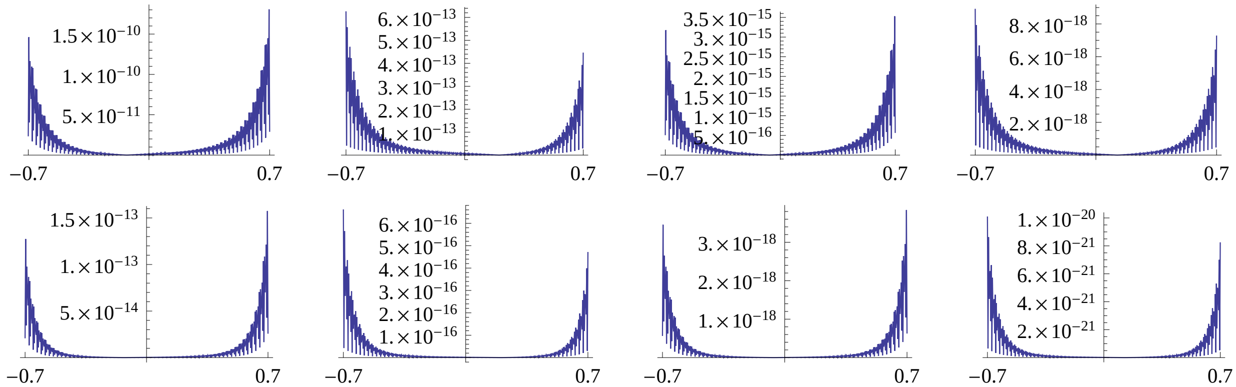

Figure 3: The graphs of $|r^p_{128,q}(f,x,\sigma)|$ for $p = 1$ (top line), $p = 2$ (bottom line) and $q = 1, 2, 3, 4$ (from left to right) on interval $[-0.7, 0.7]$ with $\sigma = 0$ for odd q and $\sigma = \pm 1$ for even q while interpolating $f(x) = \sin(x - 1)$. Parameters τ_k are the roots of the Laguerre polynomials $L^q_p(x)$.

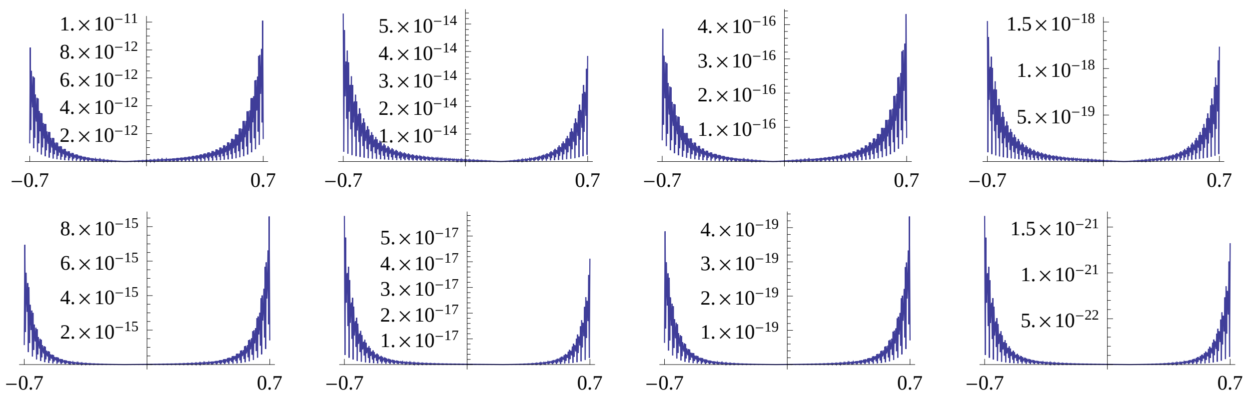

Figure 4: The graphs of $|r^p_{128,q}(f,x,\sigma)|$ for $p = 1$ (top line), $p = 2$ (bottom line) and $q = 1, 2, 3, 4$ (from left to right) on interval $[-0.7, 0.7]$ with $\sigma = 0$ for odd q and $\sigma = \pm 1$ for even q while interpolating $f(x) = \sin(x - 1)$. Parameters τ_k are derived from systems (59), (60).

Numerical experiments are performed by Wolfram's Mathematica package with high accuracy option.

7 Conclusion

We considered some trigonometric interpolations with shifted nodes and investigated their accuracies depending on the shift parameter. Two different types of interpolations were explored: the Krylov-Lanczos interpolation which performed convergence acceleration of the classical interpolation by polynomial corrections, and the rational-trigonometric-polynomial interpolation which performed convergence acceleration of the Krylov-Lanczos interpolation by rational corrections. Rational corrections, considered in this paper, contained some extra parameters which determination was important for realization of the corresponding interpolations.

We studied the convergence rates of the interpolations in different frameworks: pointwise convergence in the regions away from the endpoints $x = \pm 1$ and L_2-convergence on the entire interval $[-1, 1]$ of interpolation. In all cases, we found the exact constants of the asymptotic errors depending on the shift parameter. These estimates allowed selection of the optimal shifts for better accuracy and in case of the RTP interpolations also the optimal values of parameters in the rational corrections. We showed that optimal shifts improved the accuracy by factor $O(1/N)$ for pointwise convergence in the regions $|x| < 1$.

In particular, the results above indicate that for continuous function with discontinuous first derivative the optimal shift parameter is $\sigma = \pm 1$ which means that the first or last point of the grid must coincide with the point of discontinuity. Then, for function which is continuous with its first derivative but has discontinuous second order derivative the optimal is $\sigma = 0$. It is worth noting that similar approach is valid for interpolation of function with single singularity inside the interval $[-1, 1]$.

In case of the L_2-convergence, it turned out, that asymptotic constants were less sensitive to the shifts of interpolation nodes. Although minimization indicated some improvement but they were practically insignificant.

References

[1] J.-P. Berrut, and H. D. Mittelmann, *Optimized point shifts and poles in the linear rational pseudospectral method for boundary value problems*. Journal of Computational Physics **204**(1)(2005), 292–301.

[2] W.B. Jones, and G. Hardy, Accelerating convergence of trigonometric approximations, *Math. Comp.* **24**(111)(1970), 547–560.

[3] D. Kosloff, and H. Tal-Ezer, *A modified Chebyshev pseudospectral method with an $O(N^{-1})$ time step restriction*. Journal of Computational Physics **104**(2)(1993), 457–469.

[4] G. Plonka, *An efficient algorithm for periodic Hermite spline interpolation with shifted nodes*, Numerical Algorithms **5**(1)(1993), 51–62.

[5] G. Plonka, *Periodic spline interpolation with shifted nodes*, Journal of Approximation Theory **76**(1)(1994), 1-20.

[6] G. Plonka, *Optimal shift parameters for periodic spline interpolation*, Numerical Algorithms, **6**(2)(1994), 297–316.

[7] A. Nersessian, and N. Hovhannisyan, *Minimization of errors of the polynomial-trigonometric interpolation with shifted nodes*, Preprint of the Institute of Mathematics of the National Academy of Sciences of Armenia, 1–21, 2006, http://math.sci.am/index.php?page=13.

[8] A. Poghosyan, *L_2-convergence of some polynomial-periodic interpolations*, Reports of NAS RA, **104**(2)(2004), 83–89.

[9] A. Nersessian, A. Poghosyan, *Asymptotic error of polynomial-periodic interpolation*, Proc. International ISAAC Conference "Complex Analysis, Differential Equations and Related Topics" Yerevan 2004, pp. 88–98 (eds. G.A.Barsegian, H.G.W.Begehr, H.Ghazaryan, A.Nersessian).

[10] A. Poghosyan, *Asymptotic behavior of the Krylov-Lanczos interpolation*, Analysis and Applications **7**(2)(2009), 199–211.

[11] A. Poghosyan, *On some optimizations in trigonometric interpolation using Fourier discrete coefficients*, Armenian Journal of Mathematics **4**(2)(2012), 80–97.

[12] A. Poghosyan, *On a convergence acceleration of trigonometric interpolation*, Reports of National Academy of Sciences of Armenia **113**(2)(2013), 133–141.

[13] A. Poghosyan, *On a fast convergence of the rational-trigonometric-polynomial interpolation*, Advances in Numerical Analysis, vol. 2013, article ID 315748, 13 pages.