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Gibbs distributions of quantum systems: Cluster expansions and asymptotics of the partition function

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Contents

1 Introduction

The theory of Gibbs random fields is a relatively young domain of the probability theory. It was originated in late 1960's due to the works by Dobrushin [26, 27, 28, 29, 30], Ginibre [46, 44, 45], Lanford and Ruelle [60] , Minlos [69, 70], Minlos and Sinai [72, 73, 74], Ruelle [106] where a modern approach to the study of notions and problems of statistical mechanics at the rigorous mathematical level was developed.

There has been a great interest in the theory of Gibbs random fields due to applications in physics, in image processing, neuron networks and so on.

Starting from 1970's the Gibbs random fields were studied in the "Integral and stochastic geometry" department of the Institute of mathematics of the National Academy of Sciences of Armenia. Let us mention the works of D. Martirosian on the statistics of configurations of contour models [75, 76], of B. Nahapetian on the limit theorems for Gibbs random fields [79, 80, 82, 32] and the works of the author of the thesis on the study of the large volume behavior of the log-partition function, local limit theorem and the probabilities of large deviations for the classical systems [5, 6, 7, 8, 9, 83, 84]. An original contribution was made by R. Ambartsumian [2, 3, 4]: description of new classes of Gibbsian fields based on combinatorial (inclusion - exclusion principle) approach to the construction of point processes. (Later on the works on this topic were continued by J. Lebowitz and his group from Rutgers University [58, 56]).

We note also recent papers of B. Nahapetian in collaboration with S. Dashian where they solved the well-known problem of R. Dobrushin on description of random fields by means of one point specifications [21, 22, 23]. This allowed them to develop an alternative approach to the theory of Gibbs random fields .

In collaboration with V. Arzumanian and B. Nahapetian the author of the thesis studied the asymptotics of the partition function, the decay of correlations and limit theorems for classical lattice spin systems [83, 9, 84, 5, 6, 7].

The main technical tool for treating the problems considered in the thesis is the cluster expansions method, one of the most powerful in study of the Gibbs random fields [65, 26]. The method allows to write local characteristics of the Gibbs random field (the logarithm of the partition function, the means of local functions et.c) as an absolutely convergent series. The main term of these expansions corresponds to the non-interacting systems, while the correction terms depend on the interactions. The method is efficient for weak interactions where deeper analysis is possible. The amount of publications devoted to cluster expansions and their applications is enormous, see the surveys by Brydges [16], Pfister [89], Abdesselam and Rivasseau [1], the books of Malyshev and Minlos [65], Seiler [107], Glimm and Jaffe [47] and references therein.The cluster expansions method has important applications to classical systems [88, 71, 17], quantum systems [44, 46, 96], quantum field theory [48, 64, 10, 18], as well as to polymer systems, i.e. discrete systems with additional internal structure [49, 25, 14, 77, 110, 42, 53].

In pursue of Dobrushin's paper [26] "Sometimes it seems that the specialists in the probabilistic mathematical physics pronounce the words: "Now the cluster expansion can be applied" as some kind of magic incarnation. They mean that now we can be sure that all plausible facts can be rigorously proved. "

To apply the method of cluster expansions in a concrete situation usually one has to specify and modify the general ideas often by repeating the same arguments. There was a natural tendency to develop a unified approach. An important step forward was the article of Kotecký and Preiss with its simplified setting and its elegant condition for the convergence of the cluster expansion [57]. Their result applies to discrete systems only. At the same time all abstract (i.e. general) approaches involve restrictions that correspond to non-negative (repulsive) interactions.

The thesis presents a new approach to the abstract cluster expansions method that applies to classical and quantum systems (discrete and continuous) with general stable interactions [101].

The main result of the thesis is the asymptotic expansion of the log-partition function of a quantum gas in a bounded domain as this domain dilates to infinity. This result is based on the cluster expansion method and on a new general method of proving asymptotic expansions for the log-partition functions using estimates of two-point semiinvariants (truncated correlation functions) only.

The thesis undertakes the following strategy : with the help of the Feynman - Kac formula the quantum gases are represented as systems of interacting Brownian loops (loop gases).

Then a key notion of the decay of functions of two Brownian loops, given in terms of integral bounds, is introduced. Using the general approach to the cluster expansions bounds for two-point semiinvariants are derived to become the basic technical tool for the derivation of the asymptotic expansion of the log-partition function of the loop gases.

Various models of loop gases have been studied in [11, 33, 66, 108, 112, 12, 103, 43, 94, 95, 96, 98, 99] et. c..

The rest of the thesis is organized in the following way:

Chapter 2 is devoted to the abstract cluster expansions method [101]. It starts with settings and formulation of two general assumptions which provide the main results of Chapter 1: absolute convergence of the abstract cluster expansion and convenient representations of the first and second abstract semiinvariants. Then we derive a fundamental tree estimate in Section 2.2 and prove the theorems in Section 2.3.

In Chapter 3 we describe the decay of correlations in terms of various bounds for semiinvariants. In general setting, where the semiinvariants have a representation via Ursell functions, we give a useful bound for the abstract two-point semiinvariants, see Section 3.1.

We use this general result to study the decay of correlations in loop gases.

In Section 3.2 we briefly recall the Feynman - Kac formula [46, 43, 104, 109] which gives a convenient representation of the grand partition functions of quantum gases. We consider the cases of Maxwell - Boltzmann (MB), Bose - Einstein (BE) and Fermi - Dirac (FD) statistics. In the next Section 3.3 we introduce the corresponding models of loop gases with MB, BE and FD statistics. The case of MB statistics is the simplest and the corresponding loop gas is a system of interacting Brownian loops in $\mathbb{R}^d$ of fixed time interval β. One can think about the loop gas with MB statistics as a classical gas in $\mathbb{R}^d$ where the point particles are replaced by Brownian loops, the interaction between points by a an interaction between loops and the Lebesgue integration $\int \mathrm{d}u$ by a Wiener integration of the type $\int \mathrm{d}u \int \mathrm{d}P_\beta^{u,u}(dx)$, where $P_\beta^{u,u}(dx)$ is the conditional Wiener measure (also called Brownian Bridge measure) for Brownian loops from u to u in time β.

The loop gases corresponding to quantum gases with BE and FD statistics are the models of interacting Brownian loops in $\mathbb{R}^d$ with random time intervals that are integer multiples of β, so called winding or composite loops. One can think on these models in analogy with the loop gas obeying MB statistics with more complicated winding loop interaction and more sophisticated integration.

In Sections 3.4 we consider loop gases and define the decay property for functions of two Brownian loops [96, 93]. Combining this with the bound for the abstract two-point semiinvariants from Section 3.1, we formulate two more assumptions (separately for integrable and hard core potentials) which provide power decay of the two-point semiinvariants of the loop models. The next Section 3.5 considers concrete models of loop gases with various statistics and potentials and finds (by verifying assumptions from the previous section) the conditions on the activity and potential which provide a power decay of the corresponding two-point semiinvariants.

In Chapter 4 we consider classical gases and in this relatively simple situation we present a new approach for the derivation of the large volume asymptotics of the log-partition function of the corresponding Gibbs distribution. It was mentioned above that this approach, in contrast to the existing ones (see [92], [19]), uses bounds only for the two-point semiinvariants. As an application of this result we prove the central local limit theorem [7, 9], give a bound for the convergence rate [84] and prove the local limit theorem for the probabilities of large deviations of the particle number in a grand canonical ensemble [91, 90]. We note that similar problems for the loop gases remain open.

Chapter 5 uses a modification of the method from Chapter 4 to derive the asymptotic expansion of the log-partition function of the Gibbs distribution for interacting loop gases in a bounded domain. The following expansion is the main result of Chapter 5:

$$\ln\, Z(\Lambda_R, z) = R^2|\Lambda|\beta p(\phi, z) + R\,|\partial\,\Lambda| b(\phi, z) + 2\pi\chi(\Lambda) c(\phi, z) + o(1). \qquad (1.1)$$

Here β is the inverse temperature, $|\Lambda|$ is the area, $|\partial\Lambda|$ the length of the boundary of Λ and $\chi(\Lambda)$ is the Euler-Poincare characteristic of the domain Λ. The coefficients $p(\phi, z)$, $b(\phi, z)$ and $c(\phi, z)$ are explicitly expressed as functional integrals and are analytic functions of the activity z in a neighborhood of the origin; $p(\phi, z)$ is the pressure and $b(\phi, z)$ can be interpreted as the surface tension.

This result can be viewed as a natural generalization of the famous problem of finding the asymptotics of the partition function Tr $\exp(\beta\Delta) = \sum_{n=1}^{\infty} e^{-\beta\lambda_n}$ as $\beta \to 0$ [94]. Here λ_n are eigenvalues of the Laplacian $-\Delta$ in a bounded domain . This problem goes back to 1910's, to H. Lorentz and H. Weyl [54].

That the situation in quantum case is much more complicated illustrates the fact that, in contrast to the classical case, the log-partition function ln $Z_{id}(\Lambda_R, z)$ of the ideal (non-interacting) quantum gas has a non-trivial asymptotic expansion similar to (1.1) with constant coefficients [62]. This expansion of ln $Z_{id}(\Lambda_R, z)$, which was the only existing result in this direction, can be obtained from our expansion (1.1) as a particular case by setting $\phi = 0$. The paper [62] applies the asymptotic expansion of ln $Z_{id}(\Lambda_R, z)$ for the study of the finite size effects in the orbital magnetism of a free electron gas.

The expansion (1.1) opens a possibility to derive limit theorems for the models of interacting Brownian loops, to study the orbital magnetism as well as the diamagnetic current for interacting electron gas, in analogy with the papers [62, 63] . Other applications of (1.1) can be found in the stochastic and integral geometry [68] and in the statistics of Gibbs random fields [67].

For interacting quantum gases only the existence of the volume term of the asymptotic expansion of the log-partition function was proved [46].

We note that the geometrical factors of all the three terms of the expansion (1.1) are the same as in the well known Hadwiger's theorem from the integral geometry which states that any real-valued, additive function ψ on the space of compact convex subsets Λ in $\mathbb{R}^2$ which is convex-continuous (with respect to the Housdorff metric) and invariant with respect to the Euclidean motions is a linear combination of the area, the length of the boundary and the Euler-Poincare characteristic of Λ with constant coefficients . But in contrast to this, the expansion (1.1) is an asymptotic expansion and the connections are not clear. We note that since $\ln Z(\Lambda, z)$ as a function of convex bodies is not additive: $\ln Z(\Lambda_1 \cup \Lambda_2, z) \neq \ln Z(\Lambda_1, z) + \ln Z(\Lambda_2, z) - \ln Z(\Lambda_1 \cap \Lambda_2, z)$, the Hadwiger's theorem can not be applied to $\ln Z(\Lambda, z)$.

Finally in Section 5.4 (using different method) we prove the asymptotic expansion of the log-partition function of interacting Bose gas in a polygonal domain.

2 Cluster expansions

2.1 Formulation of results

We consider a set $\mathbb{X}$ whose elements may represent widely different objects: positions of a classical particle, polymers, i.e. a connected sets of $\mathbb{Z}^d$, closed Brownian bridges, et. c.. For the general abstract theory, we assume the structure of a measure space, $(\mathbb{X}, \mathcal{X}, \mu)$, with μ a complex measure. We denote $|\mu|$ the total variation (absolute value) of μ. Let u and q be complex measurable symmetric functions on $\mathbb{X} \times \mathbb{X}$, that are related by the equation

$$q(x,y) = e^{-u(x,y)} - 1. \tag{2.1}$$

We allow the real part of u to take the value $+\infty$, in which case $q(x,y) = -1$. In typical applications $u(x,y)$ represents the interactions between x and y, and the value $+\infty$ corresponds to a hard-core repulsion. We define the "partition function" by

$$Z = \sum_{n \geq 0} \frac{1}{n!} \int \mathrm{d}\mu(x_1) \dots \int \mathrm{d}\mu(x_n) \exp\Bigl\{ - \sum_{1 \leq i < j \leq n} u(x_i, x_j) \Bigr\}, \tag{2.2}$$

or, equivalently,

$$Z = \sum_{n \geq 0} \frac{1}{n!} \int \mathrm{d}\mu(x_1) \dots \int \mathrm{d}\mu(x_n) \prod_{1 \leq i < j \leq n} \bigl(1 + q(x_i, x_j) \bigr). \tag{2.3}$$

The term $n = 0$ of the sums is understood to be 1.

The main goal of cluster expansions is to express the partition function as the exponential of a convergent series of "cluster terms". The main difficulty is to prove the convergence. We first assume that the potential u is stable.

Assumption 1 . *There exists a nonnegative function b on $\mathbb{X}$ such that, for all n and almost all $x_1, \dots, x_n \in \mathbb{X}$,*

$$\prod_{1 \leq i < j \leq n} \bigl| 1 + q(x_i, x_j) \bigr| \leq \prod_{i=1}^{n} e^{b(x_i)}.$$

In other words, we assume the lower bound

$$\sum_{1 \leq i < j \leq n} \operatorname{Re} u(x_i, x_j) \geq - \sum_{i=1}^{n} b(x_i). \tag{2.4}$$

When the function b is constant, this is the usual definition of stability. "Almost all" means that, for given n, the set of points where the condition fails has measure zero with respect to the product measure $\otimes^n \mu$. If $\mathbb{X}$ is countable, the condition must be satisfied for all $x_1, \dots, x_n$ such that $\mu(x_i) \neq 0$.

The second condition deals with the strength of interactions.

Assumption 2 . *There exists a nonnegative function a on $\mathbb{X}$ such that for almost all $x \in \mathbb{X}$,*

$$\int \mathrm{d}|\mu|(y)\, |q(x,y)| e^{a(y)+2b(y)} \leq a(x).$$

In order to guess the correct form of a, one should consider the left side of the equation above with $a(y) \equiv 0$. The integral may depend on x; a typical situation is that x is characterized by a length $\ell(x)$, which is a positive number, so that the left side is roughly proportional to $\ell(x)$. This suggests to try $a(x) = c\ell(x)$, and one can then optimize on the value of c.

We also consider an alternate criterion that involves u rather than q.

Assumption 2' . *There exists a nonnegative function a on $\mathbb{X}$ such that for almost all $x \in \mathbb{X}$,*

$$\int \mathrm{d}|\mu|(y)\, |u(x,y)| e^{a(y)+b(y)} \leq a(x).$$

For positive u we can take $b(x) \equiv 0$; and since $1 - e^{-u} \leq u$, Assumption 2 is always better than Assumption 2'. We actually conjecture that, together with Assumption 1, a sufficient condition is

$$\int \mathrm{d}|\mu|(y)\, \min\bigl(|q(x,y)|, |u(x,y)|\bigr) e^{a(y)+b(y)} \leq a(x). \tag{2.5}$$

That is, it should be possible to combine the best of both assumptions. In this respect Assumption 2 is optimal in the case of positive potentials, and Assumption 2' is optimal in the case of hard core plus negative potentials.

We denote by $\mathcal{G}_n$ the set of all unoriented simple (i.e. without loops and with at most one edge between two different vertices) graphs and by $\mathcal{C}_n \subset \mathcal{G}_n$ the set of all connected graphs with n vertices. We write $V(G)$ for the set of vertices of a graph G. We introduce the following combinatorial function on finite sequences $(x_1, \dots, x_n)$ of elements of $\mathbb{X}$:

$$\varphi(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } n = 1, \\ \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in G} q(x_i, x_j) & \text{if } n \geq 2. \end{cases} \tag{2.6}$$

The product is over edges of G.

Theorem 2.1 (Cluster expansions) . *Suppose that Assumptions 1 and 2, or 1 and 2', hold true. We also suppose that $\int \mathrm{d}|\mu|(y) | e^{a(y)+2b(y)} < \infty$. Then we have*

$$Z = \exp\Bigl\{ \sum_{n \geq 1} \frac{1}{n!} \int \mathrm{d}\mu(x_1) \dots \mathrm{d}\mu(x_n)\, \varphi(x_1, \dots, x_n) \Bigr\}. \tag{2.7}$$

The term in the exponential converges absolutely. Furthermore, for almost all $x_1 \in \mathbb{X}$, we have the following estimate

$$\sum_{n \geq 2} \frac{1}{(n-1)!} \int \mathrm{d}|\mu|(x_2) \dots \int \mathrm{d}|\mu|(x_n)\, |\varphi(x_1, \dots, x_n)| \leq (e^{a(x_1)} - 1) e^{2b(x_1)}. \tag{2.8}$$

(Under Assumption 2', Eq. (2.8) holds with $e^{b(x_1)}$ instead of $e^{2b(x_1)}$.)

Let us turn to correlation functions. We only consider one-point and two-point correlation functions since these are the most useful and expressions become more transparent. We refer to [111] for more general functions. First, we define the *unnormalized one-point correlation function* by

$$\varsigma(x_1) = \sum_{n \geq 1} \frac{1}{(n-1)!} \int \mathrm{d}\mu(x_2) \ldots \int \mathrm{d}\mu(x_n) \prod_{1 \leq i < j \leq n} \big(1 + q(x_i, x_j)\big) \tag{2.9}$$

(the term $n = 1$ is 1 by definition). And we define the *unnormalized two-point correlation function* by

$$\varsigma(x_1, x_2) = \sum_{n \geq 2} \frac{1}{(n-2)!} \int \mathrm{d}\mu(x_3) \ldots \int \mathrm{d}\mu(x_n) \prod_{1 \leq i < j \leq n} \big(1 + q(x_i, x_j)\big) \tag{2.10}$$

(the term $n = 2$ is equal to $1 + q(x_1, x_2)$). The *normalized* correlation functions are $\varsigma(x_1)/Z$ and $\varsigma(x_1, x_2)/Z$. As is shown in Theorem 2.2, they can be expressed using the "cluster functions"

$$\sigma(x_1) = \sum_{n \geq 1} \frac{1}{(n-1)!} \int \mathrm{d}\mu(x_2) \ldots \int \mathrm{d}\mu(x_n)\, \varphi(x_1, \ldots, x_n), \tag{2.11}$$

and

$$\sigma(x_1, x_2) = \sum_{n \geq 2} \frac{1}{(n-2)!} \int \mathrm{d}\mu(x_3) \ldots \int \mathrm{d}\mu(x_n)\, \varphi(x_1, \ldots, x_n). \tag{2.12}$$

Theorem 2.2 (Cluster functions) . *Under the same assumptions as in Theorem 2.1, the cluster functions $\sigma(x_1)$ and $\sigma(x_1, x_2)$ are correctly defined. Moreover*

$$\begin{aligned} |\sigma(x_1)| &\leq e^{a(x_1) + 2b(x_1)} \\ |\sigma(x_1, x_2)| &\leq e^{2[(a(x_1)+2b(x_1)) + a(x_2) + 2b(x_2)]}. \end{aligned} \tag{2.13}$$

In statistical mechanics, the relevant expression is the *truncated two-point correlation function*

$$\frac{\varsigma(x_1, x_2)}{Z} - \frac{\varsigma(x_1)\, \varsigma(x_2)}{Z^2}.$$

When the cluster expansion converges, it is equal to $\sigma(x_1, x_2)$ by the theorem below. This function usually provides an order parameter for phase transitions and it is useful to estimate its decay properties (see Section 3.1) .

Theorem 2.3 (Correlation functions) . *Under the same assumptions as in Theorem 2.1, we have*

$$\begin{aligned} \frac{\varsigma(x_1)}{Z} &= \sigma(x_1), \\ \frac{\varsigma(x_1, x_2)}{Z} &= \sigma(x_1)\sigma(x_2) + \sigma(x_1, x_2). \end{aligned} \tag{2.14}$$

The theorems of this section are proved in Section 2.3.

2.2 The tree estimate

Let $\mathcal{T}_n \subset \mathcal{C}_n$ denote the set of trees with n vertices. The following tree estimate plays a fundamental rôle in the proof of convergence of the cluster expansions. Let n be an integer, $b_1, \ldots, b_n$ be real nonnegative numbers, and $q_{ij} = q_{ji}$, $1 \le i, j \le n$, be complex numbers. We assume that the following bound holds for any subset $I \subset \{1, \ldots, n\}$:

$$\prod_{i,j \in I, i \neq j} |1 + q_{ij}| \le \prod_{i \in I} e^{b_i}, \tag{2.15}$$

Let u_{ij} be such that $q_{ij} = e^{-u_{ij}} - 1$. We prove two distinct tree estimates, the first one involving $|q_{ij}|$ and the second one involving $|u_{ij}|$. These bounds will allow to prove the convergence under either Assumption 2 or Assumption 2'

Proposition 2.4 . *If the bound* (2.15) *holds true, we have the two bounds*

(a)

$$\Big| \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in G} q_{ij} \Big| \le \Big(\prod_{i=1}^n e^{2b_i} \Big) \sum_{G \in \mathcal{T}_n} \prod_{\{i,j\} \in G} |q_{ij}|.$$

(b)

$$\Big| \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in G} q_{ij} \Big| \le \Big(\prod_{i=1}^n e^{b_i} \Big) \sum_{G \in \mathcal{T}_n} \prod_{\{i,j\} \in G} |u_{ij}|.$$

We actually conjecture that the following estimate holds under the same hypotheses:

$$\Big| \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in G} q_{ij} \Big| \le \Big(\prod_{i=1}^n e^{b_i} \Big) \sum_{G \in \mathcal{T}_n} \prod_{\{i,j\} \in G} \min(|u_{ij}|, |q_{ij}|). \tag{2.16}$$

We prove Proposition 2.4 (a) below using Ruelle's algebraic approach, see [105] and references therein. This method is usually combined with a Banach fixed point argument for correlation functions. However, we use it differently so as to get a tree estimate.

Let $\mathcal{A}$ be the set of complex symmetric functions on $\mathcal{P}(\{1, \ldots, n\})$ the set of all subsets of $\{1, \ldots, n\}$. We introduce the following multiplication operation for $f, g \in \mathcal{A}$:

$$f * g(I) = \sum_{J \subset I} f(J) g(I \setminus J) \tag{2.17}$$

We use the standard conventions for sums and products, namely, that the empty sum is zero and empty product is 1. Together with the addition, $\mathcal{A}$ is a commutative algebra with unit $1_{\mathcal{A}}(I) = \delta_{I,\emptyset}$. We have

$$f^{*k}(I) = \sum_{J_1, \ldots, J_k \subset I, J_i \cap J_j = \emptyset, \cup J_i = I} f(J_1) \ldots f(J_k). \tag{2.18}$$

Let $\mathcal{A}_0$ be the subset of functions f such that $f(\emptyset) = 0$ ($\mathcal{A}_0$ is an ideal of the algebra $\mathcal{A}$). Notice that $f^{*k}(I) = 0$ for all $k > |I|$, $f \in \mathcal{A}_0$. Without confusing the reader, we use $|\cdot|$ for the number of elements of a finite set. We define the exponential mapping $\exp_{\mathcal{A}} : \mathcal{A}_0 \to \mathcal{A}_0 + 1_{\mathcal{A}}$ by

$$\exp_{\mathcal{A}} f = 1_{\mathcal{A}} + f + \tfrac{1}{2} f^{*2} + \cdots + \tfrac{1}{n!} f^{*n}. \tag{2.19}$$

Let Φ and Ψ be the functions defined by

$$\begin{aligned} \Phi(I) &= \sum_{G \in \mathcal{C}(I)} \prod_{\{i,j\} \in G} q_{ij}, \\ \Psi(I) &= \prod_{ij \in I, i<j} (1 + q_{ij}) = \sum_{G \in \mathcal{G}(I)} \prod_{\{i,j\} \in G} q_{ij}. \end{aligned} \tag{2.20}$$

Here, $\mathcal{G}(I)$ (resp. $\mathcal{C}(I)$) is the set of graphs (resp. connected graphs) on I. All graphs below are unoriented simple graphs (i.e. without loop and with at most one edge between two different vertices). By definition $\Phi \in \mathcal{A}_0$ and we have the relation

$$\Psi = \exp_{\mathcal{A}} \Phi. \tag{2.21}$$

We also introduce an operation that is reminiscent of differentiation:

$$D_J f(I) = \begin{cases} f(I \cup J) & \text{if } I \cap J = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{2.22}$$

This is a linear operator which satisfies

$$D_{\{i\}}(f * g) = D_{\{i\}} f * g + f * D_{\{i\}} g, \tag{2.23}$$

therefore also

$$D_{\{i\}} \exp_{\mathcal{A}} f = \exp_{\mathcal{A}} f * D_{\{i\}} f. \tag{2.24}$$

For disjoint $I, J \subset \{1, \dots, n\}$, we define

$$g_I(J) = \Big(\Psi^{*(-1)} * D_I \Psi\Big)(J). \tag{2.25}$$

Since $\Psi \in 1_{\mathcal{A}} + \mathcal{A}_0$ it has unique inverse $\Psi^{*(-1)}$ that can be proved easily by recursion.

Let $I \subset \{1, \dots, n\}$. The assumption of Proposition 2.4 implies that

$$\prod_{i \in I} \prod_{j \in I \setminus \{i\}} |1 + q_{ij}| \le \prod_{i \in I} e^{2b_i}. \tag{2.26}$$

Then there exists $i \in I$ such that

$$\prod_{j \in I \setminus \{i\}} |1 + q_{ij}| \le e^{2b_i}. \tag{2.27}$$

Such i is not unique in general. We consider a function ι that assigns the minimal of the indices $i = \iota(I)$ above to each nonempty subset $I \subset \{1, \dots, n\}$. Notice that $\iota(I) \in I$ for any subset I. It is also useful to introduce the notation $I' = I \setminus \{\iota(I)\}$.

Lemma 2.5 . *The function* $g_I(J)$ *of Eq.* (2.25) *is solution of the following equation.*

$$\begin{cases} g_\emptyset(J) = \delta_{\emptyset,J}, \\ g_I(J) = \Big(\prod_{i\in I'}(1+q_{i,\iota(I)})\Big) \sum_{K\subset J} \Big(\prod_{i\in K} q_{j,\iota(I)}\Big) g_{I'\cup K}(J\setminus K) \quad \text{if } I\neq\emptyset. \end{cases}$$

Since the equation gives $g_I(J)$ in terms of $g_K(J)$ with $|K|+|L| = |I|+|J|-1$, it is well defined inductively and it has a unique solution. Notice that $g_\emptyset(\emptyset) = 1$, and also that $g_i(\emptyset) = 1$ for any index i.

Proof. Recall the definition (2.20) of Ψ. For disjoint I, K we have

$$\begin{aligned} \Psi(I,K) &= \Big(\prod_{j\in I'\setminus K}(1+q_{j,\iota(I)})\Big)\Psi(I'\cup K) \\ &= \Big(\prod_{j\in I'}(1+q_{j,\iota(I)})\Big)\Big(\sum_{L\subset K}\prod_{k\in L} q_{k,\iota(I)}\Big)\Psi(I'\cup K). \end{aligned} \tag{2.28}$$

Then

$$\begin{aligned} g_I(J) &= \sum_{K\subset J}\Psi^{*(-1)}(J\setminus K)\Psi(I\cup K) \\ &= \Big(\prod_{j\in I'}(1+q_{j,\iota(I)})\Big)\sum_{L\subset K\subset J}\Big(\prod_{k\in L} q_{k,\iota(I)}\Big)\Psi^{*(-1)}(J\setminus K)\Psi(I'\cup K) \\ &= \Big(\prod_{j\in I'}(1+q_{j,\iota(I)})\Big)\sum_{L\subset J}\Big(\prod_{k\in L} q_{k,\iota(I)}\Big)\sum_{K'\subset J\setminus L}\Psi^{*(-1)}(J\setminus L\setminus K')\Psi(I'\cup L\cup K'). \end{aligned} \tag{2.29}$$

The last sum is equal to $g_{I'\cup L}(J\setminus L)$. One recognizes the equation of Lemma 2.5.

We now estimate the function g using another function h that satisfies an equation that is similar to that of Lemma 2.5.

$$\begin{cases} h_\emptyset(J) = \delta_{\emptyset,J}, \\ h_I(J) = e^{2b_{\iota(I)}} \sum_{K\subset J}\Big(\prod_{i\in K}|q_{i,\iota(I)}|\Big) h_{I'\cup K} J\setminus K) \quad \text{if } I\neq\emptyset. \end{cases} \tag{2.30}$$

It also has a unique solution. Since $\prod_{i\in I'}|1+q_{i,\iota(I)}| \le e^{b_{\iota(I)}}$, we can check inductively that

$$|g_I(J)| \le h_I(J) \tag{2.31}$$

for any sets I, J (with $I\cap J = \emptyset$). Now the function h can be written explicitly [71, 96]. Let $\mathcal{F}_I(J)$ be the set of forests on $I\cup J$ rooted in I. That is, a graph $G\in\mathcal{F}_I(J)$ is a forest such that each tree contains exactly one element of I, the root of the tree.

Lemma 2.6 . *The solution of Eq.* (2.30) *is*

$$h_I(J) = \Big(\prod_{i\in I\cup J} e^{2b_i}\Big)\sum_{G\in\mathcal{F}_I(J)}\prod_{\{i,j\}}|q_{ij}|.$$

Proof. Since the solution to Eq. (2.30) is unique, it is enough to check that the Ansatz of the lemma satisfies the equation. First, let us observe that both sides are multiplied by $\prod_{i\in I\cup J} e^{b_i}$. Thus it is enough to consider the case $b_i \equiv 0$.

To each forest $G \in \mathcal{F}_{I'\cup K}(J\setminus K)$ we associate a forest $\widetilde{G} \in \mathcal{F}_I(J)$ by adding a new vertex $\iota(I)$ to the set of vertices of G and edges $\{\{\iota(I), j\},\ j \in K\}$ to the set of edges of G. We denote the set of all such forests $\widetilde{G}$ by $\mathcal{F}_I(J \mid K)$. Notice that $\mathcal{F}_I(J \mid K_1) \bigcap \mathcal{F}_I(J \mid K_2) = \emptyset$ if $K_1 \neq K_2$, and $\bigcup_{K\subset J} \mathcal{F}_I(J \mid K) = \mathcal{F}_I(J)$. Therefore the sum over graphs in $\mathcal{F}_I(J)$ can be realized by first summing over the set K of indices (necessarily in J) that are connected to $\iota(I)$; then over sets of trees in $J\setminus K$, and over connections to $I'\cup K$. Explicitly,

$$\sum_{G\in\mathcal{F}_I(J)} \prod_{\{i,j\}} |q_{ij}| = \sum_{K\subset J} \Big(\prod_{i\in K} |q_{i,\iota(I)}|\Big) \sum_{G\in\mathcal{F}_{I'\cup K}(J\setminus K)} \prod_{\{i,j\}} |q_{ij}|. \tag{2.32}$$

This equation is precisely (2.30).

Proof of Proposition 2.4, bound (a). When I has a single element, by (2.21) and (2.24) the function g is equal to

$$\begin{aligned} g_1(2,\dots,n) &= \Psi^{*(-1)} * D_1\Psi(2,\dots,n) = \Psi^{*(-1)} * D_1 \exp_{\mathcal{A}} \\ &\cdot \Phi(2,\dots,n) = \Psi^{*(-1)} * \exp_{\mathcal{A}} \Phi * D_1\Phi(2,\dots,n) \\ &= D_1\Phi(2,\dots,n) = \sum_{G\in\mathcal{C}_n} \prod_{\{i,j\}} q_{ij}. \end{aligned} \tag{2.33}$$

This the left side of Proposition 2.4 (a). We have $\mathcal{F}_{\{1\}}(\{2,\dots,n\}) = \mathcal{T}_n$, the set of trees with n vertices. Thus $h_1(2,\dots,n)$ is equal to the right side of Proposition 2.4 (a), and the proof follows from Eq. (2.31).

The proof of Proposition 2.4 (b), in the absence of "hard core", i.e., when $q \neq -1$, follows from a tree identity due to Brydges, Battle and Federbush [17, 10, 16]:

$$\sum_{G\in\mathcal{C}_n} \prod_{\{i,j\}\in\gamma} (e^{-u_{i,j}} - 1) = \sum_{G\in\mathcal{T}_n} \prod_{\{i,j\}\in G} (-u_{i,j}) \int d\lambda_G(\{s_{ij}\}) \exp\left(-\sum_{i<j} s_{ij}u_{i,j}\right)$$

The full definition of the measure λ_G can be found in Ref.[16]. Here we only mention that $s_{ij} = s_{ji},\ 0 \le s_{ij} \le 1,\ \lambda_G$ depends on the tree G and is a probability measure supported on the set $s_{i,j}, 1 \le i < j \le n$, such that

$$\sum_{i<j} s_{ij} \mathrm{Re}\, u_{i,j} \ge -\sum_{i=1}^{n} b_i. \tag{A.6}$$

The extension to case where $\mathrm{Re}\, u_{i,j} = +\infty$ can be obtained using a trick due to Procacci [102]. For the proof see [101].

2.3 Proofs of the theorems

In this section we prove the theorems of Section 2.1. We associate to each $i \in \{1, \cdots, n\}$ a variable $x_i \in \mathbb{X}$ and consider only the case where Assumption 2 holds true — the case with Assumption 2' is entirely the same, one only needs to replace all $|q(x, y)|$ with $|u(x, y)|$ and all $e^{2b(\cdot)}$ with $e^{b(\cdot)}$. The proofs are based on the following tree estimate, which is a direct consequence of Proposition 2.4: for almost all $x_1, \ldots, x_n \in \mathbb{X}$,

$$|\varphi(x_1, \ldots, x_n)| \le \prod_{i=1}^{n} e^{2b(x_i)} \sum_{G \in \mathcal{T}_n} \prod_{\{i,j\} \in G} |q(x_i, x_j)|. \tag{2.34}$$

Proof of Theorem 2.1. We start by proving the bound (2.8). Let us introduce

$$\begin{aligned} K_N(x_1) &= \sum_{n=1}^{N} \frac{1}{(n-1)!} \int \mathrm{d}|\mu|(x_2) \ldots \int \mathrm{d}|\mu|(x_n) \prod_{i=1}^{n} e^{2b(x_i)} \sum_{G \in \mathcal{T}_n} \prod_{\{i,j\} \in G} |q(x_i, x_j)|, \\ K(x) &= \lim_{N \to \infty} K_N(x). \end{aligned} \tag{2.35}$$

(The term $n = 1$ is equal to $e^{2b(x_1)}$ by definition.) We show by induction that

$$K_N(x) \le e^{a(x) + 2b(x)} \tag{2.36}$$

for any N. Then $K(x) \le e^{a(x)+2b(x)}$ for almost all x, and using (2.34) we get (2.8).

The case $N = 1$ reduces to $1 \le e^{a(x)}$ and it is clear. To perform an induction in n we associate to each tree $G \in \mathcal{T}_n$ a possibly disconnected graph G' with $V(G') = \{2, \cdots, n\}$ in the following way: we delete all edges $\{1, j\}$ of the tree G with one endpoint on 1. After that operation we get a forest G' since each connected component of G' is a tree. Let $G_1, \cdots, G_m$ be the trees of the forest G'. Then setting $V_i = V(G_i)$ we have $V_1 \cup \cdots \cup V_m = \{2, \cdots, n\}$ and $V_i \cap V_j = \emptyset$ if $i \neq j$. Thus by this operation to every tree $G \in \mathcal{T}_n$ is associated a unique partition. And vice versa taking a partition $V_1 \cup \cdots \cup V_m$ of $\{2, \cdots, n\}$ and any collection of trees $G_1, \ldots, G_m, G_k \in \mathcal{T}(V_k)$ by adding a set of edges $\{1, j_k\}$, $j_k \in V_k$, $k = 1, \cdots, m$, we get a tree on $\{1, \cdots, n\}$. Here, $\mathcal{T}(V)$ denote the set of trees with V as the set of vertices. Therefore the sum over trees with n vertices can be written as a sum over forests on $\{2, \ldots, n\}$, and a sum over edges between 1 and each tree of the forest. In its turn the sum over forests on $\{2, \ldots, n\}$ can be written as a sum over all partitions $\{V_1, \cdots, V_m,\ m = 1, 2, \cdots\}$ of $\{2, \ldots, n\}$ and a sum over trees on each V_i. Explicitly,

$$\begin{aligned} K_N(x_1) = \sum_{n=1}^{N} \frac{1}{(n-1)!} \sum_{m \ge 1} \sum_{\substack{\{V_1, \ldots, V_m\} \\ \text{partition of } \{2, \ldots, n\}}} \int \mathrm{d}|\mu|(x_2) \ldots \int \mathrm{d}|\mu|(x_n) \\ \prod_{i=1}^{n} e^{2b(x_i)} \prod_{k=1}^{m} \Bigl(\sum_{\ell \in V_k} |q(x_1, x_\ell)| \sum_{G \in \mathcal{T}(V_k)} \prod_{\{i,j\} \in G} |q(x_i, x_j)| \Bigr). \end{aligned} \tag{2.37}$$

If $|V_k| = 1$ the sum over $G \in \mathcal{T}(V_k)$ is 1 by definition. The term after the sum over partitions depends on the cardinalities of the V_k's, but not on the actual labeling. Also, each $\ell \in V_k$ gives the same contribution. Therefore, taking into account that there are $\frac{(n-1)!}{n_1!\cdots n_m!}$ partitions of the set $\{2, \ldots, n\}$ into subsets $\{V_1, \ldots, V_m\}$ with fixed $|V_i| = n_i,\ i = 1, \cdots, m$, we get

$$K_N(x_1) = e^{2b(x_1)} \sum_{n=1}^{N} \sum_{m=1}^{N-1} \frac{1}{m!} \sum_{\substack{n_1,\ldots,n_m \geq 1 \\ n_1+\cdots+n_m = n-1}} \prod_{k=1}^{m} \biggl(\frac{1}{(n_k-1)!} \\ \cdot \int \mathrm{d}|\mu|(y_1) \ldots \int \mathrm{d}|\mu|(y_{n_k})\, |q(x_1, y_1)| \prod_{i=1}^{n_k} e^{2b(y_i)} \sum_{G \in \mathcal{T}_{n_k}} \prod_{\{i,j\} \in G} |q(y_i, y_j)| \biggr). \quad (2.38)$$

We obtain an upper bound by releasing the constraint $n_1 + \cdots + n_m \leq N - 1$ to $n_k \leq N - 1$, $1 \leq k \leq m$. We then get

$$\begin{aligned} K_N(x_1) &\leq e^{2b(x_1)} \exp\biggl\{ \sum_{n=1}^{N-1} \frac{1}{(n-1)!} \int \mathrm{d}|\mu(y_1) \ldots \int \mathrm{d}|\mu|(y_n)\, |q(x_1, y_1)| \\ &\qquad \cdot \prod_{i=1}^{n} e^{2b(y_i)} \sum_{G \in \mathcal{T}_n} \prod_{\{i,j\} \in G} |q(y_i, y_j)| \biggr\} \\ &= e^{2b(x_1)} \exp\biggl\{ \int \mathrm{d}|\mu|(y_1)\, |q(x_1, y_1)| K_{N-1}(y_1) \biggr\}. \end{aligned} \quad (2.39)$$

We have $K_{N-1}(y_1) \leq e^{a(y_1)+2b(y_1)}$ by the induction hypothesis. Eq. (2.35) follows from Assumption 2.

The rest of the proof is standard combinatorics. The partition function can be expanded so as to recognize the exponential of connected graphs. Namely, we start with

$$Z = 1 + \sum_{n \geq 1} \frac{1}{n!} \int \mathrm{d}\mu(x_1) \ldots \int \mathrm{d}\mu(x_n) \sum_{G \in \mathcal{G}_n} \prod_{\{i,j\}} q(x_i, x_j). \quad (2.40)$$

The graph $G \in \mathcal{G}_n$ can be decomposed into k connected graphs $G_1, \cdots, G_k$ (which are the connected components of G) whose sets of vertices $V_1, \cdots, V_k$ form a partition of $\{1, \ldots, n\}$:

As before, the contributions of the graphs $G_1, \cdots, G_k$ depend only on the cardinalities of the V_k's, hence summing first over the number m_i of vertices for each set of the partition, we get

$$\begin{aligned} Z &= 1 + \sum_{n \geq 1} \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{m_1,\ldots,m_k \geq 1 \\ m_1+\cdots+m_k = n}} \frac{1}{m_1! \ldots m_k!} \prod_{\ell=1}^{k} \biggl\{ \int \mathrm{d}\mu(x_1) \ldots \int \mathrm{d}\mu(x_{m_\ell}) \\ &\quad \cdot \sum_{G \in \mathcal{C}_{m_\ell}} \prod_{\{i,j\}} q(x_i, x_j) \biggr\} = 1 + \sum_{n \geq 1} \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{m_1,\ldots,m_k \geq 1 \\ m_1+\cdots+m_k = n}} \prod_{\ell=1}^{k} \\ &\quad \cdot \biggl\{ \frac{1}{m_\ell!} \int \mathrm{d}\mu(x_1) \ldots \int \mathrm{d}\mu(x_{m_\ell}) \varphi(x_1, \ldots, x_{m_\ell}) \biggr\}. \end{aligned} \quad (2.41)$$

The triple sum is absolutely convergent thanks to the estimate (2.8) that we have just established and the condition $\int \mathrm{d}|\mu|(y)|e^{a(y)+2b(y)} < \infty$ of Theorem 2.1. One can then interchange the sums by the dominated convergence theorem. This removes the sum over n, and this completes the proof of Theorem 2.1.

Proof of Theorem 2.2. We first prove that the cluster function $\sigma(x_1, x_2)$ given by (2.12) is correctly defined. With the help of (3.8) - (3.10) we have

$$\begin{aligned} D_{i_1,i_2}\Psi = D_{i_1}D_{i_2}\exp_{\mathcal{A}}\Phi = D_{i_1}[\exp_{\mathcal{A}}\Phi * D_{i_2}\Phi] \\ = \exp_{\mathcal{A}}\Phi * [D_{i_1}\Phi * D_{i_2}\Phi + D_{i_1,i_2}\Phi] \end{aligned} \tag{2.42}$$

Using $\Psi = \exp_{\mathcal{A}}\Phi$ and the definition (3.11) we get

$$g_{i_1,i_2} = \Psi^{*(-1)} * D_{i_1,i_2}\Psi = D_{i_1}\Phi * D_{i_2}\Phi + D_{i_1,i_2}\Phi \tag{2.43}$$

It follows from (4.10) that

$$g_{x_1,x_2}(x_3,\cdots,x_n) = D_{x_1}\varphi * D_{x_2}\varphi(x_3,\cdots,x_n) + D_{x_1,x_2}\varphi(x_3,\cdots,x_n). \tag{2.44}$$

Hence

$$|D_{x_1,x_2}\varphi(x_3,\cdots,x_n)| \le |g_{x_1,x_2}(x_3,\cdots,x_n)| + |D_{x_1}\varphi * D_{x_2}\varphi(x_3,\cdots,x_n)|. \tag{2.45}$$

This implies

$$\begin{aligned} |\sigma(x_1,x_2)| \le \sum_{n=2}^{\infty}\frac{1}{(n-2)!}\int \mathrm{d}|\mu|(x_3)\ldots\int \mathrm{d}|\mu|(x_n)|D_{x_1,x_2}\varphi(x_3,\cdots,x_n)| \\ \le \sum_{n=2}^{\infty}\frac{1}{(n-2)!}\int \mathrm{d}|\mu|(x_3)\ldots\int \mathrm{d}|\mu|(x_n)||g_{x_1,x_2}(x_3,\cdots,x_n)| \\ +\sum_{n=2}^{\infty}\frac{1}{(n-2)!}\int \mathrm{d}|\mu|(x_3)\ldots\int \mathrm{d}|\mu|(x_n)||D_{x_1}\varphi * D_{x_2}\varphi(x_3,\cdots,x_n)| = S_1 + S_2 \end{aligned} \tag{2.46}$$

By formula (2.31) $|g_{x_1,x_2}(x_3,\cdots,x_n)| \le h_{x_1,x_2}(x_3,\cdots,x_n)$. Therefore with the help of Lemma 3.4, definition of $K(x)$ given by (4.2) and the bound (4.3) we can write

$$\begin{aligned} S_1 \le \sum_{n=1}^{\infty}\frac{1}{(n-2)!}\int \mathrm{d}|\mu|(x_3)\ldots\int \mathrm{d}|\mu|(x_n)\prod_{i=1}^{n}e^{2b(x_i)}\sum_{G\in\mathcal{F}_{(x_1,x_2)}(x_3,\cdots,x_n)} \\ \cdot\prod_{\{i,j\}\in G}|q(x_i,x_j)| = \sum_{n=2}^{\infty}\frac{1}{(n-2)!}\int \mathrm{d}|\mu|(x_3)\ldots\int \mathrm{d}|\mu|(x_n)\prod_{i=1}^{n}e^{2b(x_i)} \\ \cdot\sum_{G_1\in\mathcal{T}_1(V_1)}\prod_{\{i,j\}\in G_1}|q(x_i,x_j)|\sum_{G_2\in\mathcal{T}_2(V_2)}\prod_{\{i,j\}\in G_2}|q(x_i,x_j)| = \end{aligned}$$

$$= \sum_{n=2}^{\infty} \sum_{(n_1,n_2):n_1+n_2=n-2} \Big(\frac{1}{n_1!} \prod_{i=1}^{n_1+1} e^{2b(x_i)} \sum_{G_1 \in \mathcal{T}(x_1,x_2,\cdots x_{n_1+1})} \int \mathrm{d}|\mu|(x_2) \dots$$

$$\cdot \int \mathrm{d}|\mu|(x_{n_1+1})\Big)\Big(\frac{1}{n_2!} \prod_{i=2}^{n_2+2} e^{2b(x_i)} \sum_{G_2 \in \mathcal{T}(x_2,x_3,\cdots x_{n_2+2})} \int \mathrm{d}|\mu|(x_3)$$

$$\dots \int \mathrm{d}|\mu|(x_{n_2+2}) \prod_{\{i,j\} \in G_1} |q(x_i,x_j)|\Big) = K(x_1)K(x_2) \leq$$

$$e^{a(x_1)+2b(x_1)} e^{a(x_2)+2b(x_2)}. \tag{2.47}$$

Now we prove the absolute convergence of the series S_2. With the help of (2.8) we have

$$S_2 \leq \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \int \mathrm{d}|\mu|(x_3) \dots \int \mathrm{d}|\mu|(x_n) |D_{x_1}\varphi * D_{x_2}\varphi(x_3, \cdots, x_n)|$$

$$= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \sum_{k=0}^{n-2} \sum_{J \subset \{3,\cdots,x_n\};|J|=k} \int \prod_{j \in J} \mathrm{d}|\mu|(x_j) |D_{x_1}\varphi(x_j, j \in J)|$$

$$\cdot \int \prod_{k \in \{3,\cdots,n\} \setminus J} \mathrm{d}|\mu|(x_k) |D_{x_2}\varphi(x_k, k \in \{3, \cdots, n\} \setminus J)| =$$

$$= \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \frac{1}{k!} \frac{1}{(n-2-k)!} \sum_{J \subset \{3,\cdots,x_n\};|J|=k} \int \prod_{j \in J} \mathrm{d}|\mu|(x_j) |D_{x_1}\varphi(x_j, j \in J)| \tag{2.48}$$

$$\cdot \int \prod_{k \in \{3,\cdots,n\} \setminus J} \mathrm{d}|\mu|(x_k) |D_{x_2}\varphi(x_k, k \in \{3, \cdots, n\} \setminus J)|$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \int \mathrm{d}|\mu|(y_1) \dots \int \mathrm{d}|\mu|(y_k) |D_{x_1}\varphi(y_1, \cdots, y_k))| \sum_{l=0}^{\infty} \frac{1}{l!} \int \mathrm{d}|\mu|(y_1) \dots$$

$$\cdot \int \mathrm{d}|\mu|(y_l) |D_{x_2}\varphi(y_1, \cdots, y_l))| \leq e^{a(x_1)+2b(x_1)+a(x_2)+2b(x_2)}$$

Note that interchanging of the sums $\sum_{n=2}^{\infty} \sum_{k=0}^{n-2}$ is justified because the resulting series both are absolutely convergent

Thus combining (4.13) - (4.15) we complete the proof of Theorem 2.2.

Proof of Theorem 2.3. The correlation function $\varsigma(x)$ can be expanded as a sum over graphs, that can be decomposed into a connected graph that contains 1, and other connected graphs. Taking into account the combinatorial factors, the contribution of connected graphs containing 1 yields $\sigma(x)$, and the contribution of the others yields the expression (2.7) for Z. One step involved interchanging unbounded sums, which is justified because everything is absolutely convergent, thanks to (2.8). Thus $\varsigma(x) = \sigma(x)Z$.

$$\varsigma(x_1) = \sum_{n\geq 1} \frac{1}{(n-1)!} \int \prod_{i=2}^{n} \mathrm{d}\mu(x_i) \sum_{G\in\mathcal{G}_n} \prod_{\{i,j\}\in G_1} q(x_i,x_j)$$

$$= \sum_{n\geq 1} \frac{1}{(n-1)!} \Bigg\{ \sum_{p=0}^{n-2} \sum_{\substack{W\subset\{2,\cdots,n\}\\ |W|=p}} \Big(\int \prod_{x\in W} \mathrm{d}\mu(x) \sum_{G\in\mathcal{C}(W\cup\{x_1\})} \prod_{\{i,j\}\in G} q(x_i,x_j) \Big)$$

$$\sum_{k=1}^{n-1-p} \frac{1}{k!} \sum_{\substack{n_1,\cdots,n_k\\ n_1+\cdots+n_k=n-1-p}} \sum_{\substack{\{V_1,\dots,V_k\}:|V_\ell|=n_\ell,\\ \text{partition of } \{2,\dots,n\}\setminus W}} \prod_{\ell=1}^{k} \sum_{G_\ell\in\mathcal{C}(V_\ell)} \int \prod_{x\in V_\ell} \mathrm{d}\mu(x)$$

$$\cdot \prod_{\{i,j\}\in G_\ell} q(x_i,x_j) + \int \prod_{i=2} \mathrm{d}\mu(x_i) \sum_{G\in\mathcal{C}(\{x_1,\cdots,x_n\})} \prod_{\{i,j\}\in G} q(x_i,x_j) \Bigg\} =$$

$$= \sum_{n\geq 1} \sum_{p=0}^{n-2} \frac{1}{p!} \Big(\int \prod_{i=2}^{p+1} \mathrm{d}\mu(x_i) \sum_{G\in\mathcal{C}_{(p+1)}} \prod_{\{i,j\}\in G} q(x_i,x_j) \Big) \sum_{k=1}^{n-1-p} \frac{1}{k!}$$

$$\cdot \sum_{\substack{n_1,\cdots,n_k\\ n_1+\cdots+n_k=n-1-p}} \frac{1}{n_1!\cdots n_k!} \prod_{\ell=1}^{k} \int \prod_{j=1}^{n_\ell} \mathrm{d}\mu(y_j) \sum_{G_\ell\in\mathcal{C}_{n_\ell}} \prod_{\{i,j\}\in G_\ell} q(x_i,x_j)$$

$$+ \sum_{n\geq 1} \frac{1}{(n-1)!} \int \prod_{i=2}^{n} \mathrm{d}\mu(x_i) \varphi(x_1,\dots,x_n) = \sum_{p=0}^{\infty} \frac{1}{p!} \Big(\int \prod_{i=2}^{p} \mathrm{d}\mu(x_i)$$

$$\cdot \varphi(x_1,\dots,x_{p+1}) \Big) \sum_{n\geq p+2} \sum_{k=1}^{n-1-p} \frac{1}{k!} \sum_{\substack{n_1,\cdots,n_k\\ n_1+\cdots+n_k=n-1-p}} \frac{1}{n_1!\cdots n_k!}$$

$$\cdot \prod_{\ell=1}^{k} \int \prod_{j=1}^{n_\ell} \mathrm{d}\mu(y_j) \varphi(y_1,\cdots,y_{n_l}) + \sigma(x_1) = \sigma(x_1) \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{\ell=1}^{k}$$

$$\Big(\sum_{n_\ell=1}^{\infty} \frac{1}{n_\ell!} \int \prod_{j=1}^{n_\ell} \mathrm{d}\mu(y_j) \varphi(y_1,\cdots,y_{n_\ell}) \Big) + \sigma(x_1) = \sigma(x_1)\, Z$$

The interchanging of unbounded sums is justified because everything is absolutely convergent, thanks to (2.8).

The first equation of (2.14) is proved. We now pass to the proof of the second equation. As usual we start by decomposing graphs $G \in \mathcal{G}_n$, the terms where 1 and 2 belong to the same connected graph yield $\sigma(x,y)\, Z$ for the same reasons as above. The terms where 1 and 2 belong to different connected graphs yield $\sigma(x)\sigma(y)\, Z$.

$$\varsigma(x_1,x_2) = \sum_{n\geq 2} \frac{1}{(n-2)!} \int \prod_{i=3}^{n} \mathrm{d}\mu(x_i) \sum_{G\in\mathcal{G}_n} \prod_{\{i,j\}\in G_1} q(x_i,x_j) =$$

$$= \sum_{n\ge 2} \frac{1}{(n-2)!} \Bigg\{ \sum_{p=0}^{n-3} \sum_{\substack{W\subset\{3,\cdots,n\}\\ |W|=p}} \Big(\int \prod_{x\in W} \mathrm{d}\mu(x) \sum_{G\in\mathcal{C}(W\cup\{x_1,x_2\})} \prod_{\{i,j\}\in G}$$

$$\cdot\, q(x_i,x_j) \Big) \sum_{k=1}^{n-2-p} \frac{1}{k!} \sum_{\substack{n_1,\cdots,n_k\\ n_1+\cdots+n_k=n-2-p}} \sum_{\substack{\{V_1,\ldots,V_k\}:|V_\ell|=n_\ell,\\ \text{partition of } \{3,\ldots,n\}\setminus W}} \prod_{\ell=1}^{k} \sum_{G_\ell\in\mathcal{C}(V_\ell)}$$

$$\cdot \int \prod_{x\in V_\ell} \mathrm{d}\mu(x) \prod_{\{i,j\}\in G_\ell} q(x_i,x_j) + \int \prod_{i=3} \mathrm{d}\mu(x_i)\varphi(x_1,\cdots,x_n) \Bigg\}$$

$$+ \sum_{n\ge 2} \frac{1}{(n-2)!} \Bigg\{ \sum_{p_1=0}^{n-3} \sum_{\substack{W_1\subset\{3,\cdots,n\}\\ |W_1|=p_1}} \int \prod_{x\in W_1} \mathrm{d}\mu(x) \sum_{G\in\mathcal{C}(W\cup\{x_1\})} \prod_{\{i,j\}\in G}$$

$$\cdot\, q(x_i,x_j) \sum_{p_2=0}^{n-3-p_1} \sum_{\substack{W_2\subset\{3,\cdots,n\}\setminus W_1\\ |W_2|=p_2}} \int \prod_{x\in W_2} \mathrm{d}\mu(x) \sum_{G\in\mathcal{C}(W_2\cup\{x_2\})} \prod_{\{i,j\}\in G}$$

$$\cdot\, q(x_i,x_j) \Bigg\} \Bigg[\sum_{k=1}^{n-(3+p_1+p_2)} \frac{1}{k!} \sum_{\substack{n_1,\cdots,n_k\\ n_1+\cdots+n_k=n-(3+p_1+p_2)}} \sum_{\substack{\{V_1,\ldots,V_k\}:|V_\ell|=n_\ell,\\ \text{partition of } \{3,\ldots,n\}\setminus(W_1\cup W_2)}}$$

$$\cdot \prod_{\ell=1}^{k} \sum_{G_\ell\in\mathcal{C}(V_\ell)} \int \prod_{x\in V_\ell} \mathrm{d}\mu(x) \prod_{\{i,j\}\in G_\ell} q(x_i,x_j) \Bigg] = \varsigma^A + \varsigma^B. \tag{2.49}$$

Consider first ς^A. We have

$$\varsigma^A = \sum_{n\ge 2} \sum_{p=0}^{n-3} \frac{1}{p!} \int \prod_{i=1}^{p} \mathrm{d}\mu(y_i)\varphi(x_1,x_2,y_1,\cdots,y_p)$$

$$\sum_{k=1}^{n-2-p} \frac{1}{k!} \sum_{\substack{n_1,\cdots,n_k\\ n_1+\cdots+n_k=n-2-p}} \frac{1}{n_1!\cdots n_k!} \prod_{\ell=1}^{k} \int \prod_{j=1}^{\ell} \mathrm{d}\mu(y_j)\varphi(y_1\cdots y_{n_l}) +$$

$$+ \sum_{n\ge 2} \frac{1}{(n-2)!} \int \prod_{i=3}^{n} \mathrm{d}\mu(x_i)\varphi(x_1,\cdots,x_n) = \sum_{p=0}^{\infty} \frac{1}{p!} \int \prod_{i=1}^{p} \mathrm{d}\mu(y_i)$$

$$\cdot\, \varphi(x_1,x_2,y_1,\cdots,y_p) \sum_{n\ge p+3} \sum_{k=1}^{n-2-p} \frac{1}{k!} \sum_{\substack{n_1,\cdots,n_k\\ n_1+\cdots+n_k=n-2-p}} \frac{1}{n_1!\cdots n_k!}$$

$$\cdot \prod_{\ell=1}^{k} \int \prod_{j=1}^{\ell} \mathrm{d}\mu(y_j)\varphi(y_1\cdots y_{n_\ell}) + \sigma(x_1,x_2) = \sigma(x_1,x_2)\cdot \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{\ell=1}^{k} \sum_{n_\ell=1}^{\infty} \frac{1}{n_\ell!}$$

$$\cdot \int \prod_{j=1}^{\ell} \mathrm{d}\mu(y_j)\varphi(y_1\cdots y_{n_l}) + \sigma(x_1,x_2) = \sigma(x_1,x_2) Z. \tag{2.50}$$

Next

$$\varsigma^B = \sum_{n\geq 2} \frac{1}{(n-2)!}\Bigg\{\sum_{p=0}^{n-3} \sum_{\substack{W\subset\{3,\cdots,n\}\\ |W|=p}} \sum_{\substack{p_1,p_2\geq 0\\ p_1+p_2=p}} \sum_{\substack{W_1,W_2 \text{ partition of } W\\ |W_1|=p_1,|W_2|=p_2}} \Big(\int \prod_{x\in W_1} \mathrm{d}\mu(x)$$

$$\cdot \sum_{G\in\mathcal{C}(W_1\cup\{x_1\})} \prod_{\{i,j\}\in G} q(x_i,x_j)\Big)\Big(\int \prod_{x\in W_2} \mathrm{d}\mu(x) \sum_{G\in\mathcal{C}(W_2\cup\{x_2\})} \prod_{\{i,j\}\in G}$$

$$\cdot\, q(x_i,x_j)\Big) \sum_{k=1}^{n-2-p} \frac{1}{k!} \sum_{\substack{n_1,\cdots,n_k\\ n_1+\cdots+n_k=n-2-p}} \sum_{\substack{\{V_1,\ldots,V_k\}:|V_\ell|=n_\ell,\\ \text{partition of } \{3,\ldots,n\}\setminus W}} \prod_{\ell=1}^{k} \sum_{G_\ell\in\mathcal{C}(V_\ell)} \cdot$$

$$\cdot \int \prod_{x\in V_\ell} \mathrm{d}\mu(x) \prod_{\{i,j\}\in G_\ell} q(x_i,x_j)\Bigg\} + \sum_{n\geq 2} \frac{1}{(n-2)!} \sum_{\substack{p_1,p_2\geq 0\\ p_1+p_2=n-2}}$$

$$\cdot \sum_{\substack{W_1,W_2 \text{ partition of } \{3,\cdots,n\}\\ |W_1|=p_1,|W_2|=p_2}} \Big(\int \prod_{x\in W_1} \mathrm{d}\mu(x) \sum_{G\in\mathcal{C}(W_1\cup\{x_1\})} \prod_{\{i,j\}\in G} q(x_i,x_j)\Big)$$

$$\cdot \Big(\int \prod_{x\in W_2} \mathrm{d}\mu(x) \sum_{G\in\mathcal{C}(W_2\cup\{x_2\})} \prod_{\{i,j\}\in G} q(x_i,x_j)\Big) =$$

$$= \sum_{n\geq 2}\Bigg\{\sum_{p=0}^{n-3} \frac{1}{p!} \sum_{\substack{p_1,p_2\geq 0\\ p_1+p_2=p}} \frac{p!}{p_1!p_2!}\Big(\int \prod_{x\in W_1} \mathrm{d}\mu(x) \sum_{G\in\mathcal{C}(W_1\cup\{x_1\})} \prod_{\{i,j\}\in G} q(x_i,x_j)\Big)$$

$$\cdot \Big(\int \prod_{x\in W_2} \mathrm{d}\mu(x) \sum_{G\in\mathcal{C}(W_2\cup\{x_2\})} \prod_{\{i,j\}\in G} q(x_i,x_j)\Big)$$

$$\cdot \sum_{k=1}^{n-2-p} \frac{1}{k!} \sum_{\substack{n_1,\cdots,n_k\\ n_1+\cdots+n_k=n-2-p}} \frac{1}{n_1!\cdots n_k!} \prod_{\ell=1}^{k} \int \prod_{j=1}^{\ell} \mathrm{d}\mu(y_j)\varphi(y_1\cdots y_{n_l})$$

$$+ \sum_{n\geq 2} \frac{1}{(n-2)!} \sum_{\substack{p_1,p_2\geq 0\\ p_1+p_2=n-2}} \frac{(n-2)!}{p_1!p_2!} \int \prod_{i=1}^{p_1} \mathrm{d}\mu(y_i)\varphi(x_1,y_1,\cdots,y_{p_1})$$

$$\cdot \int \prod_{i=1}^{p_2} \mathrm{d}\mu(y_i)\varphi(x_2,y_1,\cdots,y_{p_2}) = \varsigma_1^B + \varsigma_2^B. \qquad (2.51)$$

Let us treat ς_1^B. We have

$$\varsigma_1^B = \sum_{n\geq 2}\sum_{p=0}^{n-3} \sum_{\substack{p_1,p_2\geq 0\\ p_1+p_2=p}} \frac{1}{p_1!p_2!} \int \prod_{i=1}^{p_1} \mathrm{d}\mu(y_i)\varphi(x_1,y_1\cdots y_{p_l}) \int \prod_{j=1}^{p_2} \mathrm{d}\mu(y_j)$$

$$\cdot\, \varphi(x_2,y_1\cdots y_{p_2}) \sum_{k=1}^{n-2-p} \frac{1}{k!} \sum_{\substack{n_1,\cdots,n_k\\ n_1+\cdots+n_k=n-2-p}} \frac{1}{n_1!\cdots n_k!} \prod_{\ell=1}^{k} \int \prod_{j=1}^{\ell} \mathrm{d}\mu(y_j)\cdot$$

$$\cdot\,\varphi(y_1\cdots y_{n_l}) = \sigma(x_1)\sigma(x_2)\sum_{m\geq 0}\sum_{k=1}^{m+1}\frac{1}{k!}\prod_{\ell=1}^{k}\frac{1}{n_\ell!}\int\prod_{j=1}^{\ell}\mathrm{d}\mu(y_j)\varphi(y_1\cdots y_{n_l})$$
$$= \sigma(x_1)\sigma(x_2)\sum_{k\geq 1}\frac{1}{k!}\left[\sum_{n\geq 1}\frac{1}{n!}\int\prod_{j=1}^{n}\mathrm{d}\mu(y_j)\varphi(y_1\cdots y_n)\right]^k. \tag{2.52}$$

Now consider ς_2^B. We have

$$\varsigma_2^B = \sum_{n\geq 2}\sum_{\substack{p_1,p_2\geq 0\\ p_1+p_2=n-2}}\frac{1}{p_1!p_2!}\int\prod_{i=1}^{p_1}\mathrm{d}\mu(y_i)\varphi(x_1,y_1,\cdots,y_{p_1})$$
$$\cdot\int\prod_{i=1}^{p_2}\mathrm{d}\mu(y_i)\varphi(x_2,y_1,\cdots,y_{p_2}) = \sigma(x_1)\sigma(x_2). \tag{2.53}$$

Hence

$$\varsigma^B = \sigma(x_1)\sigma(x_2)\sum_{k\geq 1}\frac{1}{k!}\left[\sum_{n\geq 1}\frac{1}{n!}\int\prod_{j=1}^{n}\mathrm{d}\mu(y_j)\varphi(y_1\cdots y_n)\right]^k + \sigma(x_1)\sigma(x_2)$$
$$= \sigma(x_1)\sigma(x_2)\exp\left[\sum_{n\geq 1}\frac{1}{n!}\int\prod_{j=1}^{n}\mathrm{d}\mu(y_j)\varphi(y_1\cdots y_n)\right] = \sigma(x_1)\sigma(x_2)\,Z. \tag{2.54}$$

Thus the second equation of (2.14) is proved. Combining (4.17), (4.18) and (4.22) we complete the proof of Theorem 2.3.

3 Decay of correlations

In this section we study the decay of correlations in quantum gases. We start with general results on decay of correlations in the frame of abstract approach. Then we discuss the Feynman-Kac representation of quantum gases as a model of interacting Brownian loops, introduce the space of finite configurations of loops with corresponding measure and write the grand partition function in terms of composite (winding) loops. In the last subsection we derive useful bounds for two-point truncated correlation functions which are the main technical tool for the analysis of the asymptotic expansion of the log- partition function of quantum gases.

3.1 Bound of abstract semiinvariant

In statistical mechanics the two-point truncated correlation function plays a special role. This function usually provides an order parameter for phase transitions and it is useful to estimate its decay properties.

Theorem 3.1 (Decay of correlations) . *If Assumptions 1 and 2 of Theorem 2.1 hold true, we have for almost all $x, y \in \mathbb{X}$,*

$$|\sigma(x,y)| \quad \leq \quad e^{a(y)+2b(y)} \sum_{m\geq 0} \int \mathrm{d}|\mu|(x_1)\dots \int \mathrm{d}|\mu|(x_m) \prod_{i=0}^{m} |q(x_i,x_{i+1})| e^{a(x_i)+2b(x_i)} \quad (3.1)$$

with $x_0 \equiv x$ and $x_{m+1} \equiv y$. The term of the series corresponding to $m = 0$ is $|q(x, y)|e^{a(x)+2b(x)}$ by definition. If Assumptions 1 and 2' hold true, we have the same bound but with $|u(\cdot, \cdot)|$ instead of $|q(\cdot, \cdot)|$, and $e^{b(\cdot)}$ instead of $e^{2b(\cdot)}$.

In typical situations the functions $q(x, y)$ and $u(x, y)$ depend on the difference $x - y$ (this assumes that $\mathbb{X}$ has additional structure, namely that of a group). The estimates for $|\sigma(x, y)|$ are given by convolutions.

Notice that Theorem 3.1 does not say anything about the convergence of the series on the right side of (2,13), it may equal to $+\infty$. In many applications there is a small parameter (usually it is activity z) which provides the absolute convergence of the right side of (5.1). One can say that Theorem 3.1 is the first step in obtaining finer estimates for the two-point truncated correlation functions of concrete systems . Theorems 3.2, 3.7, 3.8 and 3.9 below illustrate this argument.

Proof of Theorem 3.1. From the tree estimate 2.34, we have

$$\begin{aligned} |\sigma(x_1,x_2)| \leq & \sum_{n\geq 2} \frac{1}{(n-2)!} \int \mathrm{d}|\mu|(x_3)\dots \int \mathrm{d}|\mu|(x_n) \Big(\prod_{i=1}^{n} e^{2b(x_i)}\Big) \\ & \cdot \sum_{G\in\mathcal{T}_n} \prod_{\{i,j\}} |\zeta(x_i,x_j)|. \end{aligned} \quad (3.2)$$

The expression above involves a sum over trees $G \in \mathcal{T}_n$ of arbitrary size that connect 1 and 2. Any such tree decomposes into a path (line) $\tau(m) = (i_0, i_1, \cdots, i_m, i_{m+1})$, $i_0 \equiv 1, i_{m+1} \equiv 2$, of $m+1$ edges that connects 1 and 2, ($m + 1 \geq 1$ is called the length of the path) and $m+2$ trees rooted in the vertices of the connecting line. Let $\mathcal{T}_n(\tau(m))$ be the set of all trees $G \in \mathcal{T}_n$ which contain a fixed path $\tau(m)$ as a subgraph: $\tau(m) \subset G$. A graph G_1 we call call a subgraph of a graph G and write $G_1 \subset G$ if $V(G_1) \subset V(G)$ and $E(G_1) \subset E(G)$ where $E(G)$ is the set of edges of the graph G. It is clear that

$$\mathcal{T}_n = \bigcup_{m=0}^{n-2} {\bigcup}' \mathcal{T}_n(\tau(m))$$

where $\bigcup'$ is taken over all possible paths $(i_0, i_1, \cdots, i_m, i_{m+1})$ of length $m+1$ that connect 1 and 2. We note also that $\mathcal{T}_n(\tau(m_1)) \cap \mathcal{T}_n(\tau(m_2)) = \emptyset$ if $m_1 \neq m_2$ as well as $\mathcal{T}_n(\tau_1(m)) \cap \mathcal{T}_n(\tau_2(m)) = \emptyset$ for any two different paths $\tau_1(m)$ and $\tau_2(m)$ of the same length. Any $G \in \mathcal{T}_n(\tau(m))$ we decompose into $m+2$ trees $G_0, \cdots, G_{m+1}$, rooted respectively in the vertices $i_0, i_1, \cdots, i_m, i_{m+1}$, by deleting the $m+1$ edges of $\tau(m)$. The contribution of the trees $G_0, \cdots, G_{m+1}$ rooted in the vertices of the path $\tau(m)$ does not depend on the actual labeling. Therefore we have

$$\begin{aligned} |\sigma(x_1, x_2)| \leq & \sum_{n \geq 2} \frac{1}{(n-2)!} \int \mathrm{d}|\mu|(x_3) \ldots \int \mathrm{d}|\mu|(x_n) \Big(\prod_{i=1}^{n} e^{2b(x_i)} \Big) \sum_{G \in \mathcal{T}_n} \prod_{\{i,j\}} \\ & \cdot |q(x_i, x_j)| \leq \sum_{n \geq 2} \frac{1}{(n-2)!} \sum_{m=0}^{n-2} \sum_{(i_0, \cdots, i_{m+1})} \sum_{\substack{n_0, \cdots, n_{m+1} \geq 0 \\ n_0 + \cdots + n_{m+1} = n-m-2}} \\ & \cdot \frac{(n-m-2)!}{n_0! \cdots n_{m+1}!} \prod_{\ell=0}^{m+1} e^{2b(x_{i_\ell})} \int \mathrm{d}|\mu|(y_1) \ldots \int \mathrm{d}|\mu|(y_{n_\ell}) \prod_{i=1}^{n_\ell} e^{2b(y_i)} \\ & \cdot \sum_{G_\ell \in \mathcal{T}_{n_\ell + 1}} \prod_{\{i,j\}} |q(x_i, x_j)| \int \mathrm{d}|\mu|(x_{i_1}) \ldots \int \mathrm{d}|\mu|(x_{i_m}) \prod_{k=0}^{m} |q(x_{i_k}, x_{i_{k+1}})| \end{aligned} \tag{3.3}$$

Since the number of different paths of length $m+1$ is $\frac{(n-2)!}{(n-m-2)!}$, using (4.2) we find that

$$\begin{aligned} |\sigma(x_1, x_2)| \leq & \sum_{n \geq 2} \sum_{m=0}^{n-2} \sum_{\substack{n_0, \cdots, n_{m+1} \geq 0 \\ n_0 + \cdots + n_{m+1} = n-m-2}} \prod_{\ell=0}^{m+1} e^{2b(x_{i_\ell})} \frac{1}{n_\ell!} \int \mathrm{d}|\mu|(y_1) \ldots \int \mathrm{d}|\mu|(y_{n_\ell}) \\ & \cdot \prod_{i=1}^{n_\ell} e^{2b(y_i)} \sum_{G_\ell \in \mathcal{T}_{n_\ell + 1}} \prod_{\{i,j\}} |q(x_i, x_j)| \int \mathrm{d}|\mu|(x_{i_1}) \ldots \int \mathrm{d}|\mu|(x_{i_m}) \prod_{k=0}^{m} \\ & \cdot |q(x_{i_k}, x_{i_{k+1}})| \leq \sum_{m=0}^{\infty} \prod_{\ell=0}^{m+1} K(x_{i_\ell}) \int \mathrm{d}|\mu|(x_{i_1}) \ldots \int \mathrm{d}|\mu|(x_{i_m}) \\ & \cdot \prod_{k=0}^{m} |q(x_{i_k}, x_{i_{k+1}})| \end{aligned} \tag{3.4}$$

The result follows from the bound (2.36) for K. Theorem 3.1 is proved

To have an absolute convergence of the right hand side of the last inequality we replace Assumption 2 by the following assumption

Assumption 3. *There exists a nonnegative function a on $\mathbb{X}$ and a number p, $0 < p < 1$ such that for almost all $x \in \mathbb{X}$,*

$$\int \mathrm{d}|\mu|(y)\, |q(x,y)| e^{a(y)+2b(y)} \hat{a}(y) \le pa(x) \tag{3.5}$$

where $\hat{a}(y) = \max(a(y), 1)$.

Assumption 3' is defined as Assumption 3 but with $u(x,y)$ instead of $q(x,y)$ and with $e^b(.)$ instead of $e^{2b}(.)$

Theorem 3.2 (Integral bound) . *If Assumptions 1 and 3, or 1 and 3' hold true then for almost all $x \in \mathbb{X}$*

$$\int \mathrm{d}|\mu|(y)\, |\sigma(x,y)| \le e^{a(x)+2b(x)} a(x) \frac{p}{1-p}. \tag{3.6}$$

Proof of Theorem 3.2. For $m = 0, 1, \cdots$ we set

$$A_m(x) = \int_{\mathbb{X}} \mathrm{d}|\mu|(y) e^{a(y)+2b(y)} \int_{\mathbb{X}^m} \prod_{i=1}^{m} \mathrm{d}|\mu|(x_i) e^{a(x_i)+2b(x_i)} \prod_{i=0}^{m} |q(x_i, x_{i+1})| \tag{3.7}$$

where $x_0 = x, x_{m+1} = y$ and

$$A_0(x) = \int_{\mathbb{X}} \mathrm{d}|\mu|(y) e^{a(y)+2b(y)} |q(x,y)| \tag{3.8}$$

By Assumption 3

$$\begin{aligned} A_m(x) = &\int_{\mathbb{X}^m} \prod_{i=1}^{m} \mathrm{d}|\mu|(x_i) e^{a(x_i)+2b(x_i)} \prod_{i=0}^{m-1} |q(x_i, x_{i+1})| \int_{\mathbb{X}} \mathrm{d}|\mu|(y) e^{a(y)+2b(y)} \\ &\cdot |q(x_m, y)| \le p \int_{\mathbb{X}^{m-1}} \prod_{i=1}^{m-1} \mathrm{d}|\mu|(x_i) e^{a(x_i)+2b(x_i)} \prod_{i=0}^{m-2} |q(x_i, x_{i+1})| \\ &\cdot \int_{\mathbb{X}} \mathrm{d}|\mu|(x_m) e^{a(x_m)+2b(x_m)} a(x_m) |q(x_{m-1}, x_m)| \le \\ &\cdots \le p^{m+1} a(x) \end{aligned} \tag{3.9}$$

Then by Theorem 3.1, with the help of Monotone Convergence Theorem we get

$$\int \mathrm{d}|\mu|(y) |\sigma(x,y)| \le e^{a(x)+2b(x)} \sum_{m \ge 0} A_m(x) \le e^{a(x)+2b(x)} a(x) \frac{p}{1-p}.$$

This completes the proof of Theorem 3.2.

3.2 Feynman-Kac representation of the quantum gas

The remarkable *Feynman-Kac formula* [40, 46, 86] gives a convenient representation of the statistical operator $e^{-\beta H}$ where the Hamiltonian H of the system is the sum of the usual kinetic energy operator and of an interaction generated by a pair potential (see formula (3.17) below). Thus one can reduce the truly quantum mechanical problem involving unbounded noncommuting operators to a problem very similar to the classical one, involving only scalar functions.

Using Feynman-Kac representation Ginibre studied the reduced density matrices of quantum gases in his pioneering work [46].

This work is an excellent mathematical introduction to the Wiener measure, the Feynman-Kac formula, and its applications . See also [43, 104, 109] .

Consider a gas of N identical spinless particles in a bounded domain $\Lambda \subset \mathbb{R}^d$ interacting through a two-body potential ϕ. We assume that ϕ is a real even function, piecewise continuous and stable with stability constant $B \geq 0$. We will consider also stable hard core potentials: $\phi(u) = +\infty$ for $|u| \leq c$, where c is the radius of a hard core. In this case we assume that outside of the core ϕ is piecewise continuous and summable.

We will consider cases where the particles obey Maxwell-Boltzmann (MB), Bose-Einshtein (BE) and Fermi-Dirac (FD) statistics. Then $L^2(\Lambda^N)$, the N-fold tensor product of $L^2(\Lambda)$, is the Hilbert space which describe the quantum states of N particles obeying MB statistics.

The Hilbert spaces appropriate for the description of N particles obeying FD (Fermi gas or fermions) and BE (Bose gas or bosons) statistics are the spaces $L^2_-(\Lambda^N)$ (resp. $L^2_+(\Lambda^N)$) of square-integrable complex functions that are antisymmetric (resp. symmetric) with respect to their arguments. We denote these spaces by $L^2_\varepsilon(\Lambda^N)$ with $\varepsilon = -1$ for FD and $\varepsilon = +1$ for BE statistics respectively. Evidently $L^2_\varepsilon(\Lambda^N)$ are subspaces of $L^2(\Lambda^N)$.

The Hamiltonian of the system is given by

$$H_N(\Lambda) = -\sum_{i=1}^{N} \Delta_i + U \tag{3.10}$$

where Δ_i the Laplacian for the i-th variable with Dirichlet boundary conditions and U a multiplication operator, which is taken as a sum of pair potentials $U(u_1, \ldots, u_n) = \sum_{1\leq i<j\leq N} \phi(u_i - u_j)$. We can take the Hamiltonian to be a self-adjoint extension of the symmetric operator

$$-\sum_{i=1}^{N} \Delta_i + \sum_{1\leq i<j\leq N} \phi(u_i - u_j)$$

(see for details [46]).

To treat all the cases simultaneously we take the Hilbert space $L^2(\Lambda^N)$. Therefore dealing with operators in $L^2(\Lambda^N)$ which are relative to the BE or FD Statistics we have to multiply them by the corresponding projectors.

Let $S_\varepsilon = S_{\varepsilon,N}$ denote the projectors onto the subspaces $L^2_\varepsilon(\Lambda^N)$ which are defined by

$$S_\varepsilon f(u_1, \cdots, u_N) = \frac{1}{N!} \sum_{\pi \in \mathcal{I}_N} \varepsilon(\pi) f(u_{\pi(1)}, \cdots, u_{\pi(N)}), \quad f \in L^2(\Lambda^N) \tag{3.11}$$

where $\mathcal{I}_N$ is the permutation group of N elements, $\varepsilon(\pi)$ is equal to the signature of the permutation π for fermions, $\varepsilon(\pi) \equiv 1$ for bosons.

We consider quantum systems in the grand canonical formalism, where the number of particles is not fixed. The appropriate Hilbert space is the direct sum of the spaces $L^2(\Lambda^N)$ which is called the Fock space:

$$\mathfrak{F}(\Lambda) = \mathfrak{F}(L^2(\Lambda)) = \bigoplus_{N=0}^{\infty} L^2(\Lambda^N) \tag{3.12}$$

where $L^2(\Lambda^0) = \mathbb{C}$, is the space of complex numbers [15, 46, 105].

We briefly describe the notion of the conditional Wiener measure $P_\beta^{u,v}$, $u, v \in \mathbb{R}^d$, $\beta > 0$. Let Ω_β be the space of all functions from $[0, \beta]$ into one point compactification $\widehat{\mathbb{R}^d}$ of $\mathbb{R}^d$, in other words $\Omega_\beta = \prod_{t \in [0,\beta]} \widehat{\mathbb{R}^d}$. Equip Ω_β with the product topology (the topology of pointwise convergence) then Ω_β is a compact Hausdorff space. Let

$$p_t(u) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{u^2}{2t}}, \ u \in \mathbb{R}^d. \tag{3.13}$$

Let $F(u_1, \cdots, u_n)$, $u_1, \cdots, u_n \in \mathbb{R}^d$ be a bounded continuous function and let $0 < t_1 < \cdots < t_n < \beta$. Consider functions of the type $f(x) = F(x(t_1), \cdots, x(t_n))$, $x \in \Omega_\beta$. Such functions, for all $0 < t_1 < \cdots < t_n < \beta$ and F, are called simple functions, they are continuous, separate points hence by Stone - Weierstrass theorem form a sup-norm dense subalgebra $C_{fin}(\Omega_\beta)$ of the algebra $C(\Omega_\beta)$ of continuous functions on Ω_β.

Define $P_\beta^{u,v}$ as a linear functional on $C_{fin}(\Omega_\beta)$ by

$$P_\beta^{u,v}(f) = \int \prod_{l=1}^{n} \mathrm{d}u_l F(u_1, \cdots, u_n) \prod_{i=0}^{n} p_{t_{i+1}-t_i}(u_{i+1} - u_i) \tag{3.14}$$

where $t_0 = 0$, $t_{n+1} = \beta$, $u_0 = u, u_{n+1} = v$. One can check that $P_\beta^{u,v}$ is a positive bounded linear form on $C_{fin}(\Omega_\beta)$ which can be extended by continuity to a norm continuous positive linear form on $C(\Omega_\beta)$.

Finally, by the Riesz representation theorem, there exists a regular Borel measure on $C(\Omega_\beta)$, also denoted by $P_\beta^{u,v}$, such that

$$P_\beta^{u,v}(f) = \int \mathrm{d}P_\beta^{u,v}(x) f(x) \tag{3.15}$$

for all $f \in C(\Omega_\beta)$. This measure is called the conditional Wiener measure on Ω_β. For the details see [104, 46, 15]. The space Ω_β is too large so that there are Borel sets which are not Baire. Fortunately the measure $P_\beta^{u,v}$ is concentrated on the subset $C([0, \beta], \mathbb{R}^d) \subset \Omega_\beta$ of continuous functions from $[0, \beta]$ to $\mathbb{R}^d$ (see Theorem 1 in Appendix 1, [45]). The elements of $C([0, \beta], \mathbb{R}^d)$ we will call elementary trajectories and β its length. Moreover $P_\beta^{u,v}$ is supported by the subspace of the α-Hölder, $\alpha < \frac{1}{2}$, continuous functions from Ω_β (see Theorem 1.2 in [46]).

We define interaction between two elementary trajectories x and y by

$$\hat{\phi}(x - y) = \int_0^\beta \mathrm{d}t \, \phi(x(t) - y(t)) \quad x, y \in C([0, \beta], \mathbb{R}^d). \tag{3.16}$$

The Feynman-Kac formula states that under our conditions on the pair interaction ϕ the statistical operator $e^{-\beta H_N(\Lambda)}$ is an integral operator on $L^2(\Lambda^N)$ with a kernel K given by

$$K(u_1, \cdots, u_N; v_1, \cdots, v_N) = \int \prod_{i=1}^N \mathrm{d}P_\beta^{u_i,v_i}(x_i) 1_\Lambda(x_i) \exp\Big\{- \sum_{1 \le i < j \le N} \hat{\phi}(x_i - x_j)\Big\} \tag{3.17}$$

where $1_\Lambda(x) = 1$ if $x(t) \in \Lambda$ for all $0 \le t \le \beta$; it is zero otherwise. This kernel is a continuous function of $u_1, \cdots, u_N$ and $v_1, \cdots, v_N$ provided $u_i \ne u_j, \ i \ne j$, or $|u_i - u_j| > c$ in case of a hard core with radius c. It is bounded by $e^{N\beta B}(2\pi\beta)^{-N\frac{d}{2}} \prod_{i=1}^N \exp(-\frac{(u_i - v_i)^2}{2\beta})$ and represents a bounded self adjoint positive trace class operator in $L^2(\Lambda^N)$ with

$$Tre^{-\beta H_N(\Lambda)} \le |\Lambda|^N (2\pi\beta)^{-N\frac{d}{2}} e^{N\beta B} \tag{3.18}$$

([46] Section 1.3 , [45], Appendix 1, see also [109, 104], [15], Section 6.3.2).

The state of a quantum system in a grand canonical ensemble, whose parameters are the activity z and the inverse temperature β, is described by a density matrix, a positive trace class operator $D(\Lambda)$ on the Fock space $\mathfrak{F}(\Lambda)$ with $TrD(\Lambda) = 1$. In the MB case it is given by

$$D(\Lambda) = Z(\Lambda)^{-1} \bigoplus_{n=0}^\infty \frac{z^n}{n!} \exp(-\beta H_n(\Lambda)) \tag{3.19}$$

where the normalizing factor

$$Z(\Lambda) = \sum_{n=0}^\infty \frac{z^n}{n!} Tr \exp(-\beta H_n(\Lambda)) \tag{3.20}$$

is called the grand canonical partition function. The density matrix of a system with BE or FD statistics is

$$D_\varepsilon(\Lambda) = Z_\varepsilon(\Lambda)^{-1} \bigoplus_{n=0}^\infty z^n S_\varepsilon \exp(-\beta H(\Lambda)) \tag{3.21}$$

with the grand canonical partition function

$$Z_\varepsilon(\Lambda) = \sum_{n=0}^\infty z^n Tr S_\varepsilon \exp(-\beta H(\Lambda)). \tag{3.22}$$

Note that the density matrices $D(\Lambda)$ and $D_\varepsilon(\Lambda)$ define the corresponding finite-volume Gibbs states.

Theorem 3.3 . *The operators $D(\Lambda)$ and $D_\varepsilon(\Lambda)$ are positive operators on $\mathfrak{F}(\Lambda)$. $D(\Lambda)$ and $D_\varepsilon(\Lambda)$ are trace class operators respectively for all z and for $z \leq e^{-\beta B}$.*

Moreover the Feynman-Kac representations of the grand partition functions for MB respectively BE and FD quantum gases are given by

$$Z(\Lambda) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} \prod_{i=1}^{n} \mathrm{d}u_i \int \prod_{i=1}^{n} \mathrm{d}P_\beta^{u_i,u_i}(x_i) 1_\Lambda(x_i) \exp\Big\{- \sum_{1\leq i<j\leq n} \hat{\phi}(x_i - x_j)\Big\}, \tag{3.23}$$

respectively by

$$Z_\varepsilon(\Lambda) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{\pi\in\mathcal{I}_n} \varepsilon(\pi) \int_{\Lambda^n} \prod_{i=1}^{n} \mathrm{d}u_i \int \prod_{i=1}^{n} \mathrm{d}P_\beta^{u_i,u_{\pi(i)}}(x_i) \cdot \exp\Big\{- \sum_{1\leq i<j\leq n} \hat{\phi}(x_i - x_j)\Big\}. \tag{3.24}$$

Proof of Theorem 3.3. Evidently both operators are positive.

Using the Feynman-Kac formula (5.6) and the stability of the potential ϕ we have

$$\begin{aligned} Z(\Lambda) &= \sum_{n=1}^{\infty} \frac{z^n}{n!} Tr \exp(-\beta H_n(\Lambda)) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} \prod_{i=1}^{n} \mathrm{d}u_i \int \prod_{i=1}^{n} \mathrm{d}P_\beta^{u_i,u_i}(x_i) 1_\Lambda(x_i) \\ &\cdot \exp\Big\{- \sum_{1\leq i<j\leq n} \hat{\phi}(x_i - x_j)\Big\} \leq \sum_{n=1}^{\infty} \frac{(ze^{\beta B}|\Lambda|(2\pi\beta)^{-\frac{d}{2}})^n}{n!} \\ &= \exp[ze^{\beta B}|\Lambda|(2\pi\beta)^{-\frac{d}{2}}] < \infty. \end{aligned} \tag{3.25}$$

Next we treat $Z_\varepsilon(\Lambda)$ invoking again the Feynman-Kac formula. First consider the operator $S_\varepsilon \exp(-\beta H(\Lambda))$. For any $f \in L^2(\Lambda^n)$,

$$\begin{aligned} &S_\varepsilon \exp(-\beta H(\Lambda)) f(u_1, \cdots, u_n) = \frac{1}{n!} \sum_{\pi\in\mathcal{I}_n} \varepsilon(\pi) \exp(-\beta H(\Lambda)) f(u_{\pi(1)}, \cdots, u_{\pi(n)}) \\ &= \frac{1}{n!} \sum_{\pi\in\mathcal{I}_n} \varepsilon(\pi) \int_{\Lambda^n} \prod_{i=1}^{n} \mathrm{d}v_i K(u_{\pi(1)}, \cdots, u_{\pi(n)}; v_1, \cdots, v_n) f(v_1, \cdots, v_n) \\ &= \frac{1}{n!} \sum_{\pi\in\mathcal{I}_n} \varepsilon(\pi) \int_{\Lambda^n} \prod_{i=1}^{n} \mathrm{d}v_i f(v_1, \cdots, v_n) \int \prod_{i=1}^{n} \mathrm{d}P_\beta^{u_{\pi(i)},v_i}(x_i) 1_\Lambda(x_i) \\ &\cdot \exp\Big\{- \sum_{1\leq i<j\leq n} \hat{\phi}(x_i - x_j)\Big\} = \frac{1}{n!} \sum_{\pi\in\mathcal{I}_n} \varepsilon(\pi) \int_{\Lambda^n} \prod_{i=1}^{n} \mathrm{d}v_i f(v_1, \cdots, v_n) \\ &\cdot \int \prod_{i=1}^{n} \mathrm{d}P_\beta^{u_i,v_{\pi(i)}}(x_i) 1_\Lambda(x_i) \exp\Big\{- \sum_{1\leq i<j\leq n} \hat{\phi}(x_i - x_j)\Big\} \\ &= \int_{\Lambda^N} \prod_{i=1}^{n} \mathrm{d}v_i \frac{1}{n!} \sum_{\pi\in\mathcal{I}_n} \varepsilon(\pi) K(u_1, \cdots, u_n; v_{\pi(1)}, \cdots, v_{\pi(n)}) f(v_1, \cdots, v_n). \end{aligned}$$

Thus $S_\varepsilon \exp(-\beta H(\Lambda))$ is an integral operator with the kernel

$$K_\varepsilon(u_1, \cdots, u_n; v_1, \cdots, v_n) = \frac{1}{n!} \sum_{\pi \in \mathcal{I}_n} \varepsilon(\pi) \int \prod_{i=1}^{n} \mathrm{d}P_\beta^{u_i, v_{\pi(i)}}(x_i) 1_\Lambda(x_i) \cdot \exp\Big\{- \sum_{1 \le i < j \le n} \hat{\phi}(x_i - x_j)\Big\}. \tag{3.26}$$

Then

$$Z_\varepsilon(\Lambda) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{\pi \in \mathcal{I}_n} \varepsilon(\pi) \int_{\Lambda^n} \prod_{i=1}^{n} \mathrm{d}u_i \int \prod_{i=1}^{n} \mathrm{d}P_\beta^{u_i, u_{\pi(i)}}(x_i) 1_\Lambda(x_i) \cdot \exp\Big\{- \sum_{1 \le i < j \le n} \hat{\phi}(x_i - x_j)\Big\} \tag{3.27}$$

This is the Feynman-Kac representations of the grand partition function for bosons and fermions.

To estimate $Z_\varepsilon(\Lambda)$ we rewrite it in terms of composite (winding) loops by collecting the elementary trajectories of length β in composite loops.

Every permutation of I_n can be written as a product of disjoint cycles. This representation is unique up to the order of the elements in the cycle since the multiplication of disjoint cycles is commutative and any rotation of a given cycle (the choice of its starting point) specifies the same cycle .

A cycle decomposition of a permutation can be viewed as a class of a permutation group.

Two permutations of I_n belong to the same class (are conjugate in I_n) if and only if they consist of the same number of disjoint cycles of the same lengths. The number of permutations in a class with k cycles of length $n_1, \cdots, n_k$ with $n_i \ge 1$, $i = 1, \cdots, k$ and $\sum_{i=1}^{k} n_i = n$ is

$$\frac{n!}{k! \prod_1^k n_i}.$$

(see for example [112, 46].

Now note that the integrand in (3.27)

$$\prod_{i=1}^{n} 1_\Lambda(x_i) \exp\Big\{- \sum_{1 \le i < j \le n} \hat{\phi}(x_i - x_j)\Big\}$$

is a symmetric function of $x_1, \cdots, x_n$. The contributions of any two elements from the same class differ only by the labelling of the integration variables. Given a permutation from the class defined by $n_1, \cdots, n_k$ with $\sum_{i=1}^{k} n_i = n$, we collect n elementary trajectories $\{x_1, \cdots, x_n\}$ into k composite loops $\{X_1, \cdots, X_k\}$ of the respective time intervals

$n_1\beta, \cdots, n_k\beta$. Therefore

$$Z_\varepsilon(\Lambda) = \sum_{n=0}^{\infty} z^n \sum_{k=1}^{n} \frac{1}{k!} \sum_{n_1,\cdots,n_k\geq 1, \sum n_i = n} \int_{\Lambda^n} \prod_{i=1}^{k} \mathrm{d}u_i \int \prod_{i=1}^{k} \mathrm{d}P_{n_i\beta}^{u_i,u_i}(X_i) \frac{1}{n_i} 1_\Lambda(X_i) e^{-U(X_1,\cdots,X_k)}$$
$$= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\Lambda^k} \prod_{i=1}^{k} \mathrm{d}u_i \sum_{n=k}^{\infty} \sum_{n_1,\cdots,n_k\geq 1, \sum n_i = n} \int \prod_{i=1}^{k} \mathrm{d}P_{n_i\beta}^{u_i,u_i}(X_i) \frac{z^{n_i}}{n_i} 1_\Lambda(X_i)$$
$$\cdot\, e^{-U(X_1,\cdots,X_k)} = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\Lambda^k} \prod_{i=1}^{k} \mathrm{d}u_i \prod_{i=1}^{k} \sum_{n_i=1}^{\infty} \frac{z^{n_i}}{n_i} \int \mathrm{d}P_{n_i\beta}^{u_i,u_i}(X_i) 1_\Lambda(X_i) e^{-U(X_1,\cdots,X_k)} \tag{3.28}$$

where $1_\Lambda(X_i) = 1$ if $X_i(t) \in \Lambda$ for any $0 \leq t \leq n_i\beta$ and is zero otherwise,

$$U(X_1, \cdots, X_k) = \sum_{0\leq i<j\leq k} \sum_{x_i\in X_i, x_j \in X_j} \hat{\phi}(x_i - x_j). \tag{3.29}$$

Let us show that the series in the last two lines of (3.28) is absolutely convergent. Using the stability of the potential ϕ we can write

$$1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\Lambda^k} \prod_{i=1}^{k} \mathrm{d}u_i \prod_{i=1}^{k} \sum_{n_i=1}^{\infty} \frac{z^{n_i}}{n_i} \int \mathrm{d}P_{n_i\beta}^{u_i,u_i}(X_i) 1_\Lambda(X_i) e^{-U(X_1,\cdots,X_k)}$$
$$\leq 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\Lambda^k} \prod_{i=1}^{k} \mathrm{d}u_i \prod_{i=1}^{k} (2\pi\beta)^{-\frac{d}{2}} \sum_{n_i=1}^{\infty} \frac{(ze^{\beta B})^{n_i}}{n_i^{1+\frac{d}{2}}}$$
$$\leq \exp\Big[|\Lambda|(2\pi\beta)^{-\frac{d}{2}} \zeta\Big(1+\frac{d}{2}\Big)\Big] \tag{3.30}$$

provided $ze^{\beta B} \leq 1$. This justifies the interchanging of unbounded sums above and completes the proof of Theorem 3.3. Here $\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s}$ is the Riemann zeta function.

3.3 Loop gases

In this section we will rewrite the grand-canonical partition functions (3.20) and (3.22) in terms of respectively elementary and composite (winding) loops. We consider the space of composite loops and define the canonical σ-algebra and measure.

Let $\beta > 0$ be the inverse temperature and let $C([0,\beta],\mathbb{R}^d)$ be the space of all Brownian trajectories of length β in $\mathbb{R}^d$. The elements of $C([0,\beta],\mathbb{R}^d)$ we call *elementary trajectories*. With the help of elementary trajectories we construct so-called composite Brownian loops in $\mathbb{R}^d$ [46].

For j=1,2,... we denote by $\mathcal{X}_{j\beta}$ the space of composite loops of fixed length jβ in $\mathbb{R}^d$:

$$\mathcal{X}_{j\beta} = \{X \in C([0, j\beta], \mathbb{R}^d) \mid X(0) = X(j\beta)\}.$$

and we put $|X| = j$ for $X \in \mathcal{X}_{j\beta}$. Note that in topology of uniform convergence $\mathcal{X}_{j\beta}$ is a Polish space with Borel σ-algebra $\mathcal{B}(\mathcal{X}_{j\beta})$.

The underlying one particle space $\mathcal{X}$ of the loop gas with BE or FD statistics is the space of composite loops in $\mathbb{R}^d$ and is defined as a topological sum of the spaces $\mathcal{X}_{j\beta}$: $\mathcal{X} = \bigcup_{j=1}^{\infty} \mathcal{X}_{j\beta}$. We define the Borel σ-algebra in $\mathcal{X}$ by $\mathcal{B}(\mathcal{X}) = \bigcup_{j=1}^{\infty} \mathcal{B}(\mathcal{X}_{j\beta})$. This means that $B \in \mathcal{B}(\mathcal{X})$ iff $B \cap \mathcal{X}_{j\beta} \in \mathcal{B}(\mathcal{X}_{j\beta})$ for each $j = 1, 2 \cdots$.

We say that an elementary trajectory x is an *elementary constituent* of a composite loop $X \in \mathcal{X}$ and write $x \in X$ if there exists i, $0 \leq i < |X|$ such that $x(t) = X(t + i\beta)$, $t \in [0, \beta]$.

Notice that the one particle space of the loop gas in MB case is the subspace $\mathcal{X}_\beta \subset \mathcal{X}$.

For any $u \in \mathbb{R}^d$ let

$$\mathcal{X}_{j\beta}^u = \{X \in \mathcal{X}_{j\beta} \mid X(0) = X(j\beta) = u\}$$

and let

$$\mathcal{X}^u = \bigcup_{j=1}^{\infty} \mathcal{X}_{j\beta}^u.$$

Without confusing the reader, we often write shortly $\mathcal{X}_{j\beta}^u$, $P_{j\beta}^u$ for $\mathcal{X}_{j\beta}^{u,u}$ and $P_{j\beta}^{u,u}$ respectively. To define the underlying reference measure on $\mathcal{X}$ we use the natural bijection τ between $\mathcal{X}$ and $\mathcal{X}^0 \times \mathbf{R}^\nu$ given by

$$\tau(X^0, u) = X^0 + u, X^0 \in \mathcal{X}^0, u \in \mathbb{R}^d.$$

Evidently τ is a bijection with

$$\tau^{-1}(X) = (X - X(0), X(0)).$$

Let $P_{j\beta}^u$ be the non-probabilistic (non-normalized) Brownian bridge measure on $\mathcal{X}_{j\beta}^u$ with

$$P_{j\beta}^u(\mathcal{X}_{j\beta}^u) = (\pi j\beta)^{\frac{-d}{2}}, u \in \mathbb{R}^d.$$

We define a measure $P_{\varepsilon,z}^u$ on $\mathcal{X}^u, u \in \mathbb{R}^d$, by the formula

$$P_{\varepsilon,z}^u = \sum_{j=1}^{\infty} \frac{\varepsilon^{j-1} z^j}{j} P_{j\beta}^u \tag{3.31}$$

where $\varepsilon = +1$ or -1 which corresponds to the case of bosons and fermions respectively. Here we assume that $0 < z \leq 1$.

The sets

$$(\mathcal{X}^u)^+ = \sum_{j=1}^{\infty} \mathcal{X}_{(2j-1)\beta}^u, \quad (\mathcal{X}^u)^- = \sum_{j=1}^{\infty} \mathcal{X}_{2j\beta}^u$$

give the Hahn decomposition of the space $\mathcal{X}^u$ with respect to the signed measure $P_{-,z}^u$:

$$\mathcal{X}^u = (\mathcal{X}^u)^+ + (\mathcal{X}^u)^-.$$

Here $(\mathcal{X}^u)^+$ and $(\mathcal{X}^u)^-$ are disjoint, $(\mathcal{X}^u)^+$ is positive and $(\mathcal{X}^u)^-$ is negative with respect to the signed measure $P_{-,z}^u$ (see [59], for example).

The Jordan decomposition of the measure $P^u_{-,z}$ is given by the formula

$$P^u_{-,z} = (P^u_{-,z})^+ - (P^u_{-,z})^-$$

where $(P^u_{-,z})^+$ and $(P^u_{-,z})^-$ are positive measures defined by.

$$(P^u_{-,z})^+ = \sum_{j=1}^{\infty} \frac{z^{2j-1}}{2j-1} P^u_{(2j-1)\beta}, \quad (P^u_{-,z})^- = \sum_{j=1}^{\infty} \frac{z^{2j}}{2j} P^u_{2j\beta}.$$

We note that the total variation $|P^u_{-,z}| = (P^u_{-,z})^+ + (P^u_{-,z})^-$ of the signed measure $P^u_{-,z}$ coincides with the measure $P^u_{+,z}$ which is finite measure with

$$P^u_{+,z}(\mathcal{X}) \leq P^u_{+,1}(\mathcal{X}) = (\pi\beta)^{-\frac{d}{2}} \sum_{j=1}^{\infty} \frac{1}{j^{1+\frac{d}{2}}} < \infty.$$

The following lemma shows that "typical" loops are localized.

Lemma 3.4 . *Let*

$$M(R) = \{X \in \mathcal{X} |\, |X(t) - X(s)| > R \quad \textit{for some} \quad s,t \in [0, |X|\beta]\,\}, \quad R \geq 0.$$

For all $z,\ 0 < z < 1$*,*

$$P^u_{+,z}(M(R)) \leq C(d)\beta^{-\frac{d}{2}} \exp[-C(\beta, z)R], \quad u \in \mathbb{R}^d,$$

where

$$C(\beta, z) = \left(\frac{|\ln z|}{64\beta}\right)^{\frac{1}{2}}.$$

In particular, if $z \leq e^{-\frac{1}{2}}$*,*

$$P^u_{+,z}(M(R)) \leq C(d)\zeta\left(\frac{d}{2}+1\right)\beta^{-\frac{d}{2}} \exp\left(-\frac{R}{8\sqrt{2\beta}}\right) \tag{3.32}$$

where ζ *is the Riemann zeta function.*

Proof. For any $u, v \in \mathbb{R}^d$ we set

$$M(\varepsilon, \delta) = \{x \in \mathcal{X}^{uv}_\beta \;\big|\; |x(t) - x(s)| > \varepsilon \text{ for some } s,t \in [0,\beta],\ |s-t| \leq \delta\}. \tag{3.33}$$

Then (see [46], formulae (1.14), (1.31))

$$P^{uv}_\beta(M(\varepsilon, \delta)) \leq C(d)\frac{4\beta}{\delta}(\pi\beta)^{-\frac{d}{2}} \left(\frac{\varepsilon}{8\sqrt{\delta}}\right)^{d-1} \exp\left(-\frac{\varepsilon^2}{128\delta}\right). \tag{3.34}$$

We denote all the constants by C indicating, if necessary, the dependence on parameters.

Let $M(R, j\beta) = \mathcal{X}_{j\beta} \bigcap M(R)$, $j = 1, 2, \ldots$. Then

$$P^u_{+,z}(M(R)) = \sum_{j=1}^{\infty} \frac{z^j}{j} \cdot P^u_{j\beta} M(R, j\beta).$$

It follows from (3.34) that

$$P^u_{j\beta}(M(R,j\beta)) \le C(d)(\pi j\beta)^{-\frac{d}{2}}\Big(\frac{R}{8\sqrt{j\beta}}\Big)^{d-1}\exp\Big(-\frac{R^2}{128j\beta}\Big)$$

Obviously $\Big(\frac{R}{8\sqrt{j\beta}}\Big)^{d-1}\exp\Big(-\frac{R^2}{256j\beta}\Big)$, as a function of $\xi=\frac{R}{8\sqrt{j\beta}}$, is bounded uniformly by a constant $C=C(d)$. Therefore for any $\tau>0$ and all z, $0<z\le\exp(-64\beta\tau^2)$,

$$e^{\tau R}P^u_{+,z}(M(R)) \;\le\; e^{\tau R}C(d)(\pi\beta)^{-\frac{d}{2}}\cdot\sum_{j=1}^{\infty}\frac{z^j}{j^{\frac{d}{2}+1}}\;\exp\Big(-\frac{R^2}{256j\beta}\Big)\le C(d)$$
$$\cdot(\pi\beta)^{-\frac{d}{2}}\sum_{j=1}^{\infty}\frac{z^j}{j^{\frac{d}{2}+1}}e^{64\tau^2 j\beta}\exp\left[-\frac{1}{4}\left(\frac{R}{8\sqrt{j\beta}}-16\tau\sqrt{j\beta}\right)^2\right]\le C(d)(\pi\beta)^{-\frac{d}{2}}$$
$$\cdot\sum_{j=1}^{\infty}\frac{[z\exp(64\tau^2\beta)]^j}{j^{\frac{d}{2}+1}}\le C(d)(\pi\beta)^{-\frac{d}{2}}\cdot\sum_{j=1}^{\infty}\frac{1}{j^{\frac{d}{2}+1}}=C(d)(\pi\beta)^{-\frac{d}{2}}\zeta\left(\frac{d}{2}+1\right). \tag{3.35}$$

If $0<z<1$ we choose $\tau=\left(\frac{|\ln z|}{64\beta}\right)^{\frac{1}{2}}$ and get the first bound of Lemma 1.

Let $z\sqrt{e}\le 1$ then setting in (3.35) $\tau=\frac{1}{8\sqrt{2\beta}}$ we get (3.32). Lemma 1 is proved.

Corollary 3.5 . *The following bound is useful in applications*

$$P^u_{+,z}(M(R))\le C(d)\zeta\left(\frac{d}{2}+1\right)\beta^{-\frac{d}{2}}\exp\left(-\frac{R}{8\sqrt{2\beta}}\right)\qquad u\in\mathbb{R}^d,\ l>0.$$

Corollary 3.6 . *In the case of MB statistics*

$$P^u_{\beta}(M(R,\beta))\le C(d,l)\beta^{-\frac{l-d}{2}}(1+R)^{-l},\quad u\in\mathbb{R}^d,\ l>0.$$

with $M(R,\beta)$ given by (3.33).

Proof of Corollary 3.5. From (3.32) with the help of the bound $R^{-l}\le 2^l(1+R)^{-l}$, $R\ge 1$, we get

$$P^u_{+,z}(M(R))\le C(d)\zeta\left(\frac{d}{2}+1\right)\beta^{-\frac{d}{2}}\exp\left(-\frac{R}{8\sqrt{2\beta}}\right)\left(\frac{R}{8\sqrt{2\beta}}\right)^l(16\sqrt{2\beta})^l$$
$$\cdot(1+R)^{-l}\le C(d,l)\zeta\left(\frac{d}{2}+1\right)\beta^{\frac{l-d}{2}}(1+R)^{-l}.$$

Here we used the following bound $\tau^l e^{-\tau}\le C(l)\ \tau>0$.

Corollary 3.6 can be proved by repeating the same arguments.

We define the intensity measure $\rho_{\varepsilon,z}$ on $\mathcal{B}(\mathcal{X})$ by

$$\rho_{\varepsilon,z}=(P^0_{\varepsilon,z}\times\lambda)\circ\tau^{-1} \tag{3.36}$$

where λ is the Lebesgue measure on $\mathbb{R}^d$. In the case of fermions the variation $|\rho_{-,z}|$ of the signed measure $\rho_{-,z}$ is equal to $\rho_{+,z}$.

The restriction of the σ-finite measure $\rho_{\varepsilon,z}$ to $\mathcal{X}_{j\beta}$ coincides with $(\frac{\varepsilon^{j-1}z^j}{j}P^0_{j\beta}\times\lambda)\circ\tau_j^{-1}$, where τ_j is the restriction of τ to $\mathcal{X}^0_{j\beta}\times\mathbb{R}^d$. For $A\in\mathcal{B}(\mathcal{X})$ we denote by $\rho_{\varepsilon,z,A}$ the restriction of $\rho_{\varepsilon,z}$ to $(A,\mathcal{B}\cap A)$.

The intensity measure for the loop gas with the MB statistics is the measure

$$z\rho = z(P^0_\beta\times\lambda)\circ\tau^{-1} = \rho_{\varepsilon,z,\mathcal{X}_\beta}.$$

Let $\mathcal{M}=\mathcal{M}(\mathcal{X})$ be the set of all finite subsets (finite configurations) of $\mathcal{X}$. Note that a configuration $\mu=\{x_1,\cdots,x_n\}\in\mathcal{M}$ can be identified with the finite sum of Dirac measures of its elements: $\mu=\delta_{x_1}+\cdots+\delta_{x_n}$. Thus the configuration space of our system $\mathcal{M}$ can be identified with the space of all finite simple point measures on $\mathcal{X}$, provided with its canonical σ-field.

On the space $\mathcal{M}$ we define the following σ-finite measure $W_{\rho_{\varepsilon,z}}$ by

$$W_{\rho_{\varepsilon,z}}(h)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathcal{X}^n}\mathrm{d}\rho_{\varepsilon,z}(X_1)\cdots\mathrm{d}\rho_{\varepsilon,z}(X_n)h(X_1,...,X_n) \tag{3.37}$$

where $h:\mathcal{M}\to\mathbb{R}_+$. Here the first term in the sum is $h(\emptyset)$.

In the case of fermions one can easily check that the total variation $|W_{\rho_{-,z}}|$ of the signed measure $W_{\rho_{-,z}}$ is equal to $W_{\rho_{+,z}}$.

Let Λ be a bounded region in $\mathbb{R}^d$. We define

$$\mathcal{X}(\Lambda)=\{X\in\mathcal{X}|X(t)\in\Lambda,\ \forall t\in[0,\beta|X|]\}.$$

Similarly we can define the set $\mathcal{M}(\mathcal{X}(\Lambda))=\mathcal{M}(\Lambda)$ of finite configurations of loops "living" in Λ:

$$\mathcal{M}(\Lambda)=\{\omega\in\mathcal{M}\,|\,\omega\subset\mathcal{X}(\Lambda)\}.$$

Note that $W_{\rho_{\varepsilon,z,\mathcal{X}(\Lambda)}}=W_{\rho_{\varepsilon,z,\Lambda}}$ is a finite measure on $\mathcal{M}(\Lambda)$ with total mass $\exp(\rho_{\varepsilon,z}(\mathcal{X}(\Lambda)))$. The corresponding probability measure $\exp(-\rho_{\varepsilon,z}(\mathcal{X}(\Lambda)))W_{\rho_{\varepsilon,z,\Lambda}}$ we call *the ideal loop gas in Λ with activity (intensity) z obeying BE or FD statistics* or *the Poisson process in $\mathcal{X}(\Lambda)$ with intensity $\rho_{\varepsilon,z}$*.

The energy $U(\omega)$ of a finite configuration ω of composite loops is given by

$$U(\omega)=\sum_{X\in\omega}U_1(X)+\frac{1}{2}\sum_{X,Y\in\omega}U_2(X,Y) \tag{3.38}$$

where

$$U_1(X)=\frac{1}{2}\sum_{x,y\in X}\hat{\phi}(x-y) \tag{3.39}$$

and

$$U_2(X,Y)=\sum_{x\in X,y\in Y}\hat{\phi}(x-y). \tag{3.40}$$

We define the Boltzmann factor $f : \mathcal{M} \to \mathbb{R}_+$ as usual by

$$f(\omega) = \exp(-U(\omega)), \quad \omega \in \mathcal{M}. \tag{3.41}$$

The Gibbs measure on $\mathcal{M}(\Lambda)$ for a bounded region Λ is given by

$$Q_\varepsilon(\Lambda, z) = \frac{\exp(-U)}{Z_\varepsilon(\Lambda, z)} W_{\varepsilon,z,\Lambda} \tag{3.42}$$

where the grand partition function

$$Z_\varepsilon(\Lambda, z) = W_{\varepsilon,z,\Lambda}(f). \tag{3.43}$$

More explicitly

$$Z_\varepsilon(\Lambda, z) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{X}_\Lambda^n} \rho_{\varepsilon,z}(\mathrm{d}X_1)...\rho_{\varepsilon,z}(\mathrm{d}X_n) \exp(-U(X_1, ..., X_n)) \tag{3.44}$$

and

$$Q_\varepsilon(\Lambda, z)(\varphi) = Z_\varepsilon(\Lambda, z)^{-1} \int_{\mathcal{M}} \mathrm{d}W_{\varepsilon,z,\Lambda}(\omega)\varphi(\omega)\exp(-U(\omega)) \tag{3.45}$$

for any measurable $\varphi : \mathcal{M} \to \mathbb{R}_+$.

Note that the condition (c) on the interaction ϕ implies the finiteness of $Z_\varepsilon(\Lambda, z)$ for all $0 < z \leq 1$. Indeed

$$|Z_\varepsilon(\Lambda, z)| \leq \exp\left[\int_{\mathcal{X}(\Lambda)} \mathrm{d}\rho_{+,z}(X)\right] \leq \exp\left[|\Lambda| P^0_{+,z}(\mathcal{X}^0)\right] < \infty.$$

We call $Q_\varepsilon(\Lambda, z)$ *the loop gas in* Λ *with intensity* z *and interaction* ϕ *obeying BE or FD statistics.*

3.4 Decay of correlations in loop gases

In this section we present estimates of the two-point truncated correlation functions $\sigma_\Lambda(x, y)$ for a model of low density interacting Brownian loops. The first non-trivial problem is how to define a decay property for the function $q(x, y)$ of two Brownian loops that is consistent with the decay property of the interaction $\hat{\phi}(x-y)$. It is well known [46] that if one integrates $q(x, y)$ with respect to the measure $\rho_{\varepsilon,z}$ over loops y which visit the exterior of a ball of radius $R + r$ in $\mathbb{R}^d$, while the loop x stays all the time in a ball of radius R with the same center, then the integral has a power decay in r. We take this property as a definition of the power decay of a function of two loops. We show that $\sigma_\Lambda(x, y)$ has the power decay under suitable conditions on the classical interaction ϕ [96]. This decay property permits, in particular, the study of large volume asymptotics of the log-partition function which we carry out in the next section.

The cases of absolutely integrable interaction potentials and hard core potentials are treated separately.

The analysis of the decay of correlations in a loop gas is based on the general Theorem 3.1. To apply this theorem we take

$$\mathbb{X} = \mathcal{X}(\Lambda), \mathrm{d}\mu(x) = \mathrm{d}\rho_{\varepsilon,z}(x)\, e^{-U_1(x)},\; q(x,y) = e^{-U_2(x,y)} - 1 \tag{3.46}$$

where U_1 and U_2 is given respectively by (3.39) and (3.40).

Assumption 4 . *There exists a nonnegative function a on $\mathcal{X}$ and a number p, $0 < p < 1$, such that for a given $l > 0$ and all $r > 0$,*

$$\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}\rho_{+,z}(y)\, |q(x,y)| e^{a(y)+2b(y)} \hat{a}(y) \leq p\, a(x)\Big(1 + \frac{r}{2}\Big)^{-l}. \tag{3.47}$$

for almost all $x \in \mathcal{X}(B_0(R))$, ($\hat{a}(y) = \max(a(y), 1)$) .

Assumption 4' is defined like Assumptions 4 with the $u(x,y)$ instead of $q(x,y)$ and $e^{b(\cdot)}$ instead of $e^{2b(\cdot)}$.

The following result describes the decay property of the two-point truncated correlation functions of the system of interacting Brownian loops.

Theorem 3.7 (Main bound. Integrable potential) . *If Assumptions 1, 3 and 4 hold true, then for all positive R and r and for almost all $x \in \mathcal{X}(B_0(R))$,*

$$\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y)|\sigma_\Lambda(x,y)| \leq C(l,p) e^{a(x)+2b(x)} a(x)(1+r)^{-l}. \tag{3.48}$$

In case where Assumptions 1, 3' and 4' hold true the same result is valid with $b(x)$ instead of $2b(x)$.

We notice that the bound (3.48) for $\sigma_\Lambda(x,y)$ is uniform in Λ therefore Theorem 3.7 is very useful in study of the behavior of the log-partition function in thermodynamic limit.

It is an open problem to prove that a modified Assumption 4 with an exponential decay on the right side (instead of power decay) implies the same type decay for σ.

Proof of Theorem 3.7. For any $r > 0$ we take Λ so large that $B_0(R+r) \subset \Lambda$ and we take $\mathbb{X} = \mathcal{X}(\Lambda)$. Since Assumption 3 is stronger then Assumption 2, Theorems 2.1, 2.3 and 3.1 as well as Theorem 3.2 hold true.

By Theorem 3.1 and Lebesgue Monotone Convergence Theorem

$$\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y)|\sigma_\Lambda(x,y)| \leq e^{a(x)+2b(x)} \sum_{m\geq 0} \int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y) e^{a(y)+2b(y)}$$

$$\cdot \int_{\mathbb{X}^m} \prod_{i=1}^m \mathrm{d}|\mu|(x_i) e^{a(x_i)+2b(x_i)} \prod_{i=0}^m |q(x_i, x_{i+1})|.$$

Now using Theorem 3.1 and monotone convergence theorem, we can write

$$\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y)|\sigma_\Lambda(x,y)| \leq e^{a(x)+2b(x)} \sum_{m=0}^{\infty} D_m(x,R,r) \tag{3.49}$$

where

$$D_m(x,R,r) = \int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y) e^{a(y)+2b(y)} \int_{\mathcal{X}^m} \prod_{i=1}^{m} \mathrm{d}|\mu|(x_i) e^{a(x_i)+2b(x_i)} \cdot \prod_{i=0}^{m} |q(x_i, x_{i+1})|, \ m \geq 1 \tag{3.50}$$

and

$$D_0(x,R,r) = \int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y) e^{a(y)+2b(y)} |q(x,y)|. \tag{3.51}$$

Let us prove by induction that for any $x \in \mathcal{X}(B_0(R))$ and $r > 0$,

$$D_m(x,R,r) \leq a(x)(m+1)p^{m+1}\Big(1 + \frac{r}{2(m+1)}\Big)^{-l}. \tag{3.52}$$

Indeed, according to Assumption 4, (3.52) holds true for $m = 0$. Suppose that (3.52) holds true for all $m \leq n-1$. Then using Proposition 6.4 we have

$$\begin{aligned}
D_n(x,R,r) &= \int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y) e^{a(y)+2b(y)} \int_{\mathcal{X}^m} \prod_{i=1}^{n} \mathrm{d}|\mu|(x_i) e^{a(x_i)+2b(x_i)} \prod_{i=0}^{n} |q(x_i, x_{i+1})| \\
&= \int_{\mathcal{X}(B_0(R+\frac{r}{n+1}))} \mathrm{d}|\mu|(x_1) e^{a(x_1)+2b(x_1)} |q(x,x_1)| \int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y) \\
&\cdot e^{a(y)+2b(y)} \int_{\mathcal{X}^{m-1}} \prod_{i=2}^{n} \mathrm{d}|\mu|(x_i) e^{a(x_i)+2b(x_i)} \prod_{i=1}^{n} |q(x_i, x_{i+1})| \\
&+ \int_{\mathcal{X}^c(B_0(R+\frac{r}{n+1}))} \mathrm{d}|\mu|(x_1) e^{a(x_1)+2b(x_1)} |q(x,x_1)| \int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y) \\
&\cdot e^{a(y)+2b(y)} \int_{\mathcal{X}^{m-1}} \prod_{i=2}^{n} \mathrm{d}|\mu|(x_i) e^{a(x_i)+2b(x_i)} \prod_{i=1}^{n} |q(x_i, x_{i+1})| \\
&\leq \int_{\mathcal{X}(B_0(R+\frac{r}{n+1}))} \mathrm{d}|\mu|(x_1) e^{a(x_1)+2b(x_1)} |q(x,x_1)| \\
&\cdot D_{n-1}\Big(x_1, R + \frac{r}{n+1}, \frac{nr}{n+1}\Big) + \int_{\mathcal{X}^c(B_0(R+\frac{r}{n+1}))} \mathrm{d}|\mu|(x_1) e^{a(x_1)+2b(x_1)} \\
&\cdot |q(x,x_1)| A_{n-1}(x_1) \leq \int_{\mathcal{X}(B_0(R+\frac{r}{n+1}))} \mathrm{d}|\mu|(x_1) e^{a(x_1)+2b(x_1)} |q(x,x_1)| a(x_1) n p^n \\
&\cdot \Big(1 + \frac{r}{2(n+1)}\Big)^{-l} + \int_{\mathcal{X}^c(B_0(R+\frac{r}{n+1}))} \mathrm{d}|\mu|(x_1) e^{a(x_1)+2b(x_1)} |q(x,x_1)| a(x_1) p^n \\
&\leq a(x)(n+1)p^{n+1}\Big(1 + \frac{r}{2(n+1)}\Big)^{-l}.
\end{aligned} \tag{3.53}$$

Thus

$$\begin{aligned}
\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y) |\sigma_\Lambda(x,y)| &\leq e^{a(x)+2b(x)} a(x) \sum_{m=1}^{\infty} m p^m \Big(1 + \frac{r}{2m}\Big)^{-l} \\
&\leq C(l,p) e^{a(x)+2b(x)} a(x) (1+r)^{-l}
\end{aligned} \tag{3.54}$$

where $C(l,p) = 2^l \sum_{m=1}^{\infty} m^{l+1} p^m < \infty$. Theorem 3.7 is proved.

The following result shows how to get a simpler bound for $\sigma_\Lambda(x,y)$ by an additional integration in x.

Theorem 3.8 (Decay of double integral) . *Let $h : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ be a measurable function. Assume that there exists a measurable function $\alpha(x^0)$, $x^0 \in \mathcal{X}^0$ such that for all positive R, $r > 0$ and for almost all $x \in \mathcal{X}(B_0(R))$,*

$$\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y)|h(x,y)| \leq \alpha(x^0)(1+r)^{-l} \tag{3.55}$$

and

$$\int_{\mathcal{X}} \mathrm{d}|\mu|(y)|h(x,y)| \leq \alpha(x^0). \tag{3.56}$$

If the function $\alpha(x^0)$ satisfies the condition

$$\int_{M(R)} \mathrm{d}|\mu|(x^0)\alpha(x^0) \leq C(1+R)^{-l} \tag{3.57}$$

with $C = C(\alpha) > 0$ and $M(R)$ given by Lemma 3.4, then

$$\int_{\mathcal{X}^0} \mathrm{d}P^0_{+,z}(x^0) \int_{\mathcal{X}^c(B_0(R))} \mathrm{d}|\mu|(y)|h(x,y)| \leq C(\alpha,\, l)(1+R)^{-l}. \tag{3.58}$$

Proof of Theorem 3.8. Denoting the left side of (3.58) by $I(R)$ we can write

$$I(R) = \int_{\mathcal{X}^0(B_0(\frac{R}{3}))} \mathrm{d}P^0_{+,z}(x^0) \int_{\mathcal{X}^c(B_0(R))} \mathrm{d}|\mu|(y)|h(x,y)| + \int_{\left(\mathcal{X}^0(B_0(\frac{R}{3}))\right)^c} \mathrm{d}P^0_{+,z}(x^0)$$
$$\cdot \int_{\mathcal{X}^c(B_0(R))} \mathrm{d}|\mu|(y)|h(x,y)| = I_1(R) + I_2(R). \tag{3.59}$$

By (3.59)

$$I_1(R) \leq \int_{\mathcal{X}^0(B_0(\frac{R}{3}))} \mathrm{d}P^0_{+,z}(x^0) \int_{\mathcal{X}^c(B_0(R))} \mathrm{d}|\mu|(y)|h(x,y)|$$
$$\leq C\big(1+R\big)^{-l} \int_{\mathcal{X}^0} \mathrm{d}P^0_{+,z}(x^0)\alpha(x^0). \tag{3.60}$$

Notice that (3.57) implies

$$\int_{\mathcal{X}^0} \mathrm{d}P^0_{+,z}(x^0)\alpha(x^0) < \infty. \tag{3.61}$$

Hence we find that

$$I_2(R) \leq \int_{\left(\mathcal{X}^0(B_0(\frac{R}{3}))\right)^c} \mathrm{d}P^0_{+,z}(x^0) \int_{\mathcal{X}} \mathrm{d}|\mu|(y)|h(x,y)| \leq C(\alpha,\, l)(1+R)^{-l} \tag{3.62}$$

Theorem 3.8 is proved.

We need a modification of Assumption 4 which can be applied to the case of hard core interactions.

Assumption 5 . *There exists a nonnegative function a on $\mathcal{X}$ and a number p, $0 < p < 1$, such that for a given $l > 0$ and all $r > 2c$, where c is the radius of a hard core,*

$$\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}\rho_{+,z}(y)\,|q(x,y)|e^{a(y)+2b(y)}\hat{a}(y) \le p\,a(x)\Big(1+\frac{r}{2}\Big)^{-l}. \tag{3.63}$$

for almost all $x \in \mathcal{X}(B_0(R))$.

Assumption 5' is defined like Assumptions 5 with the $u(x,y)$ instead of $q(x,y)$ and $e^{b(\cdot)}$ instead of $e^{2b(\cdot)}$.

Theorem 3.9 (Main bound. Hard core) . *If Assumptions 1, 3 and 5 hold true, then for all positive R, $r > 2c$ and for almost all $x \in \mathcal{X}(B_0(R))$,*

$$\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y)|\sigma_\Lambda(x,y)| \le C(l,p)e^{a(x)+2b(x)}a(x)(1+r)^{-l}. \tag{3.64}$$

In case where Assumptions 1, 3' and 5' hold true the same result is valid with $b(x)$ instead of $2b(x)$.

Proof of Theorem 3.9. In case of hard core interaction, for fixed $r > 0$ the estimate (3.52) is valid only for $m < \frac{r}{2c}$ where c is the radius of the hard core. We modify the proof of Theorem 3.7 in the following way.

Let $r > 4c$ and let $N_r = \Big[\big(\frac{r}{2c}\big)^{\frac{1}{2}}\Big]$ where $[\cdot]$ stands for the integer part. Then

$$\begin{aligned}\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y)|\sigma(x,y)| &\le e^{a(x)+2b(x)}\sum_{m\ge 0}\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y)e^{a(y)+2b(y)} \\ &\cdot \int_{\mathcal{X}^m}\prod_{i=1}^m \mathrm{d}|\mu|(x_i)e^{a(x_i)+2b(x_i)}\prod_{i=0}^m |q(x_i,x_{i+1})| \\ &\le e^{a(x)+2b(x)}\sum_{m=0}^{N_r-1} D_m(x,R,r) + e^{a(x)+2b(x)}\sum_{m=N_r}^{\infty} \\ &\cdot A_m(x) = I_1(x,R,r)+I_2(x,R,r). \end{aligned}\tag{3.65}$$

Consider $I_1(x,R,r)$.

$$\begin{aligned} I_1(x,R,r) &\le e^{a(x)+2b(x)}a(x)\sum_{m=0}^{N_r}(m+1)p^{m+1}\Big(1+\frac{r}{2(m+1)}\Big)^{-l} \le e^{a(x)+2b(x)} \\ &\cdot a(x)\sum_{m=1}^{\infty} mp^m\Big(1+\frac{r}{2m}\Big)^{-l} \le C(l,p)e^{a(x)+2b(x)}a(x)(1+r)^{-l} \end{aligned}\tag{3.66}$$

with $C(l,p) = 2^l\sum_{m=1}^\infty m^{l+1}p^m < \infty$.

Passing to $I_2(x,R,r)$ we note that $D_m(x,R,r) \le A_m(x)$, $m = 0,1,\cdots$, then

$$\begin{aligned} I_2(x,R,r) &\le e^{a(x)+2b(x)}a(x)\sum_{m=N_r+1}^{\infty} p^{m+1} \le e^{a(x)+2b(x)}a(x)\frac{p^{\left(\frac{r}{2c}\right)^{\frac{1}{2}}}}{1-p} \\ &\le C(l,p)e^{a(x)+2b(x)}a(x)(1+r)^{-l}. \end{aligned}\tag{3.67}$$

Thus

$$\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}|\mu|(y)|\hat{Z}(x,y)| \leq C(l,p)e^{a(x)+2b(x)}a(x)(1+r)^{-l}. \tag{3.68}$$

Theorem 3.9 is proved.

3.5 Decay of correlations in loop gases II

We apply the general results of the previous section to the loop gases with different potentials. According to the Theorems 3.7 and 3.9, to prove the theorems of this section, we need only to check the Assumptions 1 - 5. We consider the cases:

a. The case of repulsive integrable potential

b. The case of stable integrable potential

c. Stable potentials with hard core.

In all the cases (a) - (c) we choose the abstract space $\mathbb{X}$ to be the space $\mathcal{X}(\Lambda)$ of composite Brownian loops in Λ, the measure $\mathrm{d}\mu(X) = \mathrm{d}\rho_{\varepsilon,z}(X)e^{-U_1(X)}$ with $\mathrm{d}|\mu|(X) = \mathrm{d}\rho_{+,z}(X)e^{-U_1(X)}$.

We assume that the particles interact via two-body potential ϕ. We consider the following conditions on ϕ:

(1) ϕ is an even function on $\mathbb{R}^d \setminus \{0\}$,

(2) ϕ is stable with stability constant $B \geq 0$: for any different $u_1, \cdots, u_n \in \mathbb{R}^d$,

$$\sum_{i<j}^{n} \phi(u_i - u_j) \geq -nB. \tag{3.69}$$

(3) ϕ has spherically symmetric hard core of radius c, i.e., $\phi(u) = +\infty$ for $|u| \leq c$; ϕ is continuous outside of hard core,

We consider the following decay properties of ϕ at infinity. Let

$$\phi_l(u) = \phi(u)(1+|u|)^l, \;\; l \geq 0, \;\; \phi_0 \equiv \phi. \tag{3.70}$$

(4) ϕ has the following decay at infinity:

$$||\phi_l||_1 = \int_{\mathbb{R}^d} \mathrm{d}u|\phi_l(u)| < \infty, \;\; l \geq 0 \tag{3.71}$$

(4')

$$p_{c,l}(\phi) = \int_{|u|>c} du|\phi_l(u)| < +\infty, \;\; l \geq 0, \tag{3.72}$$

3.5.1 Repulsive integrable potentials

We start with the case of repulsive integrable potentials.

Theorem 3.10 (Repulsive integrable potentials) . *Let the pair interaction $\phi \geq 0$ satisfy conditions (1) and (4). Let $z < e^{-3}$ be from the interval*

$$zC(d,l)||\phi_l||_1\zeta\Big(\frac{d}{2}+1\Big)\beta^{1-\frac{d}{2}}\Big(1+\beta^{\frac{l}{4}-1}\Big) \leq p < 1. \tag{3.73}$$

Then for all positive $R,\ r$ and for almost all $x \in \mathcal{X}(B_0(R))$,

$$\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}\rho_{+,z}(Y)|\sigma_\Lambda(X,Y)| \leq C(l,p)e^{|X|}|X|(1+r)^{-l}, \tag{3.74}$$

for any bounded domain $\Lambda \subset \mathbb{R}^d$.

Proof of Theorem 3.10. We use Theorem 3.7 and we need to check Assumptions 1, 3, and 4 with $\mathbb{X} = \mathcal{X}(\Lambda)$, $\mathrm{d}\mu(X) = \rho_{+,z}(X)e^{-U_1(X)},\ q(X,Y) = e^{-U_2(X,Y)}$.

Since $\phi \geq 0$, Assumption 1 is fulfilled obviously with $b \equiv 0$.

Next we check Assumptions 3 with $a(X) = |X|$. Since $1 - e^u \leq u$ for positive u, using the fact that $e^{-\frac{x}{2}}x^2 \leq 4$ we find that

$$\int_{\mathcal{X}(\Lambda)} \mathrm{d}\rho_{+,z}(Y)e^{-U_1(Y)}e^{|Y|}|Y||q(X,Y)| \leq \int_{\mathcal{X}^0} \mathrm{d}P^0_{+,z}(Y^0)e^{|Y^0|}|Y^0| \int_{\mathbb{R}^d} \mathrm{d}vU_2(X,$$
$$\cdot Y^0 + v) = |X|\beta||\phi||_1 \int_{\mathcal{X}^0} \mathrm{d}P^0_{+,z}(Y^0)e^{|Y^0|}$$
$$\cdot |Y^0|^2 \leq 4|X|\beta||\phi||_1 \int_{\mathcal{X}^0} \mathrm{d}P^0_{+,z}(Y^0)e^{\frac{3}{2}|Y^0|}$$

Here and below, to check the validity of the assumptions, we lift the restriction that $Y \in \mathcal{X}(\Lambda)$ because we want a condition that does not depend on Λ. Hence Assumptions 3 is valid for all z satisfying

$$4ze^{\frac{3}{2}}||\phi||_1(2\pi)^{-\frac{d}{2}}\beta^{1-\frac{d}{2}}\zeta\left(\frac{d}{2}+1\right) \leq p < 1. \tag{3.75}$$

To complete this case it remains to check Assumption 4 for a repulsive potential ϕ.

Let $X \in \mathcal{X}(B_0(R))$, then we need to show that

$$\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}\rho_{+,z}(Y)e^{-U_1(Y)}e^{|Y|}|Y||q(X,Y)| \leq p\,|X|\Big(1+\frac{r}{2}\Big)^{-l}. \tag{3.76}$$

Denoting the left hand side of (3.76) by $D_0(x,R,r)$, we split it into two parts

$$D_0(x,R,r) = D_0' + D_0'' \tag{3.77}$$

where

$$D_0' = \int_{\mathcal{X}^c(B_0(R+r))\mathcal{X}(B_0^c(R+\frac{r}{2})} \mathrm{d}\rho_{+,z}(Y)e^{-U_1(Y)}\,|q(X,Y)|e^{|Y|}|Y|$$

and

$$D_0'' = \int_{\mathcal{X}^c(B_0(R+r))\mathcal{X}^c(B_0^c(R+\frac{r}{2})} \mathrm{d}\rho_{+,z}(Y)e^{-U_1(Y)}\,|q(X,Y)|e^{|Y|}|Y|.$$

By Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{d}v U_2(X, Y^0+v) &= \sum_{x\in X}\sum_{y^0\in Y^0}\int_{\mathbb{R}^d}\mathrm{d}v\int_0^\beta dt\phi(x(t)-y^0(t)-v)\\ &= |X||Y^0|\beta||\phi||_1. \end{aligned}\tag{3.78}$$

Therefore

$$\begin{aligned}D_0' &\le \int_{\mathcal{X}^c(B_0(R+r))\mathcal{X}(B_0^c(R+\frac{r}{2})} \mathrm{d}\rho_{+,z}(Y)e^{-U_1(Y)}\,U_2(X,Y)e^{|Y|}|Y| = \int_{\mathcal{X}^0}\mathrm{d}P^0_{+,z}(Y^0)\\ &\cdot e^{|Y^0|}|Y^0|\int_{\mathbb{R}^d}\mathrm{d}v 1_{\mathcal{X}^c(B_0(R+r))\mathcal{X}(B_0^c(R+\frac{r}{2})}(Y^0+v)U_2(X,Y^0+v)\\ &\le |X|\beta\int_{|v|>\frac{r}{2}}\mathrm{d}v\phi(v)\int_{\mathcal{X}^0}\mathrm{d}P^0_{+,z}(Y^0)e^{|Y^0|}|Y^0|^2\\ &\le 4|X|\beta||\phi_l||_1(2\pi\beta)^{-\frac{d}{2}}\sum_{j=1}^\infty\frac{(ze^{\frac{3}{2}})^j}{j^{\frac{d}{2}+1}}\Big(1+\frac{r}{2}\Big)^{-l}.\end{aligned}\tag{3.79}$$

Here we used the fact that

$$\int_{|v|>\frac{r}{2}}|\phi(v)| \le \Big(1+\frac{r}{2}\Big)^{-l}\int_{\mathbb{R}^d}dv|\phi(v)|(1+|v|)^l,$$

We treat now D_0''.

$$\begin{aligned}D_0'' &= \int_{\mathcal{X}^c(B_0(R+r))\mathcal{X}^c(B_0^c(R+\frac{r}{2})}\mathrm{d}\rho_{+,z}(Y)e^{-U_1(Y)}\,U_2(X,Y)e^{|Y|}|Y| \le \int_{\mathcal{X}^0}\mathrm{d}P^0_{+,z}(Y^0)\\ &\cdot 1_{diamY^0>\frac{r}{2}}(Y^0)e^{|Y^0|}|Y^0|\int_{\mathbb{R}^d}\mathrm{d}vU_2(X,Y^0+v) \le |X|\beta||\phi||_1\int_{\mathcal{X}^0}\mathrm{d}P^0_{+,z}(Y^0)\\ &\cdot 1_{diamY^0>\frac{r}{2}}(Y^0)e^{|Y^0|}|Y^0|^2 \le 4|X|\beta||\phi||_1\int_{\mathcal{X}^0}\mathrm{d}P^0_{+,z}(Y^0)1_{diamY^0>\frac{r}{2}}(Y^0)e^{\frac{3}{2}|Y^0|}.\end{aligned}\tag{3.80}$$

By Schwarz inequality

$$\begin{aligned}\int_{\mathcal{X}^0}\mathrm{d}P^0_{+,z}(Y^0)e^{\frac{3}{2}|Y^0|}1_{diamY^0\ge\frac{r}{2}}(Y^0) &\le \Big(\int_{\mathcal{X}^0}\mathrm{d}P^0_{+,z}(Y^0)e^{3|Y^0|}\Big)^{\frac{1}{2}}\\ &\cdot\Big(\int_{\mathcal{X}^0}\mathrm{d}P^0_{+,z}(Y^0)1_{diamY^0\ge\frac{r}{2}}(Y^0)\Big)^{\frac{1}{2}}.\end{aligned}$$

We observe that

$$\int_{\mathcal{X}^0}\mathrm{d}P^0_{+,z}(Y^0)e^{3|Y^0|} = (2\pi\beta)^{-\frac{d}{2}}\sum_{j=1}^\infty\frac{(ze^3)^j}{j^{\frac{d}{2}+1}},$$

and by Corollary 3.5,

$$\int_{\mathcal{X}^0}\mathrm{d}P^0_{+,z}(Y^0)1_{diamY^0\ge\frac{r}{2}}(Y^0) \le C(d,l)\zeta\Big(\frac{d}{2}+1\Big)\beta^{\frac{l-d}{2}}\Big(1+\frac{r}{2}\Big)^{-l}.\tag{3.81}$$

Hence

$$D_0'' \leq \sqrt{z}|X|C(d,l)||\phi||_1\zeta\Big(\frac{d}{2}+1\Big)\beta^{\frac{l}{4}-1}\Big(1+\frac{r}{2}\Big)^{-l}. \tag{3.82}$$

Combining (3.77), (3.79), (3.82) we get

$$D_0 \leq z|X|C(d,l)||\phi_l||_1\zeta\Big(\frac{d}{2}+1\Big)\beta^{1-\frac{d}{2}}\Big(1+\beta^{\frac{l}{4}-1}\Big)\Big(1+\frac{r}{2}\Big)^{-l}. \tag{3.83}$$

Thus Assumption 4 is fulfilled for all $z \leq e^{-3}$ which in addition satisfy

$$zC(d,l)||\phi_l||_1\zeta\Big(\frac{d}{2}+1\Big)\beta^{1-\frac{d}{2}}\Big(1+\beta^{\frac{l}{4}-1}\Big) \leq p < 1. \tag{3.84}$$

This completes the proof of Theorem 3.10.

3.5.2 Stable integrable potentials

Next we consider the case of stable integrable potentials

Theorem 3.11 (Stable integrable potentials) . *Let the pair interaction ϕ satisfy conditions (1), (2) and (4) and let $z < e^{-3(1+\beta B)}$ be from the interval*

$$zC(d,l)||\phi_l||_1\zeta\Big(\frac{d}{2}+1\Big)e^{\frac{3\beta B}{2}}\beta^{1-\frac{d}{2}}\Big(1+\beta^{\frac{l}{4}-1}\Big) \leq p < 1. \tag{3.85}$$

then for all positive $R,\ r$ and for almost all $x \in \mathcal{X}(B_0(R))$,

$$\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}\rho_{+,z}(Y)|\sigma_\Lambda(X,Y)| \leq C(l,p)e^{|X|}|X|(1+r)^{-l}, \tag{3.86}$$

for any bounded domain $\Lambda \subset \mathbb{R}^d$.

Proof of Theorem 3.11. We use again Theorem 3.7 and we want to check Assumptions 1, 3' and 4' with $\mathbb{X} = \mathcal{X}(\Lambda)$, $\mathrm{d}\mu(X) = \rho_{+,z}(X)e^{-U_1(X)}$, $u(X_1,X_2) = U_2(X_1,X_2)$. We suppose that the pair potential ϕ is stable with stability constant B, i.e. there exists a constant $B \geq 0$ such that for any n and any $u_1,\cdots,u_n \in \mathbb{R}^d$, $\sum_{1\leq i<j\leq n}\phi(u_i-u_j) \geq -Bn$. For given composite loops $X_1,\cdots,X_n$ stability implies

$$\sum_{1\leq i<j\leq n} U_2(X_i,X_j) + \sum_{i=1}^n U_1(X_i) = U(X_1,\cdots,X_n) \geq -\beta B\sum_{i=1}^n |X_i|.$$

Then Assumption 1 holds with

$$b(X) = \beta B|X| + U_1(X). \tag{3.87}$$

Notice that $b(X) \geq 0$, by the stability of ϕ. We use Assumption 3' with $a(X) = |X|$. Explicitly the assumption is that for any $X \in \mathcal{X}(\Lambda)$,

$$\int_{\mathcal{X}(\Lambda)} \mathrm{d}\rho_{+,z}(Y)|U_2(X,Y)|e^{|Y|(1+\beta B)}|Y| \leq p|X|,\ \ 0<p<1. \tag{3.88}$$

We have

$$\int_{\mathcal{X}(\Lambda)} \mathrm{d}\rho_{+,z}(Y)|U_2(X,Y)|e^{|Y|(1+\beta B)}|Y| \leq \int_{\mathcal{X}^0} \mathrm{d}P^0_{+,z}(Y)e^{|Y^0|(1+\beta B)}|Y^0| \int_{\mathbb{R}^d} dv|U_2(X,Y^0+v)|.$$

Then

$$\int_{\mathbb{R}^d} dv|U_2(X,Y^0+v)| \leq |X||Y^0|\beta||\phi||_1.$$

Repeating the arguments which were used in the proof of Theorem 3.9, we can write

$$\int_{\mathcal{X}} \mathrm{d}\rho_{+,z}(Y)|U_2(X,Y)|e^{|Y|(1+\beta B)}|Y| \leq 4|X|\beta||\phi||_1(2\pi\beta)^{-\frac{d}{2}}\sum_{j=1}^{\infty}\frac{(ze^{\frac{3}{2}(1+\beta B)})^j}{j^{\frac{d}{2}+1}}$$
$$\leq 4|X|||\phi||_1(2\pi)^{-\frac{d}{2}}\beta^{1-\frac{d}{2}}ze^{\frac{3}{2}(1+\beta B)}\zeta\Big(\frac{d}{2}+1\Big)$$

provided $ze^{1+\beta B} \leq 1$. Hence Assumption 3' is fulfilled if $ze^{1+\beta B} \leq 1$ and

$$4z||\phi||_1(2\pi)^{-\frac{d}{2}}\beta^{1-\frac{d}{2}}e^{\frac{3}{2}(1+\beta B)}\zeta\Big(\frac{d}{2}+1\Big) \leq p < 1 \tag{3.89}$$

Let us check Assumption 4':

$$\int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}\rho_{+,z}(Y)e^{|Y|(1+\beta B)}|Y||U_2(X,Y)| \leq p\,|X|\Big(1+\frac{r}{2}\Big)^{-l}. \tag{3.90}$$

We denote the right hand side of (3.90) by $\tilde{D}_0(X,R,r)$ and treat it similarly to $D_0(X,R,r)$ from the previous subsection. We split $\tilde{D}_0(X,R,r)$ in two parts $\tilde{D}_0'$ and $\tilde{D}_0''$, where

$$\tilde{D}_0' = \int_{\mathcal{X}^c(B_0(R+r))\mathcal{X}(B_0^c(R+\frac{r}{2}))} \mathrm{d}\rho_{+,z}(Y)e^{|Y|(1+\beta B)}|U_2(X,Y)||Y|$$

and

$$\tilde{D}_0'' = \int_{\mathcal{X}^c(B_0(R+r))\mathcal{X}^c(B_0^c(R+\frac{r}{2}))} \mathrm{d}\rho_{+,z}(Y)e^{|Y|(1+\beta B)}|U_2(X,Y)||Y|.$$

Then for $ze^{3(1+\beta B)}$,

$$\tilde{D}_0' \leq 4|X|\,(2\pi)^{-\frac{d}{2}}||\phi_l||_1\beta^{1-\frac{d}{2}}\sum_{j=1}^{\infty}\frac{(ze^{\frac{3}{2}(1+\beta B)})^j}{j^{\frac{d}{2}+1}}\Big(1+\frac{r}{2}\Big)^{-l}. \tag{3.91}$$

and

$$\tilde{D}_0'' \leq \sqrt{z}C(d,l)|X|||\phi||_1\zeta\Big(\frac{d}{2}+1\Big)e^{\frac{3}{2}(1+\beta B)}\beta^{1-\frac{d}{2}}\left(1+\beta^{\frac{l}{4}-1}\right) \leq p < 1. \tag{3.92}$$

Thus

$$\tilde{D}_0(X,R,r) \leq \sqrt{z}|X|||\phi_l||_1C(d,l)e^{3(1+\beta B)}\beta^{1-\frac{d}{2}}\left(1+\beta^{\frac{l}{4}-1}\right)\zeta\Big(\frac{d}{2}+1\Big)\Big(1+\frac{r}{2}\Big)^{-l} \tag{3.93}$$

Hence Assumption 4' is valid if $ze^{3(1+\beta B)}$ and in addition

$$\sqrt{z}||\phi_l||_1C(d,l)e^{3(1+\beta B)}\beta^{1-\frac{d}{2}}\left(1+\beta^{\frac{l}{4}-1}\right)\zeta\Big(\frac{d}{2}+1\Big) \leq p < 1. \tag{3.94}$$

Theorem 3.11 is proved.

3.5.3 Boltzmann gas. Hard core potentials

Stable potentials with hard core. Consider a low density quantum gas with hard core potential and MB statistics. We assume that potentials ϕ satisfies conditions (1), (2), (3) and (4'). The presence of a hard core makes the situation more complicated and involves estimates of integrals of Wiener sausages.

The Wiener sausage $S(x)$ generated by $x \in \mathcal{X}$ we define by

$$S(x) = \{u \in \mathbb{R}^d \mid |x(t) - u| \leq c \text{ for some } t \in [0, \beta]\}.$$

and denote by $|S(x)|$ the volume of $S(x)$.

Let

$$E_j(k) = \int_{\mathcal{X}^{00}} \mathrm{d}P_\beta^{00}(y^0)\, e^{k|S(y^0)|}\, |S(y^0)|^j \quad j,k = 0,1,2,\cdots . \tag{3.95}$$

It follows from Lemma 5.11 below that $E_j(k) < \infty$. We set also

$$E = \max(E_j(1), \sqrt{E_j(2)}, j = 0,1,2), \tag{3.96}$$

as well as

$$\xi = e^{2\beta B+1}E, \quad \eta = e^{2\beta B}\beta p_{c,l}(\phi) \tag{3.97}$$

where $p_{c,l}(\phi)$ is defined by (4'), formula (3,72).

Theorem 3.12 (Hard core stable potentials) . *Let the potential ϕ satisfy the conditions (1), (2), (3) and (4') and the activity z satisfy the relation*

$$zC(d,l)Ee^{2\beta B+1}(b_c^{-1} + e^{2\beta B}\beta p_{c,l}(\phi))\left(1 + \beta^{\frac{l-d}{4}}\right) \leq p < 1 \tag{3.98}$$

where b_c is the volume of a ball of radius c in $\mathbb{R}^d$. Then for all $R > 0, r > 2c$ and all $x \in \mathcal{X}(B_0(R))$, the two-point truncated correlation functions satisfy the bound

$$z\int_{\mathcal{X}^c(B_0(R+a))} d\rho(y)|\sigma_\Lambda(x,y)| \leq C(l,p)e^{1+2\beta B}e^{|S(x)|}(|S(x)|+1)(1+r)^{-l}.$$

The proof of Theorem 3.12 is based on Theorem 3.9. To apply this theorem we choose $\mathbb{X} = \mathcal{X}_\beta$, $\mu(dx) = z\rho(dx), a(x) = |S(x)| + 1$ and $b(x) = \beta B$ where B is the stability constant. Assumption 1 is satisfied evidently.

To check Assumption 3 we need the following lemma. Let

$$h_{\beta,k}(u) = \int_{\mathcal{X}_\beta^{0u}} \mathrm{d}P_\beta^{0u} e^{k|S(x)|}, \quad k \geq 0. \tag{3.99}$$

Lemma 3.13 . *For any $k \geq 0$ and $\beta > 0$,*

$$\sup_{u\in\mathbb{R}^d} h_{\beta,k}(u) < \infty \text{ and } ||h_{\beta,k}||_1 = \int_{\mathbb{R}^d} \mathrm{d}u\, h_{\beta,k}(u) < \infty. \tag{3.100}$$

Proof of Lemma 3.13. Let $M_p = M(b, \frac{\beta}{2p}),\ p = 1, 2, \cdots; b > 0$, where $M(\varepsilon, \delta)$ is defined by (3.33). It is clear that if $x \in M_p^c$ then, by the continuity of the Brownian trajectories, x is contained in the union of p balls of radius b centered at the points $x((2n-1)\beta/2p)$:

$$\{x(t), t \in [0, \beta]\} \subset \bigcup_{n=1}^{p} B_{x((2n-1)\beta/2p)}(b).$$

Therefore

$$S(x) \subset \bigcup_{n=1}^{p} B_{x((2n-1)\beta/2p)}(b + c). \tag{3.101}$$

Hence for $x \in M_p^c,\ |S(x)| \le pw$, where w is the volume of the ball in $\mathbb{R}^d$ of radius $b+c$. Note that $M_p^c \subset M_{p+1}^c,\ p = 1, 2, \cdots$ and $M_p^c \uparrow \mathcal{X}_\beta^{0u}$ as $p \to \infty$, hence $\mathcal{X}_\beta^{0u} = \sum_{p=0}^{\infty} (M_{p+1}^c \setminus M_p^c)$ with $M_0 \equiv \mathcal{X}_\beta^{0u}$. (By $\sum A_k$ we denote the union of disjoint sets A_k.)

From (3.101) we find for all $u \in \mathbb{R}^d$,

$$\begin{aligned}\int_{\mathcal{X}^{0u}} \mathrm{d}P_\beta^{0u}(x) e^{k|S(x)|} \le & \sum_{p=0}^{\infty} e^{(p+1)kw} \int_{M_{p+1}^c \setminus M_p^c} \mathrm{d}P_\beta^{0u}(x) = \sum_{p=0}^{\infty} e^{(p+1)kw} \\ & \cdot [P_\beta^{0u}(M_p) - P_\beta^{0u}(M_{p+1})] = \sum_{p=0}^{\infty} e^{(p+1)kw} P_\beta^{0u}(M_p) \\ & - \sum_{p=0}^{\infty} e^{(p+1)kw} P_\beta^{0u}(M_{p+1}) = e^{kw} P_\beta^{0u}(\mathcal{X}_\beta^{ou}) \\ & + (e^{kw} - 1) \sum_{p=1}^{\infty} e^{pkw} P_\beta^{0u}(M_p).\end{aligned} \tag{3.102}$$

Similarly

$$\int_{\mathbb{R}^d} \mathrm{d}u h_{\beta,k}(u) \le e^{kw} + (e^{kw} - 1) \sum_{p=1}^{\infty} e^{pkw} P_\beta^0(M_p) \tag{3.103}$$

where $\hat{P}_\beta^0(M_p) = \int_{\mathbb{R}^d} \mathrm{d}u P_\beta^{0u}(M_p)$. Hence

$$h_{\beta,k}(u) \le \frac{e^{kw}}{(2\pi\beta)^{\frac{d}{2}}} + \frac{C(d)}{(\pi\beta)^{\frac{d}{2}}} \left(\frac{b}{4\sqrt{2\beta}} \right)^{d-1} \sum_{p=1}^{\infty} p^{\frac{d+1}{2}} \exp\left(pkw - \frac{pb^2}{64\beta} \right). \tag{3.104}$$

The series in (3.104) converges for $\beta < b^2/64kw$, hence for such β,

$$\sup_{u \in \mathbb{R}^d} h_{\beta,k}(u) < +\infty. \tag{3.105}$$

According to Lemma 4 from Appendix A in [44] $\hat{P}_\beta^0(M_p)$ satisfies a bound similar to (3.34), therefore for the same $\beta < b^2/64kw$, the series in (3.104) converges and

$$||h_{\beta,k}||_1 < \infty. \tag{3.106}$$

We claim now that (3.105) and (3.106) hold true for any real positive β and k. To prove this we note that to any two trajectories $x_1 \in \mathcal{X}_{\beta_1}, x_2 \in \mathcal{X}_{\beta_2}$ such that $x_1(\beta_1) = x_2(0)$ we can associate a new trajectory $x_1 \star x_2 \in \mathcal{X}_{\beta_1+\beta_2}$ defined by

$$x_1 \star x_2(t) = \begin{cases} x_1(t), 0 \leq t \leq \beta_1 \\ x_2(t-\beta_1), \beta_1 \leq t \leq \beta_1 + \beta_2. \end{cases} \tag{3.107}$$

It is easy to see that

$$|S(x_1 \star x_2)| = |S(x_1)| + |S(x_2)| - w_c < |S(x_1)| + |S(x_2)| \tag{3.108}$$

where w_0 is the volume of the hard core. Vice versa, any $x \in \mathcal{X}^{0,u}_{\beta_1+\beta_2}$ can be written as $x = x_1 \star x_2,\ x_1 \in \mathcal{X}^{0,v}_{\beta_1},\ x_1 \in \mathcal{X}^{v,u}_{\beta_2}$ where $v = x(\beta_1)$ and $x_1(t) = x(t), 0 \leq t \leq \beta_1;\ x_2(t) = x(t+\beta_1), 0 \leq t \leq \beta_2$. Therefore

$$\begin{aligned} h_{\beta_1+\beta_2,k}(u) &= \int_{\mathbb{R}^d} \mathrm{d}v \int_{\mathcal{X}^{0,v}_{\beta_1}} \mathrm{d}P^{0,v}_{\beta_1}(x_1) \int_{\mathcal{X}^{v,u}_{\beta_2}} \mathrm{d}P^{v,u}_{\beta_2}(x_2) e^{k|S(x_1 \star x_2)|} \\ &\leq \int_{\mathbb{R}^d} \mathrm{d}v \int_{\mathcal{X}^{0,v}_{\beta_1}} \mathrm{d}P^{0,v}_{\beta_1}(x_1) \int_{\mathcal{X}^{0,u-v}_{\beta_2}} \mathrm{d}P^{0,u-v}_{\beta_2}(x_2) e^{|S(x_1)|+|S(x_2)|} \\ &= \int_{\mathbb{R}^d} \mathrm{d}v h_{\beta_1,k}(v) h_{\beta_2,k}(u-v) = h_{\beta_1,k} * h_{\beta_2,k}(u) \end{aligned} \tag{3.109}$$

where $*$ means convolution in $\mathbb{R}^d$. This implies $h_{\beta,k}(u) \leq h^{*n}_{\frac{\beta}{n},k}(u)$ for all $n = 1, 2, \cdots,\ u \in \mathbb{R}^d$ and

$$h_{\beta_1+\beta_2,k}(u) \leq \sup_u h_{\beta_1,k}(u) ||h_{\beta_2,k}||_1 \tag{3.110}$$

provided $\max(\beta_1, \beta_2) < b^2/64kw$. Hence $||h_{\beta,k}||_1 \leq ||h_{\frac{\beta}{n},k}||_1^n$ and

$$\sup_u h_{\beta,k}(u) \leq \sup_u h_{\frac{\beta}{n},k}(u) ||h_{\frac{\beta}{n},k}||_1^{n-1}.$$

Therefore taking n sufficiently large so that $\frac{\beta}{n} < b^2/64kw$ we establish the claim.

Lemma 3.13 is proved.

Now using Lemma 3.13 we can find that interval for z where the Assumption 3 is valid. For $m = 0, 1, \cdots$ and $j = 0, 1, 2, \cdots$ we set

$$\begin{aligned} A^{(j)}_m(x,z) = \int_{\mathbb{X}} \mathrm{d}|\mu|(y) e^{|S(y)|+2\beta B+1} \int_{\mathbb{X}^m} \prod_{i=1}^m \mathrm{d}|\mu|(x_i) e^{|S(x_i)|+2\beta B+1} \prod_{i=0}^m |q(x_i, x_{i+1})| \\ \cdot |S(y)|^j,\ \ A^{(0)}_m(x,z) \equiv A_m(x,z) \end{aligned} \tag{3.111}$$

where $x_0 = x, x_{m+1} = y$ and

$$A_0(x,z) = \int_{\mathbb{X}} \mathrm{d}|\mu|(y) e^{|S(y)|+2\beta B+1} |q(x,y)| \tag{3.112}$$

Consider $A_1^{(j)}(x,z)$ for $j=0,1,2$. By definition

$$A_1^{(j)}(x,z)=z\int_{\mathcal{X}_\beta^{00}}\mathrm{d}P_\beta^{00}(y^0)e^{|S(y^0)|+2\beta B+1}|S(y^0)|^j\int_{\mathbb{R}^d}\mathrm{d}u|q(x,y^0+u)|. \tag{3.113}$$

Using the bound $|S(x-y)|\le|S(x)||S(y)|b_c^{-1}$ (see Lemma in Appendix 2 of [44]), and the well know inequality [46]

$$|q(x,y)|\le\int_0^\beta\mathrm{d}t|\phi(x(t)-y(t))|\cdot\exp\int_0^\beta\mathrm{d}t\phi_-(x(t)-y(t)), \tag{3.114}$$

where $\phi_-(u)=\max(-\phi(u),0)$, we find

$$\begin{aligned}\int_{\mathbb{R}^d}\mathrm{d}u|q(x,y^0+u)|=&\int_{S(x-y^0)}\mathrm{d}u|q(x,y^0+u)|+\int_{\mathbb{R}^d\setminus S(x-y^0)}\mathrm{d}u|q(x,y^0+u)|\\ \le&|S(x-y^0)|+\beta e^{2\beta B}p_c(\phi)\\ \le&|S(x)||S(y^0)|b_c^{-1}+\beta e^{2\beta B}p_c(\phi).\end{aligned} \tag{3.115}$$

Substituting this into (3.113) and using notations (3.97) we get

$$\begin{aligned}A_1^{(j)}(x,z)=z\int_{\mathcal{X}}\mathrm{d}\rho(y)e^{|S(y)|+2\beta B+1}|S(y)|^j|q(x,y)|\le z\xi(|S(x)|b_c^{-1}+\eta)\\ \le z\xi(b_c^{-1}+\eta)(|S(x)|+1).\end{aligned} \tag{3.116}$$

Hence Assumption 3 holds true for all z from the interval (3.98).

Next we consider Assumption 5 Let

$$D_1^{(j)}(x,z;R,r)=\int_{\mathcal{X}^c(B_0(R+r))}z\mathrm{d}\rho(y)e^{|S(y)|+2\beta B+1}|S(y)|^j|q(x,y)|, \tag{3.117}$$

$j=0,1,\cdots$, with $D_1^{(0)}(x,z;R,r)\equiv D_1(x,z;R,r)$.

Then

$$\begin{aligned}D_1^{(j)}(x,z;R,r)=&\int_{\mathcal{X}^c(B_0(R+r))\mathcal{X}(B_0^c(R+\frac{r}{2}))}z\mathrm{d}\rho(y)e^{|S(y)|+2\beta B+1}|S(y)|^j|q(x,y)|\\ &+\int_{\mathcal{X}^c(B_0(R+r))\mathcal{X}^c(B_0^c(R+\frac{r}{2}))}z\mathrm{d}\rho(y)e^{|S(y)|+2\beta B+1}|S(y)|^j|q(x,y)|\\ \equiv&D_1'+D_1''.\end{aligned} \tag{3.118}$$

Consider first D_1'.

$$\begin{aligned}D_1'=&ze^{2\beta B+1}\int_{\mathcal{X}_\beta^{00}}\mathrm{d}P_\beta^{00}(y^0)e^{|S(y^0)|}|S(y^0)|^j\int_{\mathbb{R}^d}\mathrm{d}u1_{\mathcal{X}^c(B_0(R+r))\mathcal{X}(B_0^c(R+\frac{r}{2}))}\\ &\cdot(y^0+u)|q(x,y^0+u)|.\end{aligned}$$

Let $r>2c$, then hard core plays no role in the last integral, therefore

$$\begin{aligned}D_1'\le&ze^{2\beta B+1}e^{2\beta B}\beta\int_{|u|>\frac{r}{2}}\mathrm{d}u|\phi(u)|\int_{\mathcal{X}_\beta^{00}}\mathrm{d}P_\beta^{00}(y^0)e^{|S(y^0)|}|S(y^0)|^j\\ \le&ze^{2\beta B+1}e^{2\beta B}\beta E_j(1)p_{c,l}(\phi)\left(1+\frac{r}{2}\right)^{-l}.\end{aligned} \tag{3.119}$$

Passing to D_1'' and using (3.115) we have

$$D_1'' \leq ze^{2\beta B+1} \int_{\mathcal{X}_\beta^{00}} \mathrm{d}P_\beta^{00}(y^0) e^{|S(y^0)|} |S(y^0)|^j 1_{M(\frac{r}{2},\beta)}(y^0) \int_{\mathbb{R}^d} \mathrm{d}u |q(x, y^0+u)|$$
$$\leq ze^{2\beta B+1} |S(x)| b_c^{-1} \int_{\mathcal{X}_\beta^{00}} \mathrm{d}P_\beta^{00}(y^0) e^{|S(y^0)|} |S(y^0)|^{j+1} 1_{M(\frac{r}{2},\beta)}(y^0)$$
$$+ ze^{2\beta B+1} e^{2\beta B} \beta p_c(\phi) \int_{\mathcal{X}_\beta^{00}} \mathrm{d}P_\beta^{00}(y^0) e^{|S(y^0)|} |S(y^0)|^j 1_{M(\frac{r}{2},\beta)}(y^0). \tag{3.120}$$

By Schwarz inequality and Corollary 3.6 we have for $j = 0, 1, 2$,

$$\int_{\mathcal{X}_\beta^{00}} \mathrm{d}P_\beta^{00}(y^0) e^{|S(y^0)|} |S(y^0)|^j 1_{M(\frac{r}{2},\beta)}(y^0) \leq E \left(P_\beta^{00}(M(\frac{r}{2},\beta)) \right)^{\frac{1}{2}}$$
$$\leq EC(d,l) \beta^{\frac{l-d}{4}} \left(1 + \frac{r}{2}\right)^{-l}. \tag{3.121}$$

Combining this with (3.118) - (3.120) we get for $, j = 0, 1,$

$$D_1^{(j)}(x, z; R, r) \leq z\xi \left[\eta + C(d,l) \beta^{\frac{l-d}{4}} (|S(X)| b_c^{-1} + \eta) \right] \left(1 + \frac{r}{2}\right)^{-l} \tag{3.122}$$
$$\leq z\xi (b_c^{-1} + \eta) \left(1 + C(d,l) \beta^{\frac{l-d}{4}}\right) (|S(x)| + 1) \left(1 + \frac{r}{2}\right)^{-l}$$

where $C = C(\beta, z, d) > 0$. This implies that for all z satisfying (3.98) Assumption 5 holds true.

Then we can apply general Theorem 3.9 to complete the proof of Theorem 3.12.

Combining Theorems 3.12 and 3.8 one can get, with the help of Lemma 3.13, the following result

Corollary 3.14 . Under the conditions of Theorem 3.12 the following bound

$$z \int_{\mathcal{X}_\beta^{00}} dP_\beta^{00}(x^0) \int_{\mathcal{X}^c(B_u(R))} d\rho(y) |\sigma_\Lambda(x^0 + u, y)| \leq C(\Phi, \beta, z, d, c)(1+R)^{-l}$$

holds true for all $u \in \mathbb{R}^d$.

3.5.4 Boltzmann gas. Integrable potentials

At the end of this chapter we present a result on the decay of correlations in the Boltzmann loop gas with integrable potential which is the main object that we consider in Chapter 5.

Let $\mathcal{X} = \mathcal{X}_\beta$ be the space of simple (elementary) loops in $\mathbb{R}^d$ of the length β. We recall that the intensity measure for the Boltzman gas is $z\rho = \rho_{\varepsilon,z,\mathcal{X}_\beta}$, the restriction of the measure $\rho_{\varepsilon,z}$ on the space $\mathcal{X}_\beta$ of simple (elementary) loops of the length β, where $\rho_{\varepsilon,z,\mathcal{X}_\beta}$ is given by (3.36). The space of configurations for the Boltzmann gas is

$$\mathcal{M} = \{\omega \subset \mathcal{X} \mid |\omega| < \infty\}, \tag{3.123}$$

the space of finite collections of simple loops.

On the space $\mathcal{M}$ we consider the reference measure given by (3.37) which for the case of the Boltzmann loop gas becomes:

$$W_{z\rho}(h) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\mathcal{X}^n} \mathrm{d}\rho(x_1) \cdots \mathrm{d}\rho(x_n) h(x_1, \cdots, x_n). \tag{3.124}$$

Also the energy $U(\omega)$ of a configuration $\omega \in \mathcal{M}$ takes a simpler form

$$U(\omega) = \frac{1}{2} \sum_{x,y \in \omega} \hat{\phi}(x-y) \tag{3.125}$$

with $\hat{\phi}(x-y)$ from (3.16).

The Gibbs measure on $\mathcal{M}(\Lambda)$ for a bounded region Λ is given by

$$Q(\Lambda, z) = \frac{\exp(-U)}{Z_(\Lambda, z)} W_{z,\rho_\Lambda} \tag{3.126}$$

where the grand partition function

$$Z(\Lambda, z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\mathcal{X}_\Lambda^n} \mathrm{d}\rho(X_1) \cdots \mathrm{d}\rho(X_n) \exp(-U(X_1, ..., X_n). \tag{3.127}$$

We call $(\mathcal{M}(\Lambda), Q(\Lambda, z))$ *Boltzmann gas in* Λ *with potential* ϕ *and activity* z*.*

Theorem 3.15 (MB statistics) . *Let the potential* ϕ *satisfy the conditions (1), (2) and (4) and the fugacity* z *satisfy the relation*

$$zC(d,l)||\phi_l||_1 e^{1+\beta B} \beta^{1-\frac{d}{2}} (1 + \beta^{\frac{l}{2}}) \leq p < 1. \tag{3.128}$$

Then for all $R > 0, r > 0$ *and all* $x \in \mathcal{X}(B_0(R))$*, the two-point truncated correlation functions satisfy the bound*

$$z \int_{\mathcal{X}^c(B_0(R+r))} \mathrm{d}\rho(y) |\sigma_\Lambda(x,y)| \leq C(l,p) e^{1+\beta B} (1+r)^{-l}.$$

Proof of Theorem 3.15. The proof is carried out with the help of Theorem 3.7 with $\mathbb{X} = \mathcal{X}_\beta$, $\mathrm{d}\mu(x) = z\mathrm{d}\rho(x)$, $b(x) = \beta B$, $a(x) = 1$ and $u(x,y) = \hat{\phi}(x,y)$ where $\hat{\phi}$ is given by (3.16). Assumption 1 holds trivially. Assumption 3' becomes

$$ze^{1+\beta B} \int_{\mathcal{X}_\beta} \mathrm{d}\rho(y) \hat{\phi}(x,y) \leq p < 1$$

which is valid for z satisfying

$$ze^{1+\beta B} \beta (2\pi\beta)^{-\frac{d}{2}} ||\phi||_1 \leq p.$$

To check Assumption 4' we note that similarly to (3.118)

$$ze^{1+\beta B}\int_{\mathcal{X}^c(B_0(R+r))}|\hat{\phi}(x,y)| = ze^{1+\beta B}\int_{\mathcal{X}^c(B_0(R+r))\mathcal{X}(B_0^c(R+\frac{r}{2}))}\mathrm{d}\rho(y)|\hat{\phi}(x,y)|$$
$$+ze^{1+\beta B}\int_{\mathcal{X}^c(B_0(R+r))\mathcal{X}^c(B_0^c(R+\frac{r}{2}))}\mathrm{d}\rho(y)|\hat{\phi}(x,y)| \equiv D' + D''.$$

By repeating the arguments of the proof of Theorem 3.12, see (3.119) and (3.120), with the help of Corollary 3.6, we have

$$D' \leq ze^{1+\beta B}\beta(2\pi\beta)^{-\frac{d}{2}}||\phi_l||_1\left(1+\frac{r}{2}\right)^{-l}$$

as well as

$$D'' \leq zC(d,l)||\phi||_1 e^{1+\beta B}\beta^{\frac{2+l-d}{2}}\left(1+\frac{r}{2}\right)^{-l}.$$

Hence Assumption 4' is fulfilled for all z satisfying

$$zC(d,l)||\phi_l||_1 e^{1+\beta B}\beta^{1-\frac{d}{2}}(1+\beta^{\frac{l}{2}}) \leq p < 1.$$

This completes the proof of Theorem 3.15.

Combining Theorem 3.15 with Theorem 3.8 we get

Corollary 3.16 . *If the potential ϕ satisfies the conditions (1), (2) and (4) and z satisfies the relation*

$$zC(d,l)||\phi_l||_1 e^{1+\beta B}\beta^{1-\frac{d}{2}}(1+\beta^{\frac{l}{2}}) \leq p < 1, \tag{3.129}$$

then for all $R > 0$ the two-point truncated correlation functions satisfy the bound

$$z\int_{\mathcal{X}^0}\mathrm{d}P_\beta^{00}(x^0)\int_{\mathcal{X}^c(B_0(R))}\mathrm{d}\rho(y)|\sigma_\Lambda(x^0,y)| \leq C(1+R)^{-l}$$

with $C = C(\Phi,\beta,z,d)$.

This corollary will be the main technical tool for obtaining the asymptotic expansion of the log-partition function of a MB gas in Chapter 5.

4 Decay of correlations and limit theorems. Classical gases

In this Chapter we consider continuous and discrete classical systems. We start with a bound for two point semiinvariant which follows easily from Theorem 3.1. Next we demonstrate the method which is used in Section 5 to prove the asymptotic expansions for the log-partition functions of the loop models. The existing methods of proving asymptotic expansions for the log-partition functions are based on the estimates of all the semiinvariants of the system. The new approach uses only bounds of the two-point semiinvariants.

To clarify the method and our strategy for the case of interacting Brownian loops, we start with a brief discussion of the same problem for a much simpler case of classical gas in a bounded domain Λ. To make the exposition more transparent we consider the planar case.

Passing to classical lattice spin systems with vacuum and general many - body interaction, we present tree graph estimates (also called *strong cluster estimates*) for all n-point semiinvariants, $n \geq 2$, [5, 7, 8] and asymptotic expansion of the log-partition function for such models [6]. As an application of these results we prove central local limit theorem [9, 7], estimate the rate of convergence [84] and find the probabilities of large deviations (in the sense of H. Cramer [91, 90]) in the local limit theorem for the number of particles in a finite volume. All these problems remain open for the systems of interacting Brownian loops .

4.1 Decay of correlations in classical gases

We consider a gas of point particles that interact with a pair potential. We work in the grand-canonical ensemble where the parameters are the fugacity z and the inverse temperature β (both are real and positive numbers). The set $\mathbb{X}$ is an open bounded subset of $\mathbb{R}^d$ and $\mu(x) = z\mathrm{d}x$ with $\mathrm{d}x$ the Lebesgue measure. We actually write $\Lambda = \mathbb{X}$ so as to have more traditional notation. The interaction is given by a function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ which we take to be piecewise continuous; $u(x,y) = \beta\phi(x-y)$. We suppose that ϕ is stable, i.e. that there exists a constant $B \geq 0$ such that for any n and any $x_1, \ldots, x_n \in \mathbb{R}^d$:

$$\sum_{1 \leq i < j \leq n} \phi(x_i - x_j) \geq -Bn. \tag{4.1}$$

Our Assumption 1 holds with $b(x) \equiv \beta B$. The system is translation invariant so all $x \in \mathbb{R}^d$ are equivalent. The function of Assumptions 2 and 2' can then be taken to be a constant, $a(x) \equiv a$. We seek a condition that does not depend on the size of the system. Then integrals over y are on $\mathbb{R}^d$ instead of Λ. By translation invariance we can take $x = 0$.

Assumption 2 gives the condition

$$z\,\mathrm{e}^{2\beta B} \int_{\mathbb{R}^d} \big|\,\mathrm{e}^{-\beta\phi(y)} - 1\big|\mathrm{d}y \leq a\,\mathrm{e}^{-a}\,. \tag{4.2}$$

We obviously choose the constant a that maximizes the right side, which is $a = 1$. This condition is the one in [105]. Let us now assume that ϕ consists of a hard core of radius r and that it is otherwise integrable. Again with $a = 1$, Assumption 2' gives the condition

$$z\,\mathrm{e}^{\beta B}\Big[|\mathbb{B}|r^d + \beta\int_{|y|>r}|\phi(y)|\mathrm{d}y\Big] \leq \mathrm{e}^{-1}. \tag{4.3}$$

Here, $|\mathbb{B}| = \pi^{d/2}/\Gamma(\frac{d}{2}+1)$ is the volume of the ball in d dimensions. Without hard core it is the one in [17]. The domains of parameters where these conditions hold correspond to low fugacities and high temperatures.

The thermodynamic pressure is defined as the infinite volume limit of

$$p_\Lambda(\beta, z) = \frac{1}{|\Lambda|}\log Z. \tag{4.4}$$

Using Theorem 2.1, we have

$$p_\Lambda(\beta, z) = \frac{1}{|\Lambda|}\int_\Lambda \mathrm{d}x_1\Big[\sum_{n\geq 1}\frac{z^n}{n!}\int_\Lambda \mathrm{d}x_2\ldots\int_\Lambda \mathrm{d}x_n\varphi(x_1,\ldots,x_n)\Big] \tag{4.5}$$

Consider now any sequence of increasing domains $\Lambda_1 \subset \Lambda_2 \subset \ldots$ such that $\Lambda_n \to \mathbb{R}^d$. Thanks to the estimate (2.8), and using translation invariance, we get

$$p(\beta, z) \equiv \lim_{n\to\infty} p_{\Lambda_n}(\beta, z) = \sum_{n\geq 1}\frac{z^n}{n!}\int_{\mathbb{R}^d}\mathrm{d}x_2\ldots\int_{\mathbb{R}^d}\mathrm{d}x_n\,\varphi(0, x_2,\ldots,x_n). \tag{4.6}$$

(The term with $n = 1$ is equal to z.) This expression for the infinite volume pressure $p(\beta, z)$ should be viewed as a convergent series of analytic functions of β, z. Then $p(\beta, z)$ is analytic in β and z by Vitali theorem and no phase transition takes place in the domain of parameters where the cluster expansion is convergent.

The truncated two-point correlation function $\sigma(x)$ is given by (2.12). We consider only the case of Assumption 2 but a similar claim can be obtained with Assumption 2'. Let $c(x)$ be a function that satisfies the triangle inequality. The estimate of Theorem 3.1 yields

$$\mathrm{e}^{c(x)}\,\sigma(x) \leq \mathrm{e}^{2+4\beta B}\sum_{m\geq 0} z^m\,\mathrm{e}^{m+2\beta Bm}\Big(\mathrm{e}^{c(\cdot)}\big|\,\mathrm{e}^{-\beta U(\cdot)} - 1\big|\Big)^{*m}(x) \tag{4.7}$$

(with $f^{*0} \equiv f$). Recall that $\|f^{*n}\|_\infty \leq \|f\|_\infty\|f\|_1^{n-1}$, and let

$$C_p = \big\|e^{c(\cdot)}\big|\,\mathrm{e}^{-\beta U(\cdot)} - 1\big|\big\|_p. \tag{4.8}$$

Then we get

$$\sigma(x) \leq e^{-c(x)}\,\mathrm{e}^{2+4\beta B}\,\tfrac{C_\infty}{C_1}\big(1 - z\,\mathrm{e}^{1+2\beta B}\,C_1\big)^{-1}. \tag{4.9}$$

If the inequality (4.2) is strict, one can usually find a function $c(x)$ such that $C_1 \leq (z\,\mathrm{e}^{1+2\beta B})^{-1}$; the truncated two-point correlation function then decays faster than $\mathrm{e}^{-c(x)}$.

4.2 Asymptotic expansion of the log-partition function

The aim of this section is to demonstrate the method which is used in Section 5 to prove the asymptotic expansions for the log-partition functions of the loop models. The existing methods of proving asymptotic expansions for the log-partition functions are based on the estimates of all the semiinvariants of the system. The new approach uses only bounds of the two-point semiinvariants.

To clarify the method and our strategy for the case of interacting Brownian loops, we start with a brief discussion of the same problem for a much simpler case of classical gas in a bounded domain Λ. To make the exposition more transparent we consider the planar case.

Consider again a gas of point particles enclosed in a bounded domain $\Lambda \subset \mathbb{R}^2$. The state space of the system is the space $\mathcal{M}(\Lambda) = \{\omega \subset \Lambda \mid |\omega| < \infty\}$ of all finite subsets (configurations) of Λ with a canonical σ-algebra (see the details in [92]).

We assume that particles interact via stable pair interaction ϕ, which is differentiable and together with its derivatives is uniformly bounded and decreases at infinity sufficiently fast. The exact conditions on the potential ϕ are given at the beginning of Section 5.2. The Gibbs distribution on $\mathcal{M}(\Lambda)$ is given by formula (3.42).

We want to derive an asymptotic expansion of the log-partition function $\ln Z(\Lambda_R, z)$ as $R \to \infty$. For simplicity we consider the case where Λ is an open convex bounded subset of $\mathbb{R}^2$. We assume that the boundary $\partial\Lambda$ is one dimensional closed C^3-manifold.

We apply the abstract cluster expansion method of Chapter 2 by taking $\mathbb{X} = \mathcal{X}(\Lambda_R)$, $\mathrm{d}\mu(x) = \mathrm{d}(x), u(x,y) = \beta\phi(x,y), a(x) = 1, b(x) = \beta B$ and assuming that z is sufficiently small. We start with the cluster representation of the log-partition function, Theorem 2.1:

$$\ln Z(\Lambda_R, z) = \sum_{n\geq 0} \frac{z^n}{n!} \int_{\Lambda_R^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \varphi(x_1, \cdots, x_n) = \int_{\mathcal{M}(\Lambda_R)} \mathrm{d}W(\mu)\varphi(\mu) \tag{4.10}$$

where φ is the Ursell function given by formula (2.6) and $\Lambda_R = \{Ru \mid u \in \Lambda\}$. In the second step, with the help of the well-known formula, see (5.15) below, we rewrite $\ln Z(\Lambda_R, z)$ as:

$$\ln Z(\Lambda_R, z) = \int_{\Lambda_R} \mathrm{d}x \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(x,\mu) 1_{\mathcal{M}(\Lambda_R)}(x,\mu) \tag{4.11}$$

where

$$g_z(\mu) = z^{|\mu|} \frac{\varphi(\mu)}{|\mu|}, \text{ if } |\mu| \geq 1.$$

Now

$$\begin{aligned} \ln Z(\Lambda_R, z) = \int_{\Lambda_R} \mathrm{d}x \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(x,\mu) - \int_{\Lambda_R} \mathrm{d}x \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(x,\mu) \\ \cdot 1_{\mathcal{M}^c(\Lambda_R)}(\mu) = \int_{\Lambda_R} \mathrm{d}x\, I^A(x,R) - \int_{\Lambda_R} \mathrm{d}x\, I^B(x,R) \end{aligned} \tag{4.12}$$

where (by translation invariance)

$$I^A(x,R) \equiv \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(x,\mu) = \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(0,\mu) \tag{4.13}$$

and

$$I^B(x,R) \equiv \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(x,\mu) 1_{\mathcal{M}^c(\Lambda_R)}(\mu). \tag{4.14}$$

According to Theorem 2.2 the two-point truncated correlation function has the representation

$$\sigma_\Lambda(x,y) = \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(x,y,\mu)$$

The following bound follows directly from (4.9) with $c(x) = l \cdot \ln(1+|x|),\ \ l > d.$

The Main Bound: if the activity z is sufficiently small then for all bounded $\Lambda \subset \mathbb{R}^2$ and $x, y \in \Lambda$,

$$|\sigma_\Lambda(x,y)| \leq \int_{\mathcal{M}} \mathrm{d}W(\mu) \,\Big|\, g_z(x,y,\mu) \,\Big|\leq C(1+|x-y|)^{-l} \tag{4.15}$$

with $C = C(\phi, \beta, z, l) > 0$.

We set

$$\Lambda_{R,\delta} = \{x \in \Lambda_R \;\big|\; d(u, \partial\,\Lambda_R) < \delta R^\varepsilon\}.$$

Then

$$\int_{\Lambda_R} I^B(x,R)\mathrm{d}u = \int_{\Lambda_{R,\delta}} I^B(x,R)\mathrm{d}x + \int_{\Lambda_R \setminus \Lambda_{R,\delta}} I^B(x,R)\mathrm{d}x. \tag{4.16}$$

An application of the Main Bound with l sufficiently large (we can take $l > 16$) gives

$$\int_{\Lambda_R \setminus \Lambda_{R,\delta}} I^B(x,R)\mathrm{d}x = o(1) \tag{4.17}$$

Therefore we need to treat only the first term on the right side of (4.16) We set up at each point $r \in \partial\,\Lambda$ local coordinates (ξ, η) where ξ is along the tangent vector $\mathbf{s} = \mathbf{s}(r)$ and η is along the inward drawn unit normal $\mathbf{n} = \mathbf{n}(r)$ to $\partial\,\Lambda$ at r. Then $\partial\,\Lambda$ is given locally by $\eta = f_r(\xi)$, $|\xi| < \delta R^\varepsilon$, for $\delta > 0$ small enough where f_r is a function of class $\mathcal{C}^2$. Choosing $\delta = \frac{1}{2}[\sup\limits_{r\in\partial\,\Lambda} |\kappa(r)|]^{-1}$ and taking into account that the volume element in $\Lambda_{R,\delta}$ is equal to $(1 - tk_R(r))\mathrm{d}t\sigma(dr)$ (see, for example, [87]), we have that

$$\int_{\Lambda_{R,\delta}} I^B(x,R)\mathrm{d}x = \int_{\partial\,\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t(1 - tk_R(r)) I^B(r + t\mathbf{n}, R) \tag{4.18}$$

where $k_R(r)$ is the curvature of $\partial\Lambda_R$ at the point $r \in \partial\Lambda_R,\ \ k_1(r) \equiv k(r)$. Evidently $k_R(r) = R^{-1}k(R^{-1}r),\ \ r \in \partial\Lambda_R$.

Furthermore, we associate to each $r \in \partial\Lambda_R$ a cylinder

$$\Pi_{r,\delta R^\varepsilon} = \{(\xi,\eta) \mid |\xi| < \delta R^\varepsilon\}. \tag{4.19}$$

From now on we take $\epsilon = \frac{1}{8}$. To get rid of configurations that have points outside of the cylinder $\Pi_{r,\delta R^\varepsilon}$, we decompose I^B by decomposing $\mathcal{M}^c(\Lambda_R)$ as follows:

$$\mathcal{M}^c(\Lambda_R) = \mathcal{M}^c(\Lambda_R)\mathcal{M}(\Pi_{r,\delta R^\varepsilon}) + \mathcal{M}^c(\Lambda_R)\mathcal{M}^c(\Pi_{r,\delta R^\varepsilon}), r \in \partial\,\Lambda_R.$$

(We use + for the union of disjoint sets.) According to the Main Bound the term which corresponds to $\mathcal{M}^c(\Lambda_R)\mathcal{M}^c(\Pi_{r,\delta R^\varepsilon})$ is $o(R^{-2})$.

Let

$$\mathcal{F}^+_{r,\delta R^\varepsilon} = \{(\xi,\eta) \in \Pi_{r,\delta R^\varepsilon} \mid \eta > f_{r,R}(\xi)\},$$

where $f_{r,R}(\xi) = Rf_r\left(\frac{\xi}{R}\right)$. Then

$$\mathcal{M}^c(\Lambda_R)\mathcal{M}(\Pi_{r,\delta R^\varepsilon}) = \mathcal{M}^c(\Lambda_R)\mathcal{M}(\mathcal{F}^+_{r,\delta R^\varepsilon}) + \mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})$$

and applying once more the Main Bound, we get

$$I^B(r+t\mathbf{n},R) = \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu) 1_{\mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})}(\mu) + o(R^{-2}). \tag{4.20}$$

The configurations which have points on the both sides of the tangent give rise to the boundary term. Assuming that r is a point of the convex part of the boundary $\partial\Lambda_R$ (the concave case is treated similarly), we make the decomposition

$$\mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}) = \mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon}) + \mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}). \tag{4.21}$$

This implies

$$\begin{aligned} I^B(r+t\mathbf{n},R) &= \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu) 1_{\mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon})}(\mu) \\ &+ \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu) 1_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})}(\mu) \\ &= J(r+t\mathbf{n},R) + K(r+t\mathbf{n},R) + o(R^{-2}). \end{aligned} \tag{4.22}$$

Substituting this into (4.18) we find

$$\begin{aligned} \int_{\Lambda_{R,\delta}} I^B(x,R)\mathrm{d}x &= \int_{\partial\,\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t(1-tk_R(r)) J(r+t\mathbf{n},R) \\ &+ \int_{\partial\,\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t(1-tk_R(r)) K(r+t\mathbf{n},R) + o(1) \\ &= \mathcal{J}(R) + \mathcal{K}(R) + o(1). \end{aligned} \tag{4.23}$$

Consider first $\mathcal{J}(R)$. Let

$$\Pi_r^+ = \{(\xi,\eta) \bigm| \eta \geq 0\}. \tag{4.24}$$

To get rid of the restriction to the cylinder we use the decomposition

$$\mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon}) = \mathcal{M}^c(\Pi_r^+) - \mathcal{M}^c(\Pi_r^+)\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon}).$$

With the help of the Main Bound we find from here that

$$\mathcal{J}(R) = \mathcal{J}^b(R) - \mathcal{J}^c + o(1) \tag{4.25}$$

where the boundary term

$$\mathcal{J}^b(R) = R\int_{\partial\Lambda} \mathrm{d}\sigma(r) \int\limits_0^\infty \mathrm{d}t \int_{\mathcal{M}^c(\Pi_r^+)} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu) \tag{4.26}$$

and the first contribution to the constant term

$$\mathcal{J}^c = \int_{\partial\Lambda} \mathrm{d}\sigma(r)k(r) \int\limits_0^\infty \mathrm{d}t\, t \int_{\mathcal{M}^c(\Pi_r^+)} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu). \tag{4.27}$$

Thus combining (4.12), (4.13), (4.16), (4.18), (4.23) and (4.25) - (4.27) we have

$$\ln\, Z(\Lambda_R,z) = R^2|\Lambda| \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(0,\mu) - \mathcal{J}^b(R) + \mathcal{J}^c - \mathcal{K}(R) + o(1). \tag{4.28}$$

To treat $\mathcal{K}(R)$ we first rewrite $K(r+t\mathbf{n},R)$ from (4.23) as

$$\begin{aligned} K(r+t\mathbf{n},R) &= \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu) 1_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})}(\mu) \\ &= \int_{\mathcal{M}_+(\Pi^+_{r,\delta R^\varepsilon}(\mathcal{F}^+_{r,\delta R^\varepsilon})^c)} \mathrm{d}W(\omega)(-1)^{|\omega|+1} \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu,\omega) \\ &\cdot 1_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon}))}(\mu) \end{aligned} \tag{4.29}$$

where $\mathcal{M}_+$ denotes the set of non-empty finite configurations. This is due to the equation

$$\int_{\mathcal{M}^c(A)} \mathrm{d}W(\omega)h(\omega) = \int_{\mathcal{M}_+(A)^c} \mathrm{d}W(\omega)(-1)^{|\omega|+1} \int_{\mathcal{M}} \mathrm{d}W(\mu) h(\mu,\omega) \tag{4.30}$$

which is valid for any absolutely integrable function h.

Now note that we can neglect the contribution to $K(r+t\mathbf{n},R)$ of the configurations ω which have more then one point in $\Pi^+_{r,\delta R^\varepsilon}(\mathcal{F}^+_{r,\delta R^\varepsilon})^c$. Indeed

$$\begin{aligned} K(r+t\mathbf{n},R) &= \int_{\Pi^+_{r,\delta R^\varepsilon}(\mathcal{F}^+_{r,\delta R^\varepsilon})^c} \mathrm{d}x \int_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu,x) + \sum_{m\geq 2} \frac{(-1)^m}{m!} \\ &\cdot \int_{\left(\Pi^+_{r,\delta R^\varepsilon}(\mathcal{F}^+_{r,\delta R^\varepsilon})^c\right)^m} \mathrm{d}x_1 \cdots \mathrm{d}x_m \int_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})} \mathrm{d}W(\mu) \\ &\cdot g_z(r+t\mathbf{n},\mu,x_1,\cdots,x_m) = K_1(r+t\mathbf{n},R) + K_2(r+t\mathbf{n},R). \end{aligned} \tag{4.31}$$

By Lemma 5.3 below, for sufficiently small z,

$$|K_2(r+t\mathbf{n},R)| \le \sum_{m\ge 2} \frac{1}{m\, e\bar{u}} \left[\frac{ze^{\beta B+1}\bar{u}}{1-ze^{\beta B+1}\bar{u}}\right]^m \Big| \Pi^+_{r,\delta R^\varepsilon}(\mathcal{F}^+_{r,\delta R^\varepsilon})^c \Big|^m \le CR^{-\frac{5}{4}}. \tag{4.32}$$

This is because of the inequality

$$\Big| \Pi^+_{r,\delta R^\varepsilon}(\mathcal{F}^+_{r,\delta R^\varepsilon})^c \Big| \le CR^{-\frac{5}{8}} \tag{4.33}$$

which can be proved easily with the help of the following obvious bound

$$|f_{r,R}(\xi)| \le C|\xi|^2 R^{-1}. \tag{4.34}$$

It remains to treat $K_1(r+t\mathbf{n},R)$. By the Main Bound it is obvious that

$$K_1(r+t\mathbf{n},R) = \int_{\Pi^+_{r,\delta R^\varepsilon}(\mathcal{F}^+_{r,\delta R^\varepsilon})^c} \mathrm{d}x \int_{\mathcal{M}(\Pi^+_r)} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu,x) + o(R^{-2}). \tag{4.35}$$

Using the local coordinate system and the conditions (5.55) and (5.56) on $\phi \in C^1$ we can write

$$\begin{aligned}\int_{\mathcal{M}(\Pi^+_r)} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu,(\xi,\eta)) &= \int_{\mathcal{M}(\Pi^+_r)} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu,(\xi,0)) \\ &+ \frac{\partial}{\partial\eta}\int_{\mathcal{M}(\Pi^+_r)} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu,(\xi,\theta\eta))\eta, \;\; 0<\theta<1.\end{aligned} \tag{4.36}$$

By Lemma 5.4 the second summand on the right side of (4.36) is bounded in absolute value by a constant D which depends only on the potential ϕ and parameters β, z. This together with (4.33) implies

$$\begin{aligned}&\int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t |1-tk_R(r)| \int_{\Pi^+_{r,\delta R^\varepsilon}(\mathcal{F}^+_{r,\delta R^\varepsilon})^c} \mathrm{d}\xi \mathrm{d}\eta \\ &\cdot \Big| \frac{\partial}{\partial\eta}\int_{\mathcal{M}(\Pi^+_r)} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu,(\xi,\theta\eta))\eta \Big| = o(1).\end{aligned} \tag{4.37}$$

Combining (4.23), (4.29), (4.31), (4.32) and (4.35) - (4.37) we find

$$\begin{aligned}\mathcal{K}(R) &= \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t \int_{\Pi^+_{r,\delta R^\varepsilon}(\mathcal{F}^+_{r,\delta R^\varepsilon})^c} \mathrm{d}\xi \mathrm{d}\eta \int_{\mathcal{M}(\Pi^+_r)} \mathrm{d}W(\mu) \\ &\cdot g_z(r+t\mathbf{n},\mu,(\xi,0)) - \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) k_R(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t\, t \int_{\Pi^+_{r,\delta R^\varepsilon}(\mathcal{F}^+_{r,\delta R^\varepsilon})^c} \mathrm{d}\xi \mathrm{d}\eta \\ &\cdot \int_{\mathcal{M}(\Pi^+_r)} \mathrm{d}W(\mu) g_z(r+t\mathbf{n},\mu,(\xi,0)) + o(1) = \mathcal{K}_3(R) + \mathcal{K}_4(R) + o(1).\end{aligned} \tag{4.38}$$

It is easy to prove using Lemma 5.3 and the bound (4.24) that

$$\mathcal{K}_4(R) = o(1). \tag{4.39}$$

So we need to treat only $\mathcal{K}_3(R)$. With the help of the Main Bound we rewrite it as

$$\mathcal{K}_3(R) = \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t \int_{-\delta R^\varepsilon}^{\delta R^\varepsilon} \mathrm{d}\xi \int_{\mathcal{M}(\Pi_r^+)} \mathrm{d}W(\mu) g_z(r + t\mathbf{n}, \mu, (\xi, 0))$$

$$\cdot \int_0^{f_{r,R}(\xi)} \mathrm{d}\eta = \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) k_R(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t \int_{-\delta R^\varepsilon}^{\delta R^\varepsilon} \mathrm{d}\xi \frac{\xi^2}{2} \int_{\mathcal{M}(\Pi_r^+)} \mathrm{d}W(\mu)$$

$$\cdot\, g_z(r + t\mathbf{n}, \mu, (\xi, 0)) + \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t \int_{-\delta R^\varepsilon}^{\delta R^\varepsilon} \mathrm{d}\xi$$

$$\int_{\mathcal{M}(\Pi_r^+)} \mathrm{d}W(\mu) g_z(r + t\mathbf{n}, \mu, (\xi, 0)) \left(f_{r,R}(\xi) - k_R(r) \frac{\xi^2}{2} \right)$$

$$= \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) k_R(r) \int_0^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\xi \frac{\xi^2}{2} \int_{\mathcal{M}(\Pi_r^+)} \mathrm{d}W(\mu)$$

$$\cdot\, g_z(r + t\mathbf{n}, \mu, (\xi, 0)) + o(1) \tag{4.40}$$

From (4.38) - (4.40) it follows that

$$\mathcal{K}(R) = \mathcal{K}^c + o(1) \tag{4.41}$$

where

$$\mathcal{K}^c = \frac{1}{2} \int_{\partial\Lambda} \mathrm{d}\sigma(r) k(r) \int_0^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\xi\, \xi^2 \int_{\mathcal{M}(\Pi_r^+)} \mathrm{d}W(\mu) g_z(r + t\mathbf{n}, \mu, (\xi, 0)) \tag{4.42}$$

Thus (4.28), (4.41) and (4.42) imply the following

Theorem 4.1 (Classical gas) . *If the pair potential ϕ satisfies the conditions (5.55), (5.56) and if $\Lambda \subset \mathbb{R}^2$ is a bounded convex domain such that the boundary $\partial\Lambda$ is one dimensional C^3-manifold, then the following expansion holds true for all sufficiently small z:*

$$\ln Z(\Lambda_R, z) = R^2|\Lambda| \int_{\mathcal{M}} \mathrm{d}W(\mu) g_z(0, \mu) + R \int_{\partial\Lambda} \mathrm{d}\sigma(r) \int_0^{\infty} \mathrm{d}t \int_{\mathcal{M}^c(\Pi_r^+)} \mathrm{d}W(\mu)$$

$$\cdot\, g_z(r + t\mathbf{n}, \mu) + \int_{\partial\Lambda} \mathrm{d}\sigma(r) k(r) \int_0^{\infty} \mathrm{d}t \Bigg[t \int_{\mathcal{M}^c(\Pi_r^+)} \mathrm{d}W(\mu) \tag{4.43}$$

$$\cdot\, g_z(r + t\mathbf{n}, \mu) - \int_{-\infty}^{\infty} \mathrm{d}\xi\, \xi^2 \int_{\mathcal{M}(\Pi_r^+)} \mathrm{d}W(\mu) g_z(r + t\mathbf{n}, \mu, (\xi, 0)) \Bigg] + o(1)$$

as $R \to \infty$. If the potential in addition is rotation invariant, the expansion (4,43) can be simplified in a natural way, cf. Theorem 5.2.

We note that the method which we used to prove Theorem 4.1 appeared in [99, 94]. In contrast to the methods of earlier papers [92, 6, 7] and [19] which were based on the estimates of all semiinvariants of the corresponding Gibbs distribution, this new approach uses only bounds of two-point semiinvariants. On the other hand it proves the absence of logarithms as it was conjectured in [92], improves the results and can be applied to marked Gibbs random fields as well.

4.3 Local limit theorems

Let Y be a standard space with σ-algebra $\mathcal{Y}$. This means that Y is a subspace of a Polish space such that the trace of the corresponding Borel σ-algebra coincides with $\mathcal{Y}$. We introduce the space $\widetilde{Y} = \{Y, \emptyset\}$ and the element $\emptyset$ we call the *vacuum*. Let m be a measure on $\widetilde{Y}$ such that the restriction of m on Y is a probability measure and $m(\emptyset) = 1$. Let $\mathcal{M}$ be the set of all mappings $\eta : \mathbb{Z}^d \to \widetilde{Y}$ such that $\eta(u) \neq \emptyset$ only for finitely many $u \in \mathbb{Z}^d$:

$$\mathcal{M} = \{\eta : \mathbb{Z}^d \to \widetilde{Y} \mid |s(\eta)| < \infty\} \tag{4.44}$$

where $s(\eta) = \operatorname{supp} \eta = \{u \in \mathbb{Z}^d, \eta(u) \neq \emptyset\}$. $\mathcal{M}$ is the configuration space of our *spin model* with spin space Y and elements of $\mathcal{M}$ we call spin configurations. We will use also the set of all finite subsets of $\mathbb{Z}^d$ which we denote by $\mathcal{M}_0$.

Let

$$\mathcal{M}(\Lambda) = \{\eta : \mathbb{Z}^d \to \widetilde{Y} \mid s(\eta) \subset \Lambda\}, \ \ \Lambda \subset \mathbb{Z}^d. \tag{4.45}$$

We consider many-body potentials ϕ which are real measurable functions on $\mathcal{M} \equiv \mathcal{M}(\mathbb{Z}^d)$. A potential ϕ we call Euclidean invariant if $\phi(g\eta) = \phi(\eta)$ for arbitrary automorphism g of the lattice $\mathbb{Z}^d$, $(g\eta)(u) = \eta(g(u))$. We call ϕ translation invariant if $\phi(\eta_a) = \phi(\eta)$, for all $a \in \mathbb{Z}^d$, $\eta_a(u) = \eta(u - a)$,

Let δ be a translation invariant metric on $\mathbb{Z}^d$. For $\xi \in \mathcal{M}_0$ we denote by $L_\delta(\xi)$ the minimum of the lengths (with respect to the metric δ) of the trees constructed on ξ and possibly any other points of $\mathbb{Z}^d$ (see also [35]).

We assume that ϕ decays at infinity sufficiently fast, namely:

$$p_\delta(\phi) = \sum_{\xi \in \mathcal{M}_0, \, 0 \in \xi} \sup_{\eta : s(\eta) = \xi} e^{L_\delta(\xi)} |\phi(\eta)| < \infty \tag{4.46}$$

where the metric δ is such that

$$D_\delta(l) = \sum_{u \in \mathbb{Z}^d} (1 + |u|)^l \exp\left(-\frac{1}{2}\delta(0, u)\right) < \infty, \ \ l > d. \tag{4.47}$$

We define the energy of configuration η by

$$U(\eta) = \sum_{J \subset s(\eta)} \phi(\eta_J) \tag{4.48}$$

where $\eta_J(t) = \eta(t)$, $t \in J$ and $\eta_J(t) = \emptyset$, $t \in \mathbb{Z}^d \setminus J$.

The Gibbs distribution on the space $\mathcal{M}(\Lambda)$ of spin configurations in a finite volume $\Lambda \subset \mathbb{Z}^d$ is given by the density $Z^{-1}(\Lambda)\exp(-U(\eta))$, $\eta \in \mathcal{M}(\Lambda)$ with respect to the measure $m_\Lambda = \prod_{t\in\Lambda} m_t$, $m_t = m$, where the normalizing factor

$$Z(\Lambda) = \int_{\mathcal{M}(\Lambda)} \prod_{t\in\Lambda} \mathrm{d}m_\Lambda(\eta) e^(- U(\eta)). \tag{4.49}$$

For a parallelepiped $\Lambda \subset \mathbb{Z}^d$ we denote by $\Lambda^{(k)}$ the set of all k-faces of Λ, $k = 0, \cdots, d$ and by $|\Lambda^{(k)}|$ the total volume of all the k-faces of Λ and we write Λ_R for $\{Ru \mid u \in \Lambda\}$, $R > 0$.

Let $z(\eta) = e^{-\phi(\eta)}$, $|s(\eta)| = 1$ and let $\hat{z} = \sup_\eta |z(\eta)|$.

Theorem 4.2 (Asymptotic expansion. Lattice systems [6, 7]) . *Let the Euclidean invariant potential ϕ satisfy the conditions (4.46) and (4.47). Assume that*

$$\hat{z} C_\delta(\phi)(2eD_\delta(l) + 1) < 1 \tag{4.50}$$

where

$$C_\delta(\phi) = 2\exp(p_\delta(\phi) + e^{p_\delta(\phi)} - 1). \tag{4.51}$$

Then for any parallelepiped $\Lambda \subset \mathbb{Z}^d$,

$$\ln Z(\Lambda_R) = \sum_{k=0}^{d} a_k(\phi) R^{d-k} |\Lambda^{(d-k)}| + o(1), \quad R \to \infty. \tag{4.52}$$

The powerful technique of tree estimates from Section 2.2, which is known also as strong cluster estimates [34], [35]), was applied in the papers [9],[7] to prove a local limit theorem (l.l.t.) for the particle number in lattice spin systems with general many-body interaction.

For $\eta \in \mathcal{M}(\Lambda)$ we will use also the notation η_Λ to indicate the dependence on Λ and we set $|\eta| = |s(\eta)|$, $\eta \in \mathcal{M}(\Lambda)$. We say that $|\eta_\Lambda|$, $\eta \in \mathcal{M}(\Lambda)$ satisfies the l.l.t. if the relation

$$Pr(\eta \in \mathcal{M}(\Lambda) : |\eta| = N) = (2\pi D|\eta|)^{-\frac{1}{2}} \exp\left[-\frac{(N - E|\eta|)^2}{2D|\eta|}\right](1 + o(1)) \tag{4.53}$$

as $|\Lambda| \to \infty$, holds true uniformly with respect to the particle number $N \in \mathbb{Z}_+$, such that $N - E(\eta_\Lambda) \sim |\Lambda|^{\frac{1}{2}}$. Here and below in this Chapter 4

$$E|\eta_\Lambda| = \sum_{N\geq 0} NP(|\eta_\Lambda| = N), \quad D|\eta_\Lambda| = E(|\eta_\Lambda| - E|\eta_\Lambda|)^2. \tag{4.54}$$

The theorem formulated below gives the conditions for the potential under which the central limit theorem (c.l.t.) for the particle number yields the l.l.t.. We say that $|\eta_\Lambda|$ satisfies the c.l.t. if $D|\eta_\Lambda| \sim D|\Lambda|$, $D > 0$ and

$$Pr\left(\eta \in \mathcal{M}(\Lambda) \;\Big|\; \frac{|\eta_\Lambda| - E|\eta_\Lambda|}{(D|\eta_\Lambda|)^{\frac{1}{2}}} < x\right) \to (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \mathrm{d}t \tag{4.55}$$

where $x \in \mathbb{R}^1$, $|\Lambda| \to \infty$.

The c.l.t. for the Gibbs random fields has been obtained, for example, in [51], [78], [24] and [50].

Theorem 4.3 (c.l.t. yields l.l.t. [7, 9]) . *Let the translation invariant potential ϕ satisfy the conditions of Theorem 4.2 with $l > 8d$ in (4.47) and (4.50). Then the l.l.t. for the particle number follows from the c.l.t..*

Proof. It is sufficient to show that

$$A_\Lambda = \sup_{N \in \mathbb{Z}_+} \left| (2\pi D|\eta_\Lambda|)^{\frac{1}{2}} P(|\eta_\Lambda| = N) - \exp\left[-\frac{(N - E|\eta_\Lambda|)^2}{2D|\eta_\Lambda|} \right] \right| \to 0 \tag{4.56}$$

as $|\Lambda| \to \infty$. According to the inversion formula, we have

$$P(|\eta_\Lambda| = N) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \psi_\Lambda(t) \mathrm{d}t. \tag{4.57}$$

Here ψ_Λ is the characteristic function of a random variable $|\eta_\Lambda|$:

$$\psi_\Lambda(t) = \sum_{N \geq 0} e^{itN} P(|\eta_\Lambda| = N). \tag{4.58}$$

Further

$$\exp\left[-\frac{(N - E|\eta_\Lambda|)^2}{2D|\eta_\Lambda|} \right] = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left[-itR_\Lambda(N) - \frac{t^2}{2} \right] \mathrm{d}t \tag{4.59}$$

where

$$R_\Lambda(N) = (N - E|\eta_\Lambda|) D|\eta_\Lambda|)^{-\frac{1}{2}}. \tag{4.60}$$

The change of variable $t = \tau(D|\eta_\Lambda|)^{-\frac{1}{2}}$ in (4.57) yields

$$P(|\eta_\Lambda| = N) = (2\pi)^{-1} D|\eta_\Lambda|)^{-\frac{1}{2}} \int_{-\pi(D|\eta_\Lambda|)^{\frac{1}{2}}}^{\pi(D|\eta_\Lambda|)^{\frac{1}{2}}} Ee^{i\tau(S_\Lambda - R_\Lambda(N)} \mathrm{d}\tau \tag{4.61}$$

where

$$S_\Lambda = (D|\eta_\Lambda|)^{-\frac{1}{2}} (|\eta_\Lambda| - E|\eta_\Lambda|). \tag{4.62}$$

Let us take B and ϵ such that $0 < B \leq \pi\epsilon(D|\eta_\Lambda|)^{\frac{1}{2}}$, $0 < \epsilon < 1$. Then substituting (4.59) and (4.61) in(4.56), we obtain

$$\begin{aligned} A_\Lambda \leq (2\pi)^{-\frac{1}{2}} &\left[\int_{-B}^{B} |Ee^{i\tau S_\Lambda} - e^{-\frac{\tau^2}{2}}| \mathrm{d}\tau + \int_{|\tau| \geq B} e^{-\frac{\tau^2}{2}} \mathrm{d}\tau \right. \\ &\left. + \int_{B \leq |\tau| \leq \epsilon(D|\eta_\Lambda|)^{\frac{1}{2}}} |Ee^{i\tau S_\Lambda}| \mathrm{d}\tau + \int_{\epsilon(D|\eta_\Lambda|)^{\frac{1}{2}} \leq |\tau| \leq \pi(D|\eta_\Lambda|)^{\frac{1}{2}}} |Ee^{i\tau S_\Lambda}| \mathrm{d}\tau \right] \\ &= (2\pi)^{-\frac{1}{2}} (I_1 + I_2 + I_3 + I_4). \end{aligned} \tag{4.63}$$

Let us estimate the integral I_4. Let

$$\Lambda = \Lambda_k = \{t \in \mathbb{Z}^d,\ 0 < t^i < a_k,\ i = 1, 2, \cdot, d\} \tag{4.64}$$

where $\{a_k\}$ is a monotone increasing sequence. All edges of the cube Λ_k lying on the coordinate axes are divided on the intervals

$$J_{i,j}(r) = \{t^i \in \mathbb{Z}^1,\ jr < t^i < (j+1)r\},\ \ i = 1, \cdots, d,$$

where $r = r_k = o(a_k)$, $k \to \infty$. These partitions generate in a natural way the partition of the cube Λ_k on the subcubes which are assumed to be enumerated in some way: Λ_k^j, $j = 1, \cdots, [a_k/r]^d$ where $[\cdot]$ stands for the integer part. It is evident that

$$\Lambda_k = \bigcup_{j=1}^{[a_k/r]^d} \Lambda_k^j + Q_k \tag{4.65}$$

where $|Q_k| \leq C_d a_k^{d-1}$. Hence

$$\frac{|Q_k|}{|\Lambda_k|} \to 0,\ \ k \to \infty. \tag{4.66}$$

Denote by λ_k^j the center of the cube Λ_k^j and let $\mathcal{C} = \{\lambda_k^j,\ j = 1, \cdots, [a_k/r]^d\}$.

It is easy to see that

$$I_4 \leq (D|\eta_\Lambda|)^{\frac{1}{2}} \int_{\epsilon < t \leq \pi} |Ee^{it|\eta_\Lambda|}| \mathrm{d}t. \tag{4.67}$$

Next, using the general properties of conditional expectations, we get for $\zeta \in \mathcal{M}(\Lambda \setminus \mathcal{C})$,

$$|Ee^{it|\eta_\Lambda|}| = |EE(e^{it|\eta_\mathcal{C}|} + i\tau|\eta_{\Lambda\setminus\mathcal{C}}|/\zeta)| \leq \sup_{\zeta \in \mathcal{M}(\Lambda\setminus\mathcal{C})} |E(e^{it|\eta_\mathcal{C}|}/\zeta)|. \tag{4.68}$$

By the definition of the Gibbs random fields

$$|E(e^{it|\eta_\mathcal{C}|}/\zeta)| = \int_{\mathcal{M}(\mathcal{C})} e^{it|\eta|} e^{-U(\eta/\zeta)} \mathrm{d}m_\Lambda(\eta) \left[\int_{\mathcal{M}(\mathcal{C})} e^{-U(\eta/\zeta)} \mathrm{d}m_\Lambda(\eta)\right]^{-1} \tag{4.69}$$

where the conditional energy

$$U(\eta/\zeta) = \sum_{\substack{\emptyset \neq I \subset s(\eta) \\ J \subset s(\zeta)}} \phi(\eta_I + \zeta_J),\ \ \eta \in \mathcal{M}(\mathcal{C}). \tag{4.70}$$

Let

$$T(t, \eta, \zeta) = \begin{cases} \sum_{J \subset s(\zeta)} \phi(\eta_I + \zeta_J),\ t \in s(\eta) \\ 0,\ \text{otherwise} \end{cases} \tag{4.71}$$

and

$$V(\eta, \zeta) = \sum_{I \subset s(\eta),\, |I| \geq 2,\, J \subset s(\zeta)} \phi(\eta_I + \zeta_J) \tag{4.72}$$

Then combining (4.69) - (4.72) we obtain

$$|E(e^{i\tau|\eta_{\mathcal{C}}|}/\zeta)| = \int_{\mathcal{M}(\mathcal{C})} e^{i\tau|\eta|} \prod_{t\in s(\eta)} e^{-T(t,\eta,\zeta)} e^{-V(\eta,\zeta)} \mathrm{d}m_\Lambda(\eta)$$
$$\cdot \left[\int_{\mathcal{M}(\mathcal{C})} \prod_{t\in s(\eta)} e^{-T(t,\eta,\zeta)} e^{-V(\eta,\zeta)} \mathrm{d}m_\Lambda(\eta) \right]^{-1}. \tag{4.73}$$

Since

$$V(\eta,\zeta) \le |s(\eta)| \sum_{\substack{0\in I\subset \mathbb{Z}^d \\ diam(I)>r(\Lambda)}} \sup_{\zeta\in\mathcal{M}(I),\, |s(\zeta)|=|I|} |\phi(\zeta)| \le \frac{|\mathcal{C}|p_\delta(\phi)}{r^s(\Lambda)},\ \ s>d, \tag{4.74}$$

one can show (for details see [7, 9]) that for large k we have

$$|E(e^{it|\eta_{\mathcal{C}}|}/\zeta)| \le 2\sup_{\eta,\zeta} |e^{-V(\eta,\zeta)}| + 2\int_{\mathcal{M}(\mathcal{C})} e^{it|\eta|} \prod_{t\in s(\eta)} e^{-T(t,\eta,\zeta)} \mathrm{d}m_{\mathcal{C}}(\eta)$$
$$\cdot \left[\int_{\mathcal{M}(\mathcal{C})} \prod_{t\in s(\eta)} e^{-T(t,\eta,\zeta)} \mathrm{d}m_{\mathcal{C}}(\eta) \right]^{-1} \le 2e\frac{|\mathcal{C}|p_\delta(\phi)}{r^s(\Lambda)}$$
$$+\prod_{t\in\mathcal{C}} \Bigg\{ \Big| \sum_{\sigma=0}^{1} \int_{\widetilde{Y}} e^{i\tau\sigma} e^{-T(t,\eta(t),\zeta)} f(\eta,\sigma)\mathrm{d}m_t(\eta) \Big|$$
$$\cdot \left[\Big| \sum_{\sigma=0}^{1} \int_{\widetilde{Y}} e^{-T(t,\eta(t),\zeta)} f(\eta,\sigma)\mathrm{d}m_t(\eta) \Big| \right]^{-1} \Bigg\} \tag{4.75}$$

where

$$f(\eta,\sigma) = \begin{cases} 1, & \text{if} \quad \sigma = 1, \\ e^{T(t,\eta(t)}, & \text{if } \sigma = 0 \end{cases}.$$

Now consider a family of random variables $\{X_t,\ t\in\mathcal{C}\}$ which takes the values $\sigma = 0,1$ with probabilities

$$Pr(X_t=\sigma) = p_t(\sigma) = \int_{\widetilde{Y}} e^{-T(t,\eta(t),\zeta)} f(\eta,\sigma)\mathrm{d}m_t(\eta)$$
$$\cdot \left[\sum_{\sigma=0}^{1} \int_{\widetilde{Y}} e^{-T(t,\eta(t),\zeta)} f(\eta,\sigma)\mathrm{d}m_t(\eta) \right]^{-1}. \tag{4.76}$$

Thus from (4.75) we obtain

$$|E(e^{it|\eta_{\mathcal{C}}|}/\zeta)| \le 2e\frac{|\mathcal{C}|p_\delta(\phi)}{r^s(\Lambda)} + \prod_{t\in\mathcal{C}} \psi_{X_t}(\tau), \tag{4.77}$$

where ψ_{X_t} is the characteristic function of the random variable X_t.

It is easy to check that ψ_{X_t} satisfies the following estimate

$$\psi_{X_t}(\tau) = (1-2p_t(0)p_t(1))(1-cos\tau) \le 1-\alpha,\ \ t\in\mathcal{C}, \tag{4.78}$$

if only $\epsilon < |\tau| \leq \pi$ where $0 < \alpha < 1$.

To get the bound (4.78) we note that

$$T(t,\eta(t),\zeta) = \sum_{\emptyset \neq J \subset s(\zeta)} \phi(\eta_I + \zeta_J) + \sum_{t \in I \subset s(\eta),\, |I| \geq 2} \frac{1}{|I|} \sum_{\emptyset \neq J \subset s(\zeta)} \phi(\eta_I + \zeta_J)$$
$$\leq ||\phi|| + \frac{|\mathcal{C}| p_\delta(\phi)}{r^s(\Lambda)} \leq M p_\delta(\phi), \;\; 0 < M < \infty. \tag{4.79}$$

Using formulae (4.77) and (4.78) we obtain

$$|E(e^{it|\eta_{\mathcal{C}}|}/\zeta)| \leq 2e \frac{|\mathcal{C}| p_\delta(\phi)}{r^s(\Lambda)} + (1-\alpha)^{|\mathcal{C}|}, \tag{4.80}$$

This together with (4.67) yields

$$|I_4| \leq \pi (D|\eta_\Lambda|)^{\frac{1}{2}} \left(2e \frac{|\mathcal{C}| p_\delta(\phi)}{r^s(\Lambda)} + \frac{1}{2}(1-\alpha)^{|\mathcal{C}|} \right). \tag{4.81}$$

Now pass to the estimation of the integral I_3. The change of variables $t = 2\tau(D|\eta_\Lambda|)^{-\frac{1}{2}}$ in I_3 yields

$$I_3 = \frac{1}{2}(D|\eta_\Lambda|)^{\frac{1}{2}} \int_{\frac{B}{2}(D|\eta_\Lambda|)^{-\frac{1}{2}} \leq |t| \leq \epsilon} |Ee^{it(|\eta_\Lambda| - E|\eta_\Lambda|)}| \mathrm{d}t. \tag{4.82}$$

Then

$$Ee^{it(|\eta_\Lambda| - E|\eta_\Lambda|)} = e^{-\frac{\tau^2}{2} D|\eta_\Lambda| + |\Lambda| \tau^3 h(\tau,\Lambda)} = e^{-|\Lambda| \tau^2 (\frac{D|\eta_\Lambda|}{2|\Lambda|} - \tau h(\tau,\Lambda))} \tag{4.83}$$

where

$$|\Lambda| h(\tau,\Lambda) = \sum_{n \geq 3} \frac{\tau^{n-3}}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\tau^n} \log Ee^{it(|\eta_\Lambda| - E|\eta_\Lambda|)} \Big|_{\tau=0}. \tag{4.84}$$

It follows from the integral c.l.t. that $\frac{D|\eta_\Lambda|}{|\Lambda|} \to a > 0$ as $|\Lambda| \to \infty$. To prove that the integral I_3 tends to zero when $|V| \to \infty$, it is sufficient to show that $h(\tau,\Lambda)$ is uniformly bounded in a complex neighborhood of the origin: $|h(\tau,\Lambda)| \leq a_1$ if $|\tau| < \varepsilon$. Indeed then for sufficiently small ϵ we have

$$I_3 \leq (D|\eta_\Lambda|)^{\frac{1}{2}} \int_{\frac{B}{2}(D|\eta_\Lambda|)^{-\frac{1}{2}}}^{\epsilon} e^{-|\Lambda| t^2 \frac{a}{4}} \mathrm{d}t \leq k \int_{cB}^{\infty} e^{-u^2} \mathrm{d}u \tag{4.85}$$

with $k > 0, c > 0$. So we are going to prove the boundedness of $h(\tau,\Lambda)$. Let $\langle \sigma_1, \cdots, \sigma_n \rangle$ be the semiinvariant of the family of the random variables $\sigma_1, \cdots, \sigma_n$:

$$\langle \sigma_1, \cdots, \sigma_n \rangle = \frac{1}{i^n} \frac{\partial^n}{\partial \tau_1 \cdots \partial \tau_n} \ln E \exp\left(i \sum_{k=1}^{n} \tau_k \sigma_k \right) \Bigg|_{\tau_1 = \cdots \tau_n = 0}. \tag{4.86}$$

Then, using the multi-linearity of semiinvariants, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}\tau^n} \ln Ee^{i\tau(|\eta_\Lambda| - E|\eta_\Lambda|)} \Big|_{\tau=0} = \frac{1}{i^n} \langle \underbrace{|\eta_\Lambda|, \cdots, |\eta_\Lambda|} \rangle_{n\mathrm{times}}$$
$$= \sum_{t_1, \cdots, t_n \in \Lambda} \langle |\eta_{t_1}|, \cdots, |\eta_{t_n}| \rangle. \tag{4.87}$$

The well know formula which expresses the semiinvariants by means of moments yields

$$\langle |\eta_{t_1}|, \cdots, |\eta_{t_n}| \rangle = \sum_{\mathcal{P}} (-1)^{|\mathcal{P}|-1}(|\mathcal{P}|-1) \prod_{p\in\mathcal{P}} \langle \prod_{t\in p} |\eta_t| \rangle \tag{4.88}$$

where the sum is taken over all partitions of the set $\{t_1, \cdots, t_n\}$ into the subsets $p \in \mathcal{P}$.

It is possible to express the moments $\prod_{t\in p} |\eta_t|$ by means of correlation functions. Indeed

$$\begin{aligned}\langle \prod_{t\in p} |\eta_t| \rangle &= Z^{-1}(\Lambda) \int_{\mathcal{M}(\Lambda)} e^{-U(\eta)} \prod_{t\in p} |\eta_t| \mathrm{d}m_\Lambda(\eta) = Z^{-1}(\Lambda) \sum_{I:p\subset I\subset\Lambda} \int_{\mathcal{M}(I)} e^{-U(\eta)} \\ &\cdot \mathrm{d}m_I(\eta) = \int_{\mathcal{M}(I)} 1_{s(\eta)=p}(\eta) \mathrm{d}m_p(\eta) \sum_{I:p\subset I\subset\Lambda} Z^{-1}(\Lambda) \int_{\mathcal{M}(I)} 1_{I\setminus p}(\zeta) \\ &\cdot e^{-U(\eta+\zeta)} \mathrm{d}m_{I\setminus p}(\zeta) = \int_{\mathcal{M}(I)} 1_{s(\eta)=p}(\eta) \varrho_\Lambda(\eta) \mathrm{d}m_p(\eta) = \hat{\varrho}_\Lambda(p)\end{aligned} \tag{4.89}$$

where, as usual, 1_A stands for the indicator of the set A. Further

$$\prod_{p\in\mathcal{P}} \hat{\varrho}_\Lambda(p) = \int_{\mathcal{M}(\{t_1,\cdots,t_n\})} \prod_{p\in\mathcal{P}} \varrho(\eta_p) \mathrm{d}m_{\{t_1,\cdots,t_n\}}(\eta), \tag{4.90}$$

where $\eta_p(u) = \eta(u)$ if $u \in p$ and $\eta_p(u) = \emptyset$, otherwise.

Thus, combining (4.88) - (4.90) and using the well known relation between the correlation and truncated correlation functions, we obtain

$$\begin{aligned}\langle |\eta_{t_1}|, \cdots, |\eta_{t_n}| \rangle &= \int_{\mathcal{M}(\{t_1,\cdots,t_n\})} \sum_{\mathcal{P}} (-1)^{|\mathcal{P}|-1}(|\mathcal{P}|-1) \prod_{p\in\mathcal{P}} \varrho(\eta_p) \mathrm{d}m_{\{t_1,\cdots,t_n\}}(\eta) \\ &= \int_{\mathcal{M}(\{t_1,\cdots,t_n\})} \varrho^T(\eta_p) \mathrm{d}m_{\{t_1,\cdots,t_n\}}(\eta) = \hat{\varrho}^T_\Lambda(\{t_1, \cdots, t_n\})\end{aligned} \tag{4.91}$$

Now we need the following result

Theorem 4.4 (Strong cluster estimate [5, 7]) . *Let the translation invariant potential ϕ satisfy the conditions of Theorem 4.2 , including (4.46) and (4.47). Then*

$$|\varrho^T_\Lambda(\eta)| \le C_\delta^{-1}(\phi) \left(\frac{\hat{z} C_\delta(\phi)}{1 - \hat{z} C_\delta(\phi)} \right)^{|s(\eta)|} e^{-L_\delta(s(\eta))}. \tag{4.92}$$

With the help of this theorem we can write

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}\tau^n} \ln E e^{i\tau(|\eta_\Lambda|-E|\eta_\Lambda|)} \Big|_{\tau=0} &\le \sum_{t_1,\cdots,t_n\in\Lambda} |\hat{\varrho}^T_\Lambda(\{t_1, \cdots, t_n\})| \le \left(\frac{\hat{z} C_\delta(\phi)}{1 - \hat{z} C_\delta(\phi)} \right)^n \\ &\cdot C_\delta^{-1}(\phi) \sum_{t_1,\cdots,t_n\in\Lambda} e^{-L_\delta(\{t_1,\cdots,t_n\})} \le (\bar{C}(\delta))^n |\Lambda|.\end{aligned} \tag{4.93}$$

where $0 < \bar{C}(\delta)$.

This bound ensures the analyticity of the function $h(\tau, \Lambda)$ in a complex neighborhood of the origin where h satisfies the bound $|h(\tau, \Lambda)| \le a_1$ with a constants $a_1 > 0$ which is independent of Λ. Finally, combining (4.81) and (4.85), we find that for any $\varepsilon > 0$ and sufficiently large Λ and small $\epsilon > 0$, we have that

$$A_\Lambda \le (2\pi)^{-\frac{1}{2}} \left(\int_{-B}^{B} |Ee^{i\tau S_\Lambda} - e^{-\frac{\tau^2}{2}}| \mathrm{d}\tau + \int_{|\tau|\ge B} e^{-\frac{\tau^2}{2}} \mathrm{d}\tau + \varepsilon \right). \tag{4.94}$$

This completes the proof of Theorem 4.3

Combining Theorem 4.3 with the c.l.t. from [78] (Theorem 9.5.4) we directly obtain

Theorem 4.5 (Local limit theorem [9]) . *Under conditions of Theorem 4.3, we have that $|\eta_\Lambda|$ satisfies the l.l.t..*

4.4 Convergence rate and large deviations

The next theorem gives an estimate of the convergence rate in the l.l.t. for the number of particles in a bounded region (see Theorem 4.5).

Theorem 4.6 (Convergence rate [84]) . *Let the Euclidean invariant potential ϕ satisfy the conditions of Theorem 4.4 with $l > 8d$ in (4.47) and (4.50). Then for any parallelepiped $\Lambda \subset Z^d$ the following bound holds true:*

$$\sup_{N\in\mathbb{Z}_+} \left| (2\pi D|\eta_\Lambda|)^{\frac{1}{2}} P(|\eta_\Lambda| = N) - \exp\left[-\frac{(N - E|\eta_\Lambda|)^2}{2D|\eta_\Lambda|} \right] \right| \le \frac{C}{\sqrt{|\Lambda|}}. \tag{4.95}$$

At the end of this Chapter, we consider classical lattice Gibbs random fields in a finite volume $\Lambda \subset \mathbb{Z}^d$ with empty boundary conditions. As an application of Theorem 4.2, we present a local limit theorem for the probabilities of large deviations of the number of particles in a grand canonical ensemble in Λ as $|\Lambda| \to \infty$. The proof has two ingredients: a modification of the well known method of Cramer [20] for studying the probabilities of large deviations of the sums of independent identically distributed random variables and the asymptotic expansion of the log-partition function [6] and [7], together with the central l.l.t. for the number of particles, Theorem 4.5 . Note that classical lattice systems are a special case of spin systems where the spin space Y consists of a unique point. For simplicity we consider the planar case, $d = 2$.

Theorem 4.7 (Large deviations [91],[90]) . *Let the Euclidean invariant potential ϕ satisfy the conditions of Theorem 4.3. Let $\alpha = \alpha(\Lambda, \beta, z) = N - E|\eta_\Lambda|$. If $\alpha|\Lambda|^{-\frac{1}{2}} \ge 1$ and $\alpha = o(|\Lambda|)$, then for any parallelepiped $\Lambda \subset \mathbb{Z}^2$*

$$\Pr\left(\omega \in \mathcal{M}(\Lambda_R) \,\Big|\, |\omega| = N\right) = \frac{1}{\sqrt{2\pi|\Lambda_R|}} \sqrt{\Omega''_{\Lambda_R,z}(0)} \exp\left(-\frac{\alpha^2}{2|\Lambda_R|} \Omega''_{\Lambda_R,z}(0) \right)$$
$$\cdot \exp\left(-|\Lambda_R| \sum_{n\ge3} \frac{\alpha^n}{n!|\Lambda_R|^n} \Omega^{(n)}_{\Lambda_R,z}(0) \right) \left[1 + O\left(\frac{\alpha}{|\Lambda_R|} \right) \right]. \tag{4.96}$$

where $\Omega_{\Lambda_R,z}(x)$ is the so-called deviation function [20] which is a real analytic function in x in a neighborhood of the origin. Moreover the second and the third derivatives of $\Omega_{\Lambda_R,z}(x)$ with respect to x have the following expansions

$$\frac{\Omega''_{\Lambda_R,z}(0)}{|\Lambda_R|} = [a_0(\phi,z)R^2|\Lambda| + a_1(\phi,z)R|\partial\Lambda| + a_2(\Lambda,\phi,z) + r(\Lambda,\phi,z)]^{-1} \tag{4.97}$$

and

$$\frac{\Omega'''_{\Lambda_R,z}(0)}{|\Lambda_R|^2} = -\Big[a_0(\phi,z)R^2|\Lambda| + a_1(\phi,z)R|\partial\Lambda| + a_2(\Lambda,\phi,z) + r(\Lambda,\phi,z)\Big]^{-3} \cdot \Big[\frac{\partial a_0(\phi,z)}{\partial(\ln z)}R^2|\Lambda| + \frac{\partial a_1(\phi,z)}{\partial(\ln z)}R|\partial\Lambda| + \frac{\partial a_2(\Lambda,\phi,z)}{\partial(\ln z)} + \frac{\partial r(\Lambda,\phi,z)}{\partial(\ln z)}\Big] \tag{4.98}$$

where

$$\frac{\partial^j r(\Lambda_R,\phi,z)}{\partial(\ln z)^j} = o(1), \quad R\to\infty, \quad j=0,1.$$

Sketch of the proof. Let

$$P_{\Lambda,\xi}(N) = \Pr\Big(\omega\in\mathcal{M}(\Lambda)\ \Big|\ |\omega| = N\Big)$$

where $\xi = \ln z$. By the inversion formula, we have

$$P_{\Lambda,\xi}(N) = (2\pi)^{-\frac{1}{2}}\int_{-\pi}^{\pi}\psi_{\Lambda,\xi}(t)\mathrm{d}t \tag{4.99}$$

where the characteristic function of the particle number has the form

$$\psi_{\Lambda,\xi}(t) = \frac{Z(\Lambda,\xi+it)}{Z(\Lambda,\xi)}. \tag{4.100}$$

Under the conditions of the theorem the partition function $Z(\Lambda,\xi)$ is analytic in ξ on the whole plane (see for example [105]) and differs from zero for real ξ. Therefore $\psi_{\Lambda,\xi}(t)$ is analytic in t on the complex plane. Moreover

$$\frac{\psi_{\Lambda,\xi}(u-i\tau)}{\psi_{\Lambda,\xi}(-i\tau)} = \psi_{\Lambda,\xi+\tau}(u). \tag{4.101}$$

Hence

$$P_{\Lambda,\xi}(N) = \psi_{\Lambda,\xi}(-i\tau)e^{-\tau N}P_{\Lambda,\xi+\tau}(N). \tag{4.102}$$

Let $E_{\Lambda,\xi}|\omega|$ be the mathematical expectation and $D_{\Lambda,\xi}|\omega|$ be respectively the variance of the particle number in a volume Λ with respect to the Gibbs distribution in Λ with activity $z = e^\xi$. We define a function $h(\Lambda,\xi)$ by

$$h(\Lambda,\xi) = |\Lambda|^{-1}\left(E_{\Lambda,\xi+\tau}|\omega| - E_{\Lambda,\xi}|\omega|\right) \tag{4.103}$$

where $E_{\Lambda,\xi}|\omega|$ stands for the mathematical expectation with respect to the Gibbs distribution in Λ with activity $z = e^\xi$. One can easily check the following relations

$$E_{\Lambda,\xi}|\omega| = \frac{\mathrm{d}}{\mathrm{d}\xi}\ln Z(\Lambda,\xi), \quad \frac{\mathrm{d}}{\mathrm{d}\xi}E_{\Lambda,\xi}|\omega| = D_{\Lambda,\xi}|\omega|. \tag{4.104}$$

Hence the function $h(\Lambda, \xi)$ is analytic in τ on the real axes. On the other hand, by Lemma 2 from [31] for sufficiently large Λ and for arbitrary bounded ξ-interval, the following bound

$$|\Lambda|^{-1} D_{\Lambda,\xi}|\omega| \geq C(\phi) > 0. \tag{4.105}$$

Therefore for large Λ in some neighborhood of the origin which is independent of Λ there exists a function $g_{\Lambda,\xi}$ which is the inverse of $h(\Lambda, \xi)$. Now we define the deviation function $\Omega_{\Lambda,\xi}(x)$ by

$$\Omega_{\Lambda,\xi}(x) = g_{\Lambda,\xi}(x) \left(|\Lambda|^{-1} E_{\Lambda,\xi}|\omega| + x\right) - |\Lambda|^{-1} \ln \psi_{\Lambda,\xi}(-i g_{\Lambda,\xi}(x)). \tag{4.106}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}s} \ln \psi_{\Lambda,\xi}(s) = i E_{\Lambda,\xi+i\tau}|\omega|, \tag{4.107}$$

we get from (4.106) that $\frac{\mathrm{d}}{\mathrm{d}x}\Omega_{\Lambda,\xi}(x) = g_{\Lambda,\xi}(x)$. This implies real analyticity of the deviation function in some neighborhood of the origin which is independent of Λ, as well as the relations

$$\Omega_{\Lambda,\xi}(0) = \frac{\mathrm{d}}{\mathrm{d}x}\Omega_{\Lambda,\xi}(0) = 0, \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Omega_{\Lambda,\xi}(x) = \left(\frac{\mathrm{d}}{\mathrm{d}\tau} h_{\Lambda,\xi}(\tau) \Bigg|_{\tau = g_{\Lambda,\xi}(x)} \right)^{-1}$$

$$= \left(|\lambda|^{-1} \frac{\mathrm{d}}{\mathrm{d}\tau} E_{\Lambda,\xi+\tau}|\omega| \Bigg|_{\tau = g_{\Lambda,\xi}(x)} \right)^{-1} = \left(|\Lambda|^{-1} D_{\Lambda,\xi+g_{\Lambda,\xi}(x)}|\omega|\right)^{-1} \tag{4.108}$$

Suppose that $E_{\Lambda,\xi+\tau}|\omega| = N, \ N \in \mathbb{Z}_+$, then by l.l.t. Theorem 4.5

$$P_{\Lambda_R,\xi+\tau}(N) = (2\pi D_{\Lambda_R,\xi+\tau}(x)|\omega|)^{-\frac{1}{2}} \left(1 + O\frac{1}{\sqrt{|\Lambda_R|}} \right) \tag{4.109}$$

for $\xi + \tau < -\ln(C_\delta(\phi))$ as $R \to \infty$.

On the other hand, setting $\tau = g_{\Lambda,\xi}(x)$ we get from (4.103) and (4.106) that

$$\psi_{\Lambda,\xi}(-i\tau) \exp(-\tau E_{\Lambda,\xi+i\tau}|\omega|) = \exp(-|\Lambda|\Omega_{\Lambda,\xi}(x)). \tag{4.110}$$

Now let $\tau_\alpha = g_{\Lambda,\xi}(\frac{\alpha}{|\Lambda|})$ then using again (4.104) we find that for large Λ

$$\frac{N}{|\Lambda|} - \frac{E_{\Lambda,\xi}|\omega|}{|\Lambda|} = h_{\Lambda,\xi}\left(g_{\Lambda,\xi}\left(\frac{\alpha}{|\Lambda|} \right) \right) = \frac{E_{\Lambda,\xi+\tau_\alpha}|\omega|}{|\Lambda|} - \frac{E_{\Lambda,\xi}|\omega|}{|\Lambda|}. \tag{4.111}$$

Hence $E_{\Lambda,\xi+\tau_\alpha}|\omega| = N$. According to (4.102), (4.108) and (4.109) we have

$$P_{\Lambda_R,\xi}(N) = \left(\frac{1}{2\pi|\Lambda|} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \Omega_{\Lambda,\xi}\Big(\frac{\alpha}{|\Lambda|}\Big) \right)^{\frac{1}{2}} \exp\Big[-|\Lambda|\Omega_{\Lambda,\xi}\Big(\frac{\alpha}{|\Lambda|}\Big)\Big] \left(1 + O\Big(\frac{1}{\sqrt{|\Lambda|}}\Big) \right). \tag{4.112}$$

This implies equation (4.96).

To prove (4.97) and (4.98) we note that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Omega_{\Lambda,\xi}(0) = |\Lambda| \left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \ln Z(\Lambda, \xi) \right)^{-1} \tag{4.113}$$

To complete the proof of Theorem 4.7 it remains to apply the asymptotic expansion of the log-partition function, Theorem 4.2.

5 Asymptotic expansion of the log-partition function. Quantum gas

In this chapter we consider Boltzmann loop gas (see section 3.5.4) which is a quantum gas with Maxwell-Boltzmann statistics in the Feynman-Kac representation .

We study the asymptotics of the logarithm of the grand partition function $\ln Z(\Lambda_R, z)$ of the loop gas in a bounded domain $\Lambda_R = \{Ru \mid u \in \Lambda\}$ as $R \to +\infty$, where $z > 0$ is the activity or intensity parameter. For the convenience of the reader we recall the definition of the MB measure $W_{z\rho}$ which is defined on the space

$$\mathcal{M} = \mathcal{M}(\mathcal{X}) = \{\mu \subset \mathcal{X} \mid \ |\mu| < \infty\},\ \ \mathcal{X} = \mathcal{X}_\beta$$

of finite configurations of loops of fixed time interval β in $\mathbb{R}^2$ by

$$W_{z\rho}(h) = \int_{\mathcal{M}} \mathrm{d}W_{z\rho}(\omega)h(\omega) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\mathcal{X}} \dots \int_{\mathcal{X}} h(x_1, \dots, x_n)\mathrm{d}\rho(x_1)\dots\mathrm{d}\rho(x_n), \tag{5.1}$$

where h is any non-negative, measurable function on $\mathcal{M}$. See Chapter 3.3, formula (3.37).

The asymptotic expansions of the log-partition function presented in this section can be viewed in the spirit of the Kac problem. In [54] Kac has considered the problem of finding the asymptotics of the partition function $\mathrm{Tr}\,\exp(\beta\Delta) = \sum_{n=1}^{\infty} e^{-\beta\lambda_n}$, where λ_n are eigenvalues of the Laplacian $-\Delta$ in a bounded domain as $\beta \to 0$. This problem has a long history and goes back to Hendrik Lorentz and Herman Weyl, see [54]. Kac has shown that the eigenvalues λ_n uniquely define geometrical characteristics of the domain, such as the area, the length of the boundary, and his conjecture was that the constant term is $\frac{1}{6}(1-h)$ where h is the number of the holes of the domain. Using Feynman-Kac formula the partition function can be written as a Brownian integral with a path of a time length β constrained to a bounded domain. This leads to an equivalent formulation of the Kac problem, that is, to find the asymptotics of the Brownian integral as $\beta \to 0$. A dual problem is to study the asymptotics of the Brownian integral with a path of a fixed time length β, constrained to a bounded domain as this domain is delated to infinity. This was considered in [5] for the case of MB statistics. Note that the Brownian integral can be identified with the log partition function $\ln Z_{id}(\Lambda, z)$ of the ideal Ginibre gas.

Here we study the large volume asymptotics of the log-partition function for the loop gas with interaction.

Chapter 5 is organized as follows. In Section 5.1 we consider domains with smooth boundaries. We assume that the particles interact via stable pair potential with nice decay properties at infinity.

The following expansion is the main result of Section 5.1:

$$\ln Z(\Lambda_R, z) = R^2\beta p(\phi, z)|\Lambda| + R\, b(\phi, z)|\partial\,\Lambda| + o(R) \text{ as } R \to +\infty. \tag{5.2}$$

Here β is the inverse temperature, $|\Lambda|$ is the volume and $|\partial\Lambda|$ the surface measure of Λ. The coefficients $p(\phi, z)$ and $b(\phi, z)$ are the same as in expansion (4.3) below.

To get the constant term, in Section 5.2, we impose more restrictive conditions on the potential ϕ and consider domains Λ which are open convex bounded subsets of $\mathbb{R}^2$ with finitely many convex closed holes. We assume that the connected parts of the boundary of Λ are one dimensional closed C^3-manifolds.

Then the main result of section Section 5.2 is:

$$\ln Z(\Lambda_R, z) = R^2|\Lambda|\beta p(\phi, z) + R\,|\partial\Lambda| b(\phi, z) + 2\pi\chi(\Lambda)c(\phi, z) + o(1). \tag{5.3}$$

Here $\chi(\Lambda)$ is the Euler-Poincare characteristic of the domain Λ. The coefficients $p(\phi, z)$, $b(\phi, z)$ and $c(\phi, z)$ are explicitly expressed as functional integrals and are analytic functions of the activity z in a neighborhood of the origin; $p(\phi, z)$ is the pressure and $b(\phi, z)$ can be interpreted as the surface tension.

We consider the case $\nu = 2$ only for simplicity.

Section 5.3 is devoted to the asymptotic expansion of the log-partition function of the ideal Bose gas. The class of admissible domains is the same as in Section 5.2.

Finally Section 5.4 presents a different method of proving the asymptotic expansion which is applicable only to polygonal domains. This method, in contrast to the method adopted for the domains with smooth boundaries, permits to get all the non-decreasing terms of the expansion.

5.1 Boundary term

We consider classical stable pair interaction given by a function ϕ which is continuous, even function on $\mathbb{R}^2 \setminus \{0\}$. We assume also that

$$\int_{\mathbb{R}^2} \mathrm{d}u |\phi(u)|(1 + |u|)^l < +\infty \tag{5.4}$$

where $l \geq 0$ will be chosen later.

Let Λ be an open convex bounded subsets of $\mathbb{R}^2$ with finitely many convex closed holes. We assume that the connected parts of the boundary of Λ are one dimensional closed C^2-manifolds.

Let $\mathbf{n}(r)$ be the inward drawn unit normal to $\partial\Lambda$ at the point $r \in \partial\Lambda$ and let

$$\Pi_r^+ = \{x \in \mathbb{R}^2 \mid \langle x, \mathbf{n}(r)\rangle \geq 0\}$$

where $\langle ., .\rangle$ is the scalar product in $\mathbb{R}^2$. When there is no confusion, we denote the configuration $(\mu \cup \omega)$ by (μ, ω) as well as the configuration $(\{x\} \cup \mu)$ by (x, μ) .

The following theorem will be proved in this section.

Theorem 5.1 (Boundary term) . *If ϕ satisfies (3) with $l > 16$ and z is from the interval*

$$0 < z < C \cdot \left[\int_{\mathbb{R}^2} \mathrm{d}u \, |\phi(u)|(1+|u|)^l \right]^{-1} \tag{5.5}$$

where $C = C(\beta, l)$, then the log-partition function of the Boltzmann gas in Λ_R has the following asymptotic expansion:

$$\begin{aligned} \ln\, Z(\Lambda_R, z) = {} & zR^2|\Lambda| \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) \int_{\mathcal{M}} \mathrm{d}W_{z\rho}(\mu) \frac{\varphi(x^0,\mu)}{|\mu|+1} \\ & - zR \int_{\partial\Lambda} \mathrm{d}\sigma(r) \int_0^{+\infty} \mathrm{d}t \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) \int_{\mathcal{M}} \mathrm{d}W_{z\rho}(\mu) \\ & \cdot 1_{\mathcal{M}^c(\Pi_r^+)}(r + t\mathbf{n}(r) + x^0, \mu) \frac{\varphi(r + t\mathbf{n}(r) + x^0, \mu)}{|\mu|+1} + o(R) \end{aligned} \tag{5.6}$$

as $R \to +\infty$ where φ is the Ursell function, $\sigma(dr)$ is the arc length element and 1_A denotes the indicator function of the set A.

If ϕ is also rotation invariant then

$$\begin{aligned} \ln\, Z(\Lambda_R, z) = {} & zR^2|\Lambda| \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) \int_{\mathcal{M}} \mathrm{d}W_{z\rho}(\mu) \frac{\varphi(x^0,\mu)}{|\mu|+1} \\ & - zR|\partial\Lambda| \int_0^{+\infty} \mathrm{d}t \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) \int_{\mathcal{M}} \mathrm{d}W_{z\rho}(\mu) 1_{\mathcal{M}^c(\Pi_0^+)}(t\mathbf{n}^0 + x^0, \mu) \cdot \\ & \cdot \frac{\varphi(t\mathbf{n}^0 + x^0, \mu)}{|\mu|+1} + o(R) \text{ as } R \to +\infty. \end{aligned} \tag{5.7}$$

Here $\mathbf{n}^0$ is any fixed unit vector and $\Pi_0^+ = \{x \in \mathbb{R}^2 \mid \langle x, \mathbf{n}^0 \rangle \geq 0\}$.

To prove this theorem we undertake the following strategy. We start with the cluster representation of the log-partition function by means of the Ursell function φ:

$$\ln Z(\Lambda_R, z) = W_{z\rho_{\Lambda_R}}(\varphi) = \int_{\mathcal{M}} \mathrm{d}W_\rho(\mu) z^{|\mu|} \varphi(\mu) \text{ if } z \text{ is sufficiently small.} \tag{5.8}$$

Here $W_{z\rho}$ is given by (3.37) and g is defined below by (5.10). In the second step we rewrite (5.8) as:

$$\ln\, Z(\Lambda_R, z) = \int_{\mathcal{X}} \mathrm{d}\rho(x) \int_{\mathcal{M}} \mathrm{d}W_\rho(\mu) g_z(x,\mu) 1_{\mathcal{M}(\Lambda_R)}(x,\mu) \tag{5.9}$$

where

$$g_z(\mu) = z^{|\mu|} \frac{\varphi(\mu)}{|\mu|}, \text{ if } |\mu| \geq 1. \tag{5.10}$$

Writing explicitly the integration with respect to ρ, we obtain the representation

$$\begin{aligned} \ln Z(\Lambda_R, z) = {} & \int_{\Lambda_R} \mathrm{d}u \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) \int_{\mathcal{M}} \mathrm{d}W_\rho(\mu) 1_{\mathcal{M}(\Lambda_R)}(u + x^0, \mu) \\ & \cdot g_z(u + x^0, \mu). \end{aligned} \tag{5.11}$$

Then the main idea is the following: we write this integral as an integral over the set $\mathcal{A}$ of all configurations $(u + x^0, \mu)$ of loops with arbitrary $x^0 \in \mathcal{X}^0$, $\mu \in \mathcal{M}$, then we subtract the integral over the set $\mathcal{A}_1$ of configurations $(u + x^0, \mu)$ where at least one loop leaves the domain Λ_R (it can be $u+x^0$ or any loop from μ). The integral over $\mathcal{A}$ gives the main term of our asymptotics which is proportional to the area of Λ_R. The second integral we split in two integrals by decomposing the set $\mathcal{A}_1$ as $\mathcal{A}_1 = \mathcal{A}_2 + \mathcal{A}_3$. Here $\mathcal{A}_2$ is the set of configurations $(u+x^0, \mu)$ where at least one loop crosses the tangent and $\mathcal{A}_3$ consists of those configurations where at least one leaves Λ_R but non of them crosses the tangent. The integral over $\mathcal{A}_2$ gives us the boundary term. The analysis of the correction terms is based on the estimates for the decay of correlations in the loop gas which was obtained in Section 3.5.4. To show that the last integral over the set $\mathcal{A}_3$ is $o(R)$ as $R \to \infty$ we also translate some arguments given in section II of Ref.[62] where similar problem for one loop case is solved.

5.1.1 Proof of Theorem 5.1

We start with the cluster representation of the log-partition function:

$$\ln Z(\Lambda_R, z) = W_{z\rho_{\Lambda_R}}(\varphi) \text{ if } z \text{ is sufficiently small.} \tag{5.12}$$

In the second step we rewrite (5.12) as:

$$\ln Z(\Lambda_R, z) = \int_{\mathcal{X}} \mathrm{d}\rho(x) \int_{\mathcal{M}} \mathrm{d}W_\rho(\mu) g_z(x, \mu) 1_{\mathcal{M}(\Lambda_R)}(x, \mu). \tag{5.13}$$

Writing explicitly the integration with respect to ρ we obtain the representation

$$\ln Z(\Lambda_R, z) = \int_{\Lambda_R} \mathrm{d}u \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) \int_{\mathcal{M}} \mathrm{d}W_\rho(\mu) 1_{\mathcal{M}(\Lambda_R)}(u + x^0, \mu) \cdot g_z(u + x^0, \mu). \tag{5.14}$$

The representation (5.13) directly follows from (5.12) with the help of the following well-known formula (see, for example, [46])

$$\int_{\mathcal{M}} F(\mu) \sum_{\omega \subset \mu} h_1(\omega) h_2(\mu \setminus \omega) \mathrm{d}W_\rho(\mu) \tag{5.15}$$
$$= \int_{\mathcal{M}} \int_{\mathcal{M}} F(\mu_1 \cup \mu_2) h_1(\mu_1) h_2(\mu_2) \mathrm{d}W_\rho(\mu_1) \mathrm{d}W_\rho(\mu_2)$$

which is valid if either the functions F, h_1 and h_2 are non-negative or at least one side is absolutely convergent.

For any measurable $A \subset \mathcal{M}$ let

$$\mathcal{W}_A f(\mu) = \int_A \mathrm{d}W_\rho(\omega)) f(\mu, \omega), \ \ \mu \in \mathcal{M}. \tag{5.16}$$

We use the notation $f(\mu,\omega)$ instead of $f(\mu\cup\omega)$ and since ρ is diffuse we can consider μ and ν in (5.16) as disjoint configurations. For brevity we will write $\mathcal{W}$ instead of $\mathcal{W}_{\mathcal{M}}$.

Note that the two-point truncated correlation function can be written in this notations as

$$\sigma_\Lambda(x,y) = \int_{\mathcal{M}(\Lambda)} \mathrm{d}W_{z\rho}(\omega)\varphi(x,y,\omega) = \mathcal{W}_{\mathcal{M}(\Lambda)}\varphi(x,y) \tag{5.17}$$

We will use often the notation

$$\mathcal{W}_A|f|(\mu) = \int_A \mathrm{d}W_\rho(\omega))|f(\mu,\omega)|, \;\; \mu\in\mathcal{M}. \tag{5.18}$$

It follows from Theorem 3.1 that

$$\mathcal{W}_{\mathcal{M}(\Lambda)}|\varphi|(x,y) \le e^{2\beta B+1}\sup_{x,y}|q(x,y)|\sum_{m=1}^{\infty}[ze^{2\beta B+1}\int_{\mathcal{X}}\mathrm{d}\rho(y)|q(x,y)|]^m. \tag{5.19}$$

Since $\sup_{x,y}|q(x,y)| < e^{2\beta B}+1$ we have that uniformly in Λ,

$$\mathcal{W}_{\mathcal{M}(\Lambda)}|\varphi|(x,y) < \infty \tag{5.20}$$

provided

$$ze^{2\beta B+1}\int_{\mathcal{X}}\mathrm{d}\rho(y)|q(x,y)| < 1. \tag{5.21}$$

Now(5.14) implies

$$\begin{aligned}\ln Z(\Lambda_R,z) &= \int_{\Lambda_R}\mathrm{d}u\int_{\mathcal{X}^0}\mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}(\Lambda_R)}g_z)(u+x^0)\\ &= \int_{\Lambda_R}\mathrm{d}u\int_{\mathcal{X}^0}\mathrm{d}P^0(x^0)(\mathcal{W}g_z)(u+x^0)\\ &- \int_{\Lambda_R}\mathrm{d}u\int_{\mathcal{X}^0}\mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\Lambda_R)}g_z)(u+x^0).\end{aligned} \tag{5.22}$$

Putting

$$I^A(u,R) = \int_{\mathcal{X}^0}\mathrm{d}P^0(x^0)(\mathcal{W}g_z)(u+x^0) \tag{5.23}$$

and

$$I^B(u,R) = \int_{\mathcal{X}^0}\mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\Lambda_R)}g_z)(u+x^0), \tag{5.24}$$

we can rewrite (5.22) as

$$\ln Z(\Lambda_R,z) = \int_{\Lambda_R}\mathrm{d}u\, I^A(u,R) - \int_{\Lambda_R}\mathrm{d}u\, I^B(u,R). \tag{5.25}$$

Consider the first integral on the right-hand side of (5.25). Note that $(\mathcal{W}g_z)(u+x^0)$ does not depend on $u\in\mathbb{R}^2$ because of translation invariance of $\mathcal{M}$, W_ρ and g_z. It follows from Theorem 2.1, formula (2.8), that

$$(\mathcal{W}|g_z|)(u+x^0) \le e^{2\beta B+1}. \tag{5.26}$$

Therefore

$$\int_{\Lambda_R} \mathrm{d}u\, I^A(u,R) = R^2\,|\Lambda| \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}g_z)(x^0). \tag{5.27}$$

The integral on the right-hand side of (5.23) multiplied by β^{-1} is called **pressure**.

Now consider the second integral on the right-hand side of (5.25). Let

$$\Lambda_{R,\delta} = \{u \in \Lambda_R \;\big|\; d(u, \partial\,\Lambda_R) < \delta R^\varepsilon\}.$$

Then

$$\int_{\Lambda_R} I^B(u,R)\mathrm{d}u = \int_{\Lambda_{R,\delta}} I^B(u,R)\mathrm{d}u + \int_{\Lambda_R\setminus\Lambda_{R,\delta}} I^B(u,R)\mathrm{d}u. \tag{5.28}$$

Taking from now on $\varepsilon = \frac{1}{8}$ and $l > 16$ in condition (4) (see (3.71)), we find from Corollary 3.16 that

$$\int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(B_{\delta R^\varepsilon}(u))}|g_z|)(u+x^0) = o(R^{-2}) \tag{5.29}$$

as $R \to \infty$, uniformly in $u \in \mathbb{R}^2$. Here $B_{\delta R^\varepsilon}(u)$ is a ball of radius δR^ε centered at $u \in \mathbb{R}^2$. On account of (5.29)

$$\int_{\Lambda_R\setminus\Lambda_{R,\delta}} I^B(u,R)\mathrm{d}u = o(1).$$

Therefore

$$\int_{\Lambda_R} I^B(u,R)\mathrm{d}u = \int_{\Lambda_{R,\delta}} I^B(u,R)\mathrm{d}u + o(1). \tag{5.30}$$

To treat the integral on the right-hand side of (5.30) we set up at each point $r \in \partial\,\Lambda$ local coordinates (ξ,η) where ξ is along the tangent vector $\mathbf{s} = \mathbf{s}(r)$ and η is along the inward drawn unit normal $\mathbf{n} = \mathbf{n}(r)$ to $\partial\,\Lambda$ at r. Then $\partial\,\Lambda$ is given locally by $\eta = f_r(\xi)$, $|\xi| < \delta R^\varepsilon$, for $\delta > 0$ small enough where f_r is a function of class $\mathcal{C}^2$. Choosing $\delta = \frac{1}{2}[\sup_{r\in\partial\,\Lambda} |\kappa(r)|]^{-1}$ and taking into account that the volume element in $\Lambda_{R,\delta}$ is equal to $(1 - tk_R(r))\mathrm{d}t\sigma(dr)$ (see, for example,[87]), we have that

$$\int_{\Lambda_{R,\delta}} I^B(u,R)\mathrm{d}u = \int_{\partial\,\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t(1 - tk_R(r))I^B(r + t\mathbf{n}, R) \tag{5.31}$$

where $k_R(r)$ is the curvature of $\partial\Lambda_R$ at the point $r \in \partial\Lambda_R$, $k_1(r) \equiv k(r)$. Evidently $k_R(r) = R^{-1}k(R^{-1}r)$, $r \in \partial\Lambda_R$.

Furthermore, we associate to each $r \in \partial\Lambda_R$ the cylinder

$$\Pi_{r,\delta R^\varepsilon} = \{(\xi,\eta) \;\big|\; |\xi| < \delta R^\varepsilon\}.$$

To get rid of configurations that have loops visiting outside of the cylinder $\Pi_{r,\delta R^\varepsilon}$, we decompose I^B by decomposing $\mathcal{M}^c(\Lambda_R)$ as follows:

$$\mathcal{M}^c(\Lambda_R) = \mathcal{M}^c(\Lambda_R)\mathcal{M}(\Pi_{r,\delta R^\varepsilon}) + \mathcal{M}^c(\Lambda_R)\mathcal{M}^c(\Pi_{r,\delta R^\varepsilon}), r \in \partial\Lambda_R.$$

(We use + for the union of disjoint sets.) According to Corollary 3.16 the term which corresponds to $\mathcal{M}^c(\Lambda_R)\mathcal{M}^c(\Pi_{r,\delta R^\varepsilon})$ in this decomposition is $o(R^{-2})$, therefore

$$\begin{aligned} I^B(r+t\mathbf{n},R) &= \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\Lambda_R)}g_z)(r+t\mathbf{n}+x^0) \\ &= \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\Lambda_R)\mathcal{M}(\Pi_{r,\delta R^\varepsilon})}g_z)(u+x^0) + o(R^{-2}) \\ &= I_1^B(r+t\mathbf{n},R) + o(R^{-2}). \end{aligned} \tag{5.32}$$

To treat I_1^B, we put

$$\mathcal{F}^+_{r,\delta R^\varepsilon} = \{(\xi,\eta) \in \Pi_{r,\delta R^\varepsilon} \bigm| \eta > f_{r,R}(\xi)\},$$

where $f_{r,R}(\xi) = Rf_r\left(\frac{\xi}{R}\right)$.

Now we decompose I_1^B by decomposing $\mathcal{M}^c(\Lambda_R)\mathcal{M}(\Pi_{r,\delta R^\varepsilon})$ as follows:

$$\mathcal{M}^c(\Lambda_R)\mathcal{M}(\Pi_{r,\delta R^\varepsilon}) = \mathcal{M}^c(\Lambda_R)\mathcal{M}(\mathcal{F}^+_{r,\delta R^\varepsilon}) + \mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}).$$

Note that each configuration $\mu \in \mathcal{M}^c(\Lambda_R)\mathcal{M}(\mathcal{F}^+_{r,\delta R^\varepsilon})$ contains at least one loop touching the exterior of Λ_R in $\mathcal{F}^+_{r,\delta R^\varepsilon}$ (which can be a hole). Again using Corollary 3.16, we get

$$\int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\Lambda_R)\mathcal{M}(\mathcal{F}^+_{r,\delta R^\varepsilon})}g_z)(r+t\mathbf{n}+x^0) = o(R^{-2}).$$

Hence

$$\begin{aligned} I_1^B(r+t\mathbf{n},R) &= \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi_{r,\delta R^\varepsilon})}g_z)(r+t\mathbf{n}+x^0) \\ &\quad + o(R^{-2}) = I_2^B(r+t\mathbf{n},R) + o(R^{-2}). \end{aligned} \tag{5.33}$$

To obtain the boundary term of the expansion, we separate from the set $\mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})$ those configurations which have at least one loop crossing the tangent. Let

$$\Pi^+_{r,\delta R^\varepsilon} = \{(\xi,\eta) \in \Pi_{r,\delta R^\varepsilon} \bigm| \eta \geq 0\}.$$

Then

$$\mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}) = \mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon}) + \mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}). \tag{5.34}$$

Remark. For shortness we consider only the case where r is a point of the convex part of the boundary $\partial\Lambda_R$. The case where r belongs to the concave part of the boundary (the boundary of a hole) can be treated similarly. For example, (5.34) in the concave case will have the form:

$$\mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}) = \mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon}) - \mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}(\mathcal{F}^+_{r,\delta R^\varepsilon}).$$

From (5.34) it follows that

$$\begin{aligned} I_3^B(r+t\mathbf{n},R) &= \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi_{r,\delta R^\varepsilon})}g_z)(r+t\mathbf{n}+x^0) \\ &+ \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})}g_z)(r+t\mathbf{n}+x^0) \\ &= J(r+t\mathbf{n},R) + K(r+t\mathbf{n},R). \end{aligned} \tag{5.35}$$

Now combining (5.31) - (5.33) and (5.35) we get

$$\begin{aligned} \int_{\Lambda_{R,\delta}} I^B(u,R)\mathrm{d}u &= \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} dt(1-tk_R(r))J(r+t\mathbf{n},R) \\ &+ \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t(1-tk_R(r))K(r+t\mathbf{n},R) + o(1) \\ &= \mathcal{J}(R) + \mathcal{K}(R) + o(1). \end{aligned} \tag{5.36}$$

Below we will show that $\mathcal{J}(R)$ is the boundary term (up to the corrections) and that $\mathcal{K}(R) = o(R)$. First we treat $\mathcal{J}(R)$. To get rid of the restriction to the cylinder, we decompose $J(r+t\mathbf{n},R)$ by decomposing $\mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon})$:

$$\mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon}) = \mathcal{M}^c(\Pi^+_r) - \mathcal{M}^c(\Pi^+_r)\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon}),$$

where

$$\Pi^+_r = \{(\xi,\eta) \mid \eta \geq 0\} \tag{5.37}$$

We estimate the term corresponding to $\mathcal{M}^c(\Pi^+_r)\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon})$ by using again Corollary 3.16 and find from (5.35) and (5.36) that

$$\mathcal{J}(R) = \mathcal{J}_1(R) - \mathcal{J}_2(R) + o(1)$$

where

$$\mathcal{J}_1(R) = \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\Pi^+_r)}g_z)(r+t\mathbf{n}+x^0)$$

and

$$\mathcal{J}_2(R) = \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t\; tk_R(r) \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\Pi^+_r)}g_z)(r+t\mathbf{n}+x^0).$$

Furthermore, writing the integration $\int_0^{\delta R^\varepsilon} dt$ as the difference $\int_0^\infty dt - \int_{\delta R^\varepsilon}^\infty dt$ and using the translation invariance of g_z, W_ρ and $\mathcal{M}^c(\Pi_r^+)$ as well as Corollary 3.16, we obtain

$$\mathcal{J}(R) = \mathcal{J}^b(R) - \mathcal{J}_1^c(R) + o(1) \tag{5.38}$$

where

$$\mathcal{J}^b(R) = R\int_{\partial\Lambda} \mathrm{d}\sigma(r) \int\limits_0^\infty \mathrm{d}t \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\Pi_r^+)} g_z)(r + t\mathbf{n} + x^0) \tag{5.39}$$

and

$$\mathcal{J}_1^c(R) = \int_{\partial\Lambda} \mathrm{d}\sigma(r) \int\limits_0^\infty \mathrm{d}t\ tk(r) \int_{\mathcal{X}^0} \mathrm{d}P^0(dx^0)(\mathcal{W}1_{\mathcal{M}^c(\Pi_r^+)} g_z)(r + t\mathbf{n} + x^0). \tag{5.40}$$

By Corollary 3.16

$$\int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\Pi_r^+)}|g_z|)(r + t\mathbf{n} + x^0) \le \frac{C}{(1+t)^l},$$

therefore both integrals $\mathcal{J}^b(R)$ and $\mathcal{J}_1^c(R)$ are convergent.

Note that $\mathcal{J}^b(R)$ is the boundary term and $\mathcal{J}_1^c(R)$ is a contribution to the constant term

Now we consider $\mathcal{K}(R)$ from (5.36). In this section we show only that $\mathcal{K}(R) = o(R)$. The further analysis, which we carry out in the next section, allows to separate from $\mathcal{K}(R)$ additional contributions to the constant term. The term $K(r + t\mathbf{n}, R)$ from (5.35) can be written in the form

$$\begin{aligned}
K(r + t\mathbf{n}, R) &= \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})} g_z)(r + t\mathbf{n} + x^0) \\
&= \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)1_{\mathcal{X}(\mathcal{F}^+_{r,\delta R^\varepsilon})}(r + t\mathbf{n} + x^0)(\mathcal{W}1_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})} g_z)(r + t\mathbf{n} + x^0) \\
&+ \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})}(r + t\mathbf{n} + x^0)(\mathcal{W}1_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})} g_z)(r + t\mathbf{n} + x^0) \\
&= K^A(r + t\mathbf{n}, R) + K^B(r + t\mathbf{n}, R).
\end{aligned} \tag{5.41}$$

This implies the following decomposition of $\mathcal{K}(R)$:

$$\begin{aligned}
\mathcal{K}(R) &= \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int\limits_0^{\delta R^\varepsilon} \mathrm{d}t(1 - tk_R(r))K^A(r + t\mathbf{n}, R) \\
&+ \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int\limits_0^{\delta R^\varepsilon} \mathrm{d}t(1 - tk_R(r))K^B(r + t\mathbf{n}, R) \\
&= \mathcal{K}^A(R) + \mathcal{K}^B(R).
\end{aligned} \tag{5.42}$$

Let us estimate first $\mathcal{K}^A(R)$. By the choice of δ, $|1 - tk_R(r)| \leq 2$ for all $t \in [0, \delta R^\varepsilon]$, hence

$$\mid \mathcal{K}^A(R) \mid \leq 2 \int_{\partial \Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t \mid K^A(r + t\mathbf{n}, R) \mid . \tag{5.43}$$

It is easy to check that for any measurable and non-negative function f defined on the space $\mathcal{M}$ and any measurable Λ_1, $\Lambda_2 \subset \mathbb{R}^2$

$$\int_{\mathcal{M}} \mathrm{d}W_\rho(\mu) 1_{\mathcal{M}^c(\Lambda_1)}(\mu) 1_{\mathcal{M}(\Lambda_2)}(\mu) f(\mu) \leq \int_{\mathcal{X}^c(\Lambda_1)} \mathrm{d}\rho(x) \int_{\mathcal{M}} \mathrm{d}W_\rho(\mu) f(x, \mu). \tag{5.44}$$

Indeed, using formula (5.15), we have that

$$\begin{aligned}
&\int_{\mathcal{M}} \mathrm{d}W_\rho(\mu) 1_{\mathcal{M}^c(\Lambda_1)}(\mu) 1_{\mathcal{M}(\Lambda_2)}(\mu) f(\mu) \\
&\leq \int_{\mathcal{M}} \mathrm{d}W_\rho(\mu) 1_{\mathcal{M}^c(\Lambda_1)}(\mu) 1_{\mathcal{M}(\Lambda_2)}(\mu) f(\mu) \sum_{\bar{\mu} \subset \mu} 1_{\mathcal{M}_1^c(\Lambda_1)}(\bar{\mu}) 1_{\mathcal{M}_1(\Lambda_2)}(\bar{\mu}) \\
&\leq \int_{\mathcal{X}^c(\Lambda_1)\mathcal{X}(\Lambda_2)} \mathrm{d}\rho(x) \int_{\mathcal{M}} \mathrm{d}W_\rho(\mu) f(x, \mu) = \int_{\mathcal{X}^c(\Lambda_1)\mathcal{X}(\Lambda_2)} \mathrm{d}\rho(x) (\mathcal{W} f)(x)
\end{aligned}$$

where $\mathcal{M}_1 = \{\mu \in \mathcal{M} \mid |\mu| = 1\}$ is the subspace of one-element configurations.

Then on account of (5.43)

$$\begin{aligned}
\mid K^A(r + t\mathbf{n}, R) \mid &\leq \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) \int_{\mathcal{M}} \mathrm{d}W_\rho(\mu) 1_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})}(\mu) \\
&\cdot \mid g_z(r + t\mathbf{n} + x^0, \mu) \mid \leq \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) \int_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})} \mathrm{d}\rho(y) \\
&\cdot (\mathcal{W} \mid g_z \mid)(r + t\mathbf{n} + x^0, y).
\end{aligned}$$

As it was mentioned above, there exists a constant $C = C(\beta, z, \phi, l)$ such that $(\mathcal{W} \mid g_z \mid)(x, y) \leq C,$ for any $x, y \in \mathcal{X}$. Hence putting $\mathcal{D}(r, R) = \rho(\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon}))$ we find that

$$\mid K^A(r + t\mathbf{n}, R) \mid \leq \frac{C}{2\pi\beta} \cdot \mathcal{D}(r, R). \tag{5.45}$$

Thus we need to study $\mathcal{D}(r, R),\ r \in \partial\Lambda_R$. Obviously

$$\mathcal{D}(r, R) = \int_{-\delta R^\varepsilon}^{\delta R^\varepsilon} \mathrm{d}\xi \int_0^\infty \mathrm{d}\eta \int_{\mathcal{X}^0} \mathrm{d}P^0(y^0) 1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(y^0 + (\xi, \eta)).$$

Note that $(\xi, \eta) + y^0 \in \mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})$ iff

$$\begin{cases} \inf_{0 \leq t \leq \beta} \langle y^0(t) + \eta, \mathbf{n}(r) \rangle \geq 0, \\ \sup_{0 \leq t \leq \beta} |\widetilde{y^0(t)} + \xi| < \delta R^\varepsilon \end{cases}$$

where $\widetilde{y^0(t)} = y^0(t) - \langle y^0(t), \mathbf{n}(\mathbf{r})\rangle$ is the projection of $y^0(t)$ onto the tangent to $\partial\Lambda_R$ at the point r. On the other hand, $(\xi, \eta) + y^0 \in \mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})$ iff $\sup_{0\le t\le\beta}[f_{r,R}(\widetilde{y^0(t)} + \xi) - \langle y^0(t) + \eta, \mathbf{n}(\mathbf{r})\rangle] > 0$.

To simplify notations, where there is no confusion, we will omit below the arguments in the above mentioned conditions, for example, we will write the last condition in the form: $\sup[f_{r,R}(\tilde{y}^0 + \xi) - \langle y^0 + \eta, \mathbf{n}\rangle] > 0$.

Now putting for each $\xi \in (-\delta R^\varepsilon, \delta R^\varepsilon)$ and $r \in \partial\Lambda_R$,

$$A_R(r,\xi) = \{(\eta, y^0) \in [0, +\infty) \times \mathcal{X}^0 \bigm| -\inf\langle y^0, \mathbf{n}\rangle < \eta < \sup[f_{r,R}(\tilde{y^0} + \xi) - \langle y^0, \mathbf{n}\rangle]\},$$

we can write, using Fubini's theorem, that

$$\begin{aligned}\mathcal{D}(r,R) &= \int_{-\delta R^\varepsilon}^{\delta R^\varepsilon} \mathrm{d}\xi \int_0^\infty \mathrm{d}\eta \int_{\mathcal{X}^0} \mathrm{d}P^0(y^0) 1_{\sup|\tilde{y}^0+\xi|<\delta R^\varepsilon}(\xi, y^0) 1_{A_R(r,\xi)}(\eta, y^0) \\ &= \int_{-\delta R^\varepsilon}^{\delta R^\varepsilon} \mathrm{d}\xi \int_{\mathcal{X}^0} \mathrm{d}P^0(y^0) 1_{\sup|\tilde{y}^0+\xi|<\delta R^\varepsilon}(\xi, y^0)\{\sup[f_{r,R}(\xi + \tilde{y}^0) - \langle y^0, \mathbf{n}\rangle] + \inf\langle y^0, \mathbf{n}\rangle\}.\end{aligned}$$

Following the paper [5], we choose $\tau(\mathbf{n})$ and $\tau_R(\xi)$ so that

$$\begin{cases} \inf\langle y^0, \mathbf{n}\rangle = \langle y^0(\tau(\mathbf{n})), \mathbf{n}\rangle, \\ \sup[f_{r,R}(\xi + \tilde{y}^0) - \langle y^0, \mathbf{n}\rangle] = f_{r,R}(\xi + \widetilde{y^0(\tau_R(\xi))}) - \langle y^0(\tau_R(\xi)), \mathbf{n}\rangle \end{cases}$$

Note that $\tau(\mathbf{n})$ is P^0-almost surely unique (see Ref. [55] and reference there, a direct proof is given in Appendix A.3). Then

$$f_{r,R}(\xi + \widetilde{y^0(\tau(\mathbf{n}))}) \le \sup[f_{r,R}(\xi + \tilde{y}^0) - (y^0, \mathbf{n})] + \inf(y^0, \mathbf{n}) \le f_{r,R}(\xi + \widetilde{y^0(\tau_R(\xi))}).$$

At the same time, we have that

$$|f_{r,R}(\xi)| \le CR^{-1}|\xi|^2 \ , \ \text{if } |\xi| < \delta R^\varepsilon, \tag{5.46}$$

with a constant $C = C(\Lambda)$. Therefore, using the condition: $\sup|\xi + \tilde{y}^0| < \delta R^\varepsilon$, we obtain

$$|\sup[f_{r,R}(\xi + \tilde{y}^0) - \langle y^0, \mathbf{n}\rangle] + \inf\langle y^0, \mathbf{n}\rangle| \le CR^{-\frac{3}{4}}, \tag{5.47}$$

uniformly in $r \in \partial\Lambda_R$. This implies

$$\mathcal{D}(r,R) \le CR^{-\frac{5}{8}} \tag{5.48}$$

and therefore from (5.46) and (5.48) it follows that

$$|\ \mathcal{K}^A(R)\ | \le \frac{C}{\pi\beta} R^{\frac{1}{2}}. \tag{5.49}$$

Now we estimate $\mid \mathcal{K}^B(R) \mid$. We can rewrite $K^B(r+t\mathbf{n},R)$ from (5.42) as

$$K^B(r+t\mathbf{n},R) = \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) 1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(r+t\mathbf{n}+x^0)$$
$$\cdot \int_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})} \mathrm{d}W_\rho(\mu) g_z)(r+t\mathbf{n}+x^0,\mu).$$

Hence

$$\mid K^B(r+t\mathbf{n},R) \mid \leq \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) 1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(r+t\mathbf{n}+x^0)$$
$$\cdot (\mathcal{W} \mid g_z \mid)(r+t\mathbf{n}+x^0). \tag{5.50}$$

Using Fubini's theorem and (5.26) we have

$$\mid \mathcal{K}^B(R) \mid \leq \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t (1 - t k_R(r)) \mid K^B(r+t\mathbf{n},R) \mid \tag{5.51}$$
$$\leq 2e^{2\beta B+1} \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^\infty \mathrm{d}t \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) 1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(r+t\mathbf{n}+x^0)$$
$$= 2e^{2\beta B+1} \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) 1_{\sup|\tilde{x}^0|<\delta R^\varepsilon}(x^0) \cdot$$
$$\cdot \{\sup[f_{r,R}(\tilde{x}^0) - \langle x^0,\mathbf{n}\rangle] + \inf\langle x^0,\mathbf{n}\rangle\} \leq \frac{e^{2\beta B+1}}{\pi\beta} |\partial\Lambda| R^{\frac{1}{4}}.$$

Here we used also the estimate (5.48) with $\xi = 0$. Thus $\mathcal{K}^B(R) = o(R)$. This together with (5.42) and (5.49) gives: $\mathcal{K}(R) = o(R)$.

Now combining (5.30), (5.36) and (5.38) - (5.40) we get

$$\int_{\Lambda_R} I^B(u,R)\mathrm{d}u = \mathcal{J}^b(R) + o(R). \tag{5.52}$$

Finally (5.23), (5.25) and (5.52) imply

$$\ln\, Z(\Lambda_R,z) = R^2 |\Lambda| \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}g_z)(x^0)$$
$$- R \int_{\partial\Lambda} \mathrm{d}\sigma(r) \int_0^\infty \mathrm{d}t \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\Pi^+_r)} g_z)(r+t\mathbf{n}+x^0) + o(R).$$

To get (5.6), it remains to write explicitly the integrals $\mathcal{W}g_z$ and $\mathcal{W}1_{\mathcal{M}^c(\Pi^+_r)}g_z)$.

Now if ϕ is in addition rotation invariant then, because of rotation invariance of the Ursell function and the measure $W_{z\rho}$, the integral $(\mathcal{W}1_{\mathcal{M}^c(\Pi^+_r)}g_z)(r+t\mathbf{n}+x^0)$ does not depend on the orientation of the normal $\mathbf{n}(r)$ and the half-plane Π^+_r, $r \in \partial\Lambda_R$, hence it can be evaluated with respect to any fixed unit vector $\mathbf{n}^0$ and corresponding half-plane Π^+_0. This implies (5.7) or, in other words, the expansion (1) with

$$p(\phi,z) = \beta^{-1} \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}g_z)(x^0)$$

and

$$b(\phi, z) = -\int_0^\infty \mathrm{d}t \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)(\mathcal{W}1_{\mathcal{M}^c(\Pi_0^+)}g_z)(t\mathbf{n}^0 + x^0) \tag{5.53}$$

which completes the proof of Theorem 5.1.

5.2 Constant term.

We continue the study of the asymptotic behavior of the log-partition function of a quantum gas with MB statistics.

This section studies the third (constant) term of the asymptotics of $\ln Z(\Lambda_R, z)$ as $R \to +\infty$. The class of domains Λ to be considered in this section consists of open convex bounded subsets of $\mathbb{R}^2$ with finitely many convex closed holes such that the connected parts of the boundary of Λ are one dimensional closed C^3-manifolds.

We assume that particles interact via pair interaction ϕ, which is an even function on $\mathbb{R}^2$ and satisfies the stability condition with a constant $B \geq 0$:

$$\sum_{1 \leq i < j \leq n} \phi(u_i - u_j) \geq -Bn. \tag{5.54}$$

Moreover we assume that ϕ is differentiable and is uniformly bounded together with its derivatives so that

$$|\phi(u)| \leq M, \ \ ||\phi_l||_1 = \int_{\mathbb{R}^\nu} du|\phi_l(u)| < +\infty \tag{5.55}$$

where $\phi_l(u) = \phi(u)(1 + |u|)^l$ with $l \geq 16$, and

$$|\nabla\phi(u)| \leq M', \ \ ||\nabla\phi||_1 = \int_{\mathbb{R}^\nu} du|\nabla\phi(u)| < +\infty. \tag{5.56}$$

Let $\mathcal{X}^0 = \{x \in \mathcal{C}([0, \beta], \mathbb{R}^2) \mid x(0) = x(\beta) = 0\}$ be the space of Brownian loops in $\mathbb{R}^2$ which start and end at 0. Let P^0 be the non-normalized Brownian bridge measure on $\mathcal{X}^0$, $P^0(\mathcal{X}^0) = (2\pi\beta)^{-1}$. We will consider also modified measures $P^0_k,\ k = 1, 2, \cdots$, given by $P^0_k(dx^0) = (\sup|x^0|)^k P^0(dx^0)$. Let $\lambda = \lambda(\beta) = \max\{P^0_1(\mathcal{X}^0), P^0_2(\mathcal{X}^0)\}$.

The main result of Section 5.2 is

Theorem 5.2 (Constant term) . *If the potential ϕ satisfies the above mentioned conditions (5.54) - (5.56) and z is from the interval*

$$0 < z < [2^l\beta e^{\beta B+1}\lambda \max(M, ||\phi_l||_1, ||\nabla\phi||_1)]^{-1} \tag{5.57}$$

then for any admissible domain Λ the log-partition function has the following asymptotic expansion:

$$\ln Z(\Lambda_R, z) = R^2|\Lambda|\beta p(\phi, z) + Rb(\Lambda, \phi, z) + c(\Lambda, \phi, z) + o(1). \tag{5.58}$$

where the coefficients $p(\phi, z), b(\Lambda, \phi, z)$ and $c(\Lambda, \phi, z)$ are given explicitly by Eqs. (5.23), (5.64) and (5.133) respectively.

If ϕ is rotation invariant the coefficients $b(\Lambda, \phi, z)$ and $c(\Lambda, \phi, z)$ take a simpler form

$$\ln Z(\Lambda_R, z) = R^2|\Lambda|\beta p(\phi, z) + R\,|\partial\,\Lambda|b(\phi, z) + 2\pi\chi(\Lambda)c(\phi, z) + o(1) \tag{5.59}$$

with $b(\phi, z)$ and $c(\phi, z)$ given by Eqs. (5.135) and (5.136).

5.2.1 The proof of Theorem 5.2.

Here we briefly describe how the volume, the boundary terms as well as the first contribution to the constant term were obtained in Section 4.1.

We write $\ln Z(\Lambda_R, z)$ as an integral of the Ursell function over the finite configurations of loops in Λ_R, we then separate a loop and release all the constraints except that the separated loop starts in Λ_R. This gives the volume term. Then we take away the integral over the configurations where at least one loop leaves Λ_R. Approximating this integral by the integral over the configurations where at least one loop crosses the tangent line we obtain the boundary term as well as the first contribution $\mathcal{J}_1^c$ to the constant term (see Eq. (5.66) below).

So using the notations of Section 4.1.1 we rewrite the formula (5.36) as

$$\begin{aligned}\int_{\Lambda_R} I^B(u, R)\mathrm{d}u &= \int_{\partial\,\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t(1 - tk_R(r))J(r + t\mathbf{n}, R)\\ &+ \int_{\partial\,\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t(1 - tk_R(r))K(r + t\mathbf{n}, R) + o(1)\\ &= \mathcal{J}(R) + \mathcal{K}(R) + o(1),\end{aligned} \tag{5.60}$$

with

$$J(r + t\mathbf{n}, R) = \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)\mathcal{W}(1_{\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi_{r,\delta R^\varepsilon})}g_z)(r + t\mathbf{n} + x^0) \tag{5.61}$$

and

$$K(r + t\mathbf{n}, R) = \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)\mathcal{W}(1_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})}g_z)(r + t\mathbf{n} + x^0). \tag{5.62}$$

It was shown (see (5.38)) that

$$\mathcal{J}(R) = -Rb(\Lambda, \phi, z) - \mathcal{J}_1^c + o(1). \tag{5.63}$$

Here

$$b(\Lambda, \phi, z) = -\int_{\partial\,\Lambda} \mathrm{d}\sigma(r) \int_0^\infty \mathrm{d}t \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0)\mathcal{W}(1_{\mathcal{M}^c(\Pi^+_r)}g_z)(r + t\mathbf{n} + x^0) \tag{5.64}$$

and

$$\mathcal{J}_1^c = \int_{\partial\Lambda} \mathrm{d}\sigma(r) k(r) \int_0^\infty \mathrm{d}t\, t \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) \mathcal{W}(1_{\mathcal{M}^c(\Pi_r^+)} g_z)(r + t\mathbf{n} + x^0) \tag{5.65}$$

where $\Pi_r^+ = \{(\xi, \eta) \mid \eta > 0\}$.

The integrals on the right-hand side of Eqs. (5.64) and (5.65) are absolutely convergent for z satisfying

$$0 < z < \pi(2^{l-1} e^{\beta B+1} ||\phi_l||_1)^{-1}. \tag{5.66}$$

We observe that $Rb(\Lambda, \phi, z)$ is the boundary term and $\mathcal{J}_1^c$ is the first contributions to the constant term of the asymptotic expansion (5.58).

Now we consider the term $\mathcal{K}(R)$ from (5.60). In Section 4.1.1 we have just shown that $\mathcal{K}(R) = o(R)$. Below, analyzing this term in more details, we will separate from $\mathcal{K}(R)$ the second and the third contributions to the constant term and will show that the rest is $o(1)$.

Note that, due to the factor $1_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})}$, the integration on the right-hand side of Eq. (5.62) is over those configurations $\{r + t\mathbf{n} + x^0, \omega\}$ in $\Pi^+_{r,\delta R^\varepsilon}$ where at least one loop leaves Λ_R. There are two possibilities: either the loop $r + t\mathbf{n} + x^0$ stays in Λ_R then at least one loop from the configuration ω leaves Λ_R, or the loop $r + t\mathbf{n} + x^0$ itself leaves Λ_R then ω is any configuration in $\Pi^+_{r,\delta R^\varepsilon}$. We will treat these cases separately.

Therefore we write $K(r + t\mathbf{n}, R)$ as

$$\begin{aligned} K(r + t\mathbf{n}, R) &= \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) 1_{\mathcal{X}(\mathcal{F}^+_{r,\delta R^\varepsilon})}(r + t\mathbf{n} + x^0) \mathcal{W}(1_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})} g_z) \\ &\times (r + t\mathbf{n} + x^0) + \int_{\mathcal{X}^0} \mathrm{d}P^0(dx^0) 1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})}(r + t\mathbf{n} + x^0) \\ &\times \mathcal{W}(1_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})} g_z)(r + t\mathbf{n} + x^0) \\ &= K^A(r + t\mathbf{n}, R) + K^B(r + t\mathbf{n}, R). \end{aligned} \tag{5.67}$$

Respectively

$$\begin{aligned} \mathcal{K}(R) &= \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t (1 - t k_R(r)) K^A(r + t\mathbf{n}, R) \\ &+ \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t (1 - t k_R(r)) K^B(r + t\mathbf{n}, R) \\ &= \mathcal{K}^A(R) + \mathcal{K}^B(R). \end{aligned} \tag{5.68}$$

B. Analysis of $\mathcal{K}^A(R)$. The second contribution to the constant term.

In this subsection we will separate from $\mathcal{K}^A(R)$ the second contribution $\mathcal{J}_2^c$ to the constant term (see Eq. (5.112) below). Note that integration in $K^A(r+t\mathbf{n}, R)$ is over the configurations

$(r+t\mathbf{n}+x^0,\omega)$ where the loop $r+t\mathbf{n}+x^0$ stays in Λ_R and at least one loop from the configuration ω leaves Λ_R. To get rid of the dependence on Λ_R we decompose $\mathcal{K}^A(R)$ as

$$\begin{aligned}
K^A(r+t\mathbf{n},R) &= \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) 1_{\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(r+t\mathbf{n}+x^0)\mathcal{W}_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})} \\
&\times g_z(r+t\mathbf{n}+x^0) - \int_{\mathcal{X}^0} \mathrm{d}P^0(x^0) 1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})} \\
&\times (r+t\mathbf{n}+x^0)\mathcal{W}_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})} g_z(r+t\mathbf{n}+x^0) \\
&= K^{A_1}(r+t\mathbf{n},R) - K^{A_2}(r+t\mathbf{n},R). \qquad (5.69)
\end{aligned}$$

Now the integration in $K^{A_2}(r+t\mathbf{n},R)$ is over the configurations $\{r+t\mathbf{n}+x^0,\omega\}$ that have two or more loops visiting the domain $\Pi^+_{r,\delta R^\varepsilon} \setminus \mathcal{F}^+_{r,\delta R^\varepsilon}$. The contribution of such configurations is small (due to the bound (5.48)).

By similar arguments we approximate $K^{A_1}(r+t\mathbf{n},R)$ by the integral over the configurations $\{r+t\mathbf{n}+x^0,\omega\}$ where the loop $r+t\mathbf{n}+x^0$ stays in $\Pi^+_{r,\delta R^\varepsilon}$ and exactly one loop from ω visits the domain $\Pi^+_{r,\delta R^\varepsilon} \setminus \mathcal{F}^+_{r,\delta R^\varepsilon}$. Then we approximate the last integral by the integral where this loop crosses the parabola tangent to $\partial\Lambda_R$, but not the tangent. Thus we get the quantity $\mathcal{J}_2^c$.

The decomposition (5.69) implies

$$\mathcal{K}^A(R) = \mathcal{K}^{A_1}(R) - \mathcal{K}^{A_2}(R) \qquad (5.70)$$

where

$$\mathcal{K}^{A_1}(R) = \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t(1-tk_R(r))K^{A_1}(r+t\mathbf{n},R) \qquad (5.71)$$

and

$$\mathcal{K}^{A_2}(R) = \int_{\partial\Lambda_R} \mathrm{d}\sigma(r) \int_0^{\delta R^\varepsilon} \mathrm{d}t(1-tk_R(r))K^{A_2}(r+t\mathbf{n},R) \qquad (5.72)$$

Denote

$$S(r+t\mathbf{n}+x^0) = \mathcal{W}_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})}(g_z)(r+t\mathbf{n}+x^0).$$

With the help of the equality

$$\int_{\mathcal{M}^c(\mathcal{X}(\Lambda))} h(\omega)\mathrm{d}W_\rho(\omega) = \int_{\mathcal{M}_+(\mathcal{X}^c(\Lambda))} \mathrm{d}W_\rho(\omega)(-1)^{|\omega|+1}\mathcal{W}h(\omega) \qquad (5.73)$$

which holds true for any absolutely integrable h, we have

$$S(r+t\mathbf{n}+x^0)=\int_{\mathcal{M}_+(\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}))\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})}\mathrm{d}W_\rho(\omega)(-1)^{|\omega|+1}\mathcal{W}_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})}(g_z)$$
$$\cdot(r+t\mathbf{n}+x^0,\omega)=\int_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}\mathrm{d}\rho(y)\mathcal{W}_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})}(g_z)$$
$$\cdot(r+t\mathbf{n}+x^0,y)+\sum_{m=2}^{\infty}\frac{(-1)^m}{m!}\int_{[\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})]^m}\prod_{i=1}^{m}\mathrm{d}\rho(y_i)$$
$$\cdot\mathcal{W}_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})}(g_z)(r+t\mathbf{n}+x^0,y_1,\cdots,y_m)$$
$$=S_1(r+t\mathbf{n}+x^0)+\hat{S}_2(r+t\mathbf{n}+x^0). \tag{5.74}$$

According to this we split K^{A_1} into two parts

$$K^{A_1}(r+t\mathbf{n},R)=\int_{\mathcal{X}^0}\mathrm{d}P^0(x^0)1_{\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(r+t\mathbf{n}+x^0)S_1(r+t\mathbf{n}+x^0)$$
$$+\int_{\mathcal{X}^0}\mathrm{d}P^0(x^0)1_{\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(r+t\mathbf{n}+x^0)\hat{S}_2(r+t\mathbf{n}+x^0)$$
$$=K_1^{A_1}(r+t\mathbf{n},R)+\hat{K}^{A_1}(r+t\mathbf{n},R). \tag{5.75}$$

We estimate the series $\hat{S}_2(r+t\mathbf{n}+x^0)$ with the help of the following lemma (see Appendix A.1).

Lemma 5.3 . *Let $\bar{u}=\beta\max\{M,(2\pi\beta)^{-1}||\phi||_1\}$ then for all z from the interval*

$$0<z<(\bar{u}e^{\beta B+1})^{-1} \tag{5.76}$$

and all $\omega\in\mathcal{M}$ the following inequality holds true:

$$\mathcal{W}(|\tilde{g}_z|)(\omega)\le\frac{(|\omega|-1)!}{e\bar{u}}\left[\frac{ze^{\beta B+1}\bar{u}}{1-ze^{\beta B+1}\bar{u}}\right]^{|\omega|}. \tag{5.77}$$

Due to the inequality (5.48)

$$\rho(\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon}))\le CR^{-\frac{5}{8}} \tag{5.78}$$

where $C=C(\Lambda,\beta)$ is a positive constant. Using this bound and Lemma 5.3 we find that for all z from the interval (5.76),

$$|\hat{S}_2(r+t\mathbf{n}+x^0)|\le\frac{1}{e\bar{u}}\sum_{m=2}^{\infty}\left(\frac{z\bar{u}e^{\beta B+1}}{1-z\bar{u}e^{\beta B+1})}\right)^{m+1}\left[\rho(\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon}))\right]^m$$
$$\le CR^{-\frac{5}{4}} \tag{5.79}$$

with $C=C(\Lambda,\beta,z)$. Hence

$$|\hat{\mathcal{K}}^{A_1}(R)|\le 2\int_{\partial\Lambda_R}\sigma(dr)\int_0^{\delta R^\varepsilon}dt\mid\tilde{K}^{A_1}(r+t\mathbf{n},R)\mid\le|\partial\Lambda|CR^{-\frac{1}{8}}. \tag{5.80}$$

Now passing to $\mathcal{K}_1^{A_1}(R)$ from (5.75) we have

$$\mathcal{K}_1^{A_1}(R) = \int_{\partial\Lambda_R} \sigma(dr) \int_0^{\delta R^\varepsilon} dt \int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(r + t\mathbf{n} + x^0)$$
$$\times S_1(r + t\mathbf{n} + x^0) - \int_{\partial\Lambda_R} \sigma(dr) \int_0^{\delta R^\varepsilon} dttk_R(r) \int_{\mathcal{X}^0} P^0(dx^0)$$
$$\times 1_{\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(r + t\mathbf{n} + x^0) S_1(r + t\mathbf{n} + x^0). \tag{5.81}$$

From the bound (5.48) and Lemma 5.3 it follows that the second summand on the right-hand side of Eq. (5.81) is $o(1)$. Therefore, combining Eqs. (5.71), (5.80) and (5.81), we have

$$\mathcal{K}^{A_1}(R) = \int_{\partial\Lambda_R} \sigma(dr) \int_0^{\delta R^\varepsilon} dt K_2^{A_1}(r + t\mathbf{n}, R) + o(1) = \mathcal{K}_2^{A_1}(R) + o(1) \tag{5.82}$$

where

$$K_2^{A_1}(r + t\mathbf{n}, R) = \int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(r + t\mathbf{n} + x^0)$$
$$\times \int_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})} \rho(dy) \mathcal{W}_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})}(\tilde{g}_z)(r + t\mathbf{n} + x^0, y). \tag{5.83}$$

We can not use directly Corollary 3.16 to get rid of the restrictions to the cylinder $\Pi^+_{r,\delta R^\varepsilon}$ in Eq. (5.83). First observe that with the help of the identity (5.15) we have

$$\int_{\mathcal{X}} \rho(dy) \int_{\mathcal{M}^c(B_{\delta R^\varepsilon}(r+t\mathbf{n}))} W_\rho(d\omega) |\tilde{g}_z(r + t\mathbf{n} + x^0, y, \omega)|$$
$$= \int_{\mathcal{M}} W_\rho(d\omega) \left(\sum_{y\in\omega} 1_{\mathcal{M}^c(B_{\delta R^\varepsilon}(r+t\mathbf{n}))}(\omega \setminus \{y\}) \right) |\tilde{g}_z(r + t\mathbf{n} + x^0, \omega)|$$
$$\leq \int_{\mathcal{M}^c(B_{\delta R^\varepsilon}(r+t\mathbf{n}))} W_\rho(d\omega) |g_z(r + t\mathbf{n} + x^0, \omega)|. \tag{5.84}$$

Then applying Corollary 3.16 we get

$$K_2^{A_1}(r + t\mathbf{n}, R) = K_3^{A_1}(r + t\mathbf{n}, R) + o(R^{-2}) \tag{5.85}$$

with

$$K_3^{A_1}(r + t\mathbf{n}, R) = \int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}(\Pi^+_r)}(r + t\mathbf{n} + x^0) \int_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}$$
$$\times \rho(dy) \mathcal{W}_{\mathcal{M}(\Pi^+_r)}(\tilde{g}_z)(r + t\mathbf{n} + x^0, y). \tag{5.86}$$

Since $\phi \in C^1$, writing $y = (\xi, \eta) + y^0$ in the local coordinates, we have

$$\mathcal{W}_{\mathcal{M}(\Pi^+_r)}(\tilde{g}_z)(r + t\mathbf{n} + x^0, y) = \mathcal{W}_{\mathcal{M}(\Pi^+_r)}(\tilde{g}_z)(r + t\mathbf{n} + x^0, (\xi, \eta_0) + y^0)$$
$$+ \frac{\partial}{\partial\eta} \mathcal{W}_{\mathcal{M}(\Pi^+_r)}(\tilde{g}_z)(r + t\mathbf{n} + x^0, (\xi, \bar{\eta}_0) + y^0)$$
$$\times (\eta - \eta_0) \tag{5.87}$$

where $\eta_0 = -\inf\langle y^0, \mathbf{n}\rangle = -\inf_t\langle y^0(t), \mathbf{n}\rangle$ and $\bar\eta_0 = \eta_0 + \theta(\eta - \eta_0)$, $0 < \theta < 1$. According to this we decompose $K_3^{A_1}(r + t\mathbf{n}, R)$ as

$$K_3^{A_1}(r + t\mathbf{n}, R) = K_4^{A_1}(r + t\mathbf{n}, R) + K_5^{A_1}(r + t\mathbf{n}, R) \tag{5.88}$$

where

$$\begin{aligned} K_4^{A_1}(r + t\mathbf{n}, R) = &\int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}(\Pi_r^+)}(r + t\mathbf{n} + x^0) \int_{|\xi|\le\delta R^\varepsilon} d\xi \\ &\times \int_0^\infty d\eta \int_{\mathcal{X}^0} P^0(dy^0) 1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}((\xi,\eta) + y^0) \\ &\times \mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde g_z)(r + t\mathbf{n} + x^0, (\xi, \eta_0) + y^0). \end{aligned} \tag{5.89}$$

and

$$\begin{aligned} K_5^{A_1}(r + t\mathbf{n}, R) = &\int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}(\Pi_r^+)}(r + t\mathbf{n} + x^0) \int_{|\xi|\le\delta R^\varepsilon} d\xi \int_0^\infty d\eta(\eta - \eta_0) \\ &\times \int_{\mathcal{X}^0} P^0(dy^0) 1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}((\xi,\eta) + y^0) \\ &\times \frac{\partial}{\partial\eta}\mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde g_z)(r + t\mathbf{n} + x^0, (\xi, \bar\eta_0) + y^0). \end{aligned} \tag{5.90}$$

To estimate $K_5^{A_1}$ we use the following lemma (see the proof in Appendix A.2).

Lemma 5.4 *If $\phi \in C^1$ satisfies conditions (5.54) - (5.56), then for all z from the interval $0 < z < 2\pi[e^{\beta B+1}\max(||\phi||_1, ||\nabla\phi||_1)]^{-1}$, the derivative of the two-point truncated correlation function satisfies the following bound*

$$|\frac{\partial}{\partial\eta}\mathcal{W}(g_z)(x^0 + (\xi,\eta), y)| \le D, \;\; (\xi,\eta) \in \mathbb{R}^2 \tag{5.91}$$

where $D = D(\phi, \beta, z)$ does not depend on $x^0 + (\xi, \eta), y$.

Using Fubini's theorem we have

$$\begin{aligned} K_5^{A_1}(r + t\mathbf{n}, R) = &\int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}(\Pi_r^+)}(r + t\mathbf{n} + x^0) \int_{|\xi|\le\delta R^\varepsilon} d\xi \int_{\mathcal{X}^0} P^0(dy^0) \\ &\times 1_{|\widetilde{y^0}+\xi|\le\delta R^\varepsilon}(y^0) \frac{\partial}{\partial\eta}\mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde g_z)(r + t\mathbf{n} + x^0, (\xi, \bar\eta_0) + y^0) \\ &\times \int_{\eta_0}^{\sup[f_{r,R}(\xi+\widetilde{y^0}) - \langle y^0, \mathbf{n}\rangle]} d\eta(\eta - \eta_0). \end{aligned} \tag{5.92}$$

We recall that $\widetilde{y^0} = \widetilde{y^0(t)}$ denotes the projection of $y^0(t)$ onto the tangent to $\partial\Lambda_R : \widetilde{y^0(t)} = y^0(t) - \langle y^0(t), \mathbf{n}\rangle$. Due to the bound (5.47),

$$\{\sup[f_{r,R}(\xi + \widetilde{y^0}) - \langle y^0, \mathbf{n}\rangle] - \eta_0\}^2 \le C(\Lambda, \beta) R^{-\frac{3}{2}}. \tag{5.93}$$

Therefore applying Lemma 5.4 we get

$$|K_5^{A_1}(r+t\mathbf{n},R)| \le DC(\Lambda,\beta)R^{-\frac{11}{8}}. \tag{5.94}$$

Combining Eqs. (5.82), (5.85), (5.88) and (5.94) we find

$$\mathcal{K}^{A_1}(R) = \int_{\partial\Lambda_R} \sigma(dr) \int_0^{\delta R^\varepsilon} dt K_4^{A_1}(r+t\mathbf{n},R) + o(1) = \mathcal{K}_4^{A_1}(R) + o(1) \tag{5.95}$$

where $K_4^{A_1}(r+t\mathbf{n},R)$ is given by Eq. (5.89). Observe that

$$\mathcal{K}_4^{A_1}(R) = \int_{\partial\Lambda_R} \sigma(dr) \int_0^{\infty} dt K_4^{A_1}(r+t\mathbf{n},R) + o(1). \tag{5.96}$$

Indeed, by Corollary 3.16,

$$\begin{aligned} \mid \int_{\delta R^\varepsilon}^{\infty} dt K_4^{A_1}(r+t\mathbf{n},R) \mid \le \int_{\delta R^\varepsilon}^{\infty} dt \int_{\mathcal{X}^0} P^0(dx^0) \int_{\mathcal{X}^c(B_t(r+t\mathbf{n}))} \mathcal{W}_{\mathcal{M}(\Pi_r^+)} |\tilde{g}_z|(r+t\mathbf{n} \\ + x^0, (\xi,\eta_0)+y^0) \le CR^{-\frac{15}{8}} \end{aligned} \tag{5.97}$$

which implies Eq. (5.96).

The integral in $\mathcal{K}_4^{A_1}(R)$ over the loops $(\xi,\eta)+y^0 \in \mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})$ we approximate by the integral over the loops which cross the parabola, tangent to $\partial\Lambda_R$, without crossing the tangent line. Arguing as we did above to estimate $K_5^{A_1}(r+t\mathbf{n},R)$, we can write that for all $t>0$,

$$\begin{aligned} K_4^{A_1}(r+t\mathbf{n},R) = &\int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}(\Pi_r^+)}(r+t\mathbf{n}+x^0) \int_{|\xi|\le\delta R^\varepsilon} d\xi \int_{\mathcal{X}^0} P^0(dy^0) \\ &\times 1_{\sup|\xi+\widetilde{y^0}|\le\delta R^\varepsilon}(\xi,y^0) \mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r+t\mathbf{n}+x^0,(\xi,\eta_0)+y^0) \\ &\times \{\sup[f_{r,R}(\xi+\widetilde{y^0}) - \langle y^0,\mathbf{n}\rangle] - \eta_0\}. \end{aligned} \tag{5.98}$$

Approximating $\{\sup[f_{r,R}(\xi+\widetilde{y^0}) - \langle y^0,\mathbf{n}\rangle] - \eta_0\}$ by $\frac{1}{2}k_R(r)(\xi+\widetilde{y^0(\tau(\mathbf{n}))})^2$ we get

$$K_4^{A_1}(r+t\mathbf{n},R) = K_6^{A_1}(r+t\mathbf{n},R) + K_7^{A_1}(r+t\mathbf{n},R) \tag{5.99}$$

where

$$\begin{aligned} K_6^{A_1}(r+t\mathbf{n},R) = &\int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}(\Pi_r^+)}(r+t\mathbf{n}+x^0) \int_{|\xi|\le\delta R^\varepsilon} d\xi \int_{\mathcal{X}^0} P^0(dy^0) \\ &\times 1_{\sup|\xi+\widetilde{y^0}|\le\delta R^\varepsilon}(\xi,y^0) \mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r+t\mathbf{n}+x^0,(\xi,\eta_0)+y^0) \\ &\times \frac{1}{2}k_R(r)(\xi+\widetilde{y^0(\tau(\mathbf{n}))})^2 \end{aligned} \tag{5.100}$$

and the correction

$$\begin{aligned} K_7^{A_1}(r+t\mathbf{n},R) = &\int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}(\Pi_r^+)}(r+t\mathbf{n}+x^0) \int_{|\xi|\le\delta R^\varepsilon} d\xi \int_{\mathcal{X}^0} P^0(dy^0) \\ &\times 1_{\sup|\xi+\widetilde{y^0}|\le\delta R^\varepsilon}(\xi,y^0) \mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r+t\mathbf{n}+x^0,(\xi,\eta_0)+y^0) \\ &\times \{\sup[f_{r,R}(\xi+\widetilde{y^0}) - \langle y^0,\mathbf{n}\rangle] - \eta_0 - \frac{1}{2}k_R(r)(\xi+\widetilde{y^0(\tau(\mathbf{n}))})^2\}. \end{aligned} \tag{5.101}$$

Here $\tau(\mathbf{n})$ satisfies the condition $\inf\langle y^0, \mathbf{n}\rangle = \langle y^0(\tau(\mathbf{n})), \mathbf{n}\rangle$. As in Section 4.1.1 we also choose $\tau_R(\xi)$ so that

$$\sup[f_{r,R}(\xi + \widetilde{y^0}) - \langle y^0, \mathbf{n}\rangle] = f_{r,R}(\xi + \widetilde{y^0(\tau_R(\xi))}) - \langle y^0(\tau_R(\xi)), \mathbf{n}\rangle. \tag{5.102}$$

We want to show that

$$\int_{\partial\Lambda_R} \sigma(dr) \int_0^\infty dt K_7^{A_1}(r + t\mathbf{n}, R) = o(1). \tag{5.103}$$

Since $\partial\Lambda \in C^3$, we have that

$$|f_{r,R}(\xi)| \leq C|\xi|^2 R^{-1} \text{ and } |f_{r,R}(\xi) - \frac{1}{2}k_R(r)\xi^2| \leq C|\xi|^3 R^{-2} \tag{5.104}$$

for ξ, $|\xi| \leq \delta R^\varepsilon$. By the choice of $\tau_R(\xi)$ and $\tau(\mathbf{n})$, obviously

$$\begin{aligned} 0 \leq & \langle y^0(\tau_R(\xi)), \mathbf{n}\rangle - \langle y^0(\tau(\mathbf{n})), \mathbf{n}\rangle \\ & \leq f_{r,R}(\xi + \widetilde{y^0(\tau_R(\xi))}) - f_{r,R}(\xi + \widetilde{y^0(\tau(\mathbf{n}))}) \leq CR^{-\frac{3}{4}}, \end{aligned}$$

for all y^0, ξ with $\sup|\xi + \widetilde{y^0}| \leq \delta R^\varepsilon$. Now $\tau(\mathbf{n})$ is P^0-almost surely unique (see Appendix A.3), hence $\tau_R(\xi) \to \tau(\mathbf{n})$ as $R \to \infty$. Using Eq. (5.104) and the definitions of $\tau(\mathbf{n})$ and $\tau_R(\xi)$, we have

$$\begin{aligned} & \sup[f_{r,R}(\xi + \widetilde{y^0}) - \langle y^0, \mathbf{n}\rangle] + \inf\langle y^0, \mathbf{n}\rangle - \frac{1}{2}k_R(r)(\xi + \widetilde{y^0(\tau(\mathbf{n}))})^2 \\ & \leq f_{r,R}(\xi + \widetilde{y^0(\tau_R(\xi))}) - \frac{1}{2}k_R(r)(\xi + \widetilde{y^0(\tau(\mathbf{n}))})^2 \\ & \leq CR^{-2}|\xi + \widetilde{y^0(\tau_R(\xi))}|^3 + \frac{1}{2}k_R(r)\left[(\xi + \widetilde{y^0(\tau_R(\xi))})^2 - (\xi + \widetilde{y^0(\tau(\mathbf{n}))})^2\right] \\ & \leq CR^{-\frac{13}{8}} + \frac{1}{2}R^{-1}\bar{k}_\Lambda \mid (\xi + \widetilde{y^0(\tau_R(\xi))})^2 - (\xi + \widetilde{y^0(\tau(\mathbf{n}))})^2 \mid . \end{aligned}$$

On the other hand

$$\begin{aligned} & \sup[f_{r,R}(\xi + \widetilde{y^0}) - \langle y^0, \mathbf{n}\rangle] + \inf\langle y^0, \mathbf{n}\rangle - \frac{1}{2}k_R(r)(\xi + \widetilde{y^0(\tau(\mathbf{n}))})^2 \\ & \geq f_{r,R}(\xi + \widetilde{y^0(\tau(\mathbf{n}))}) - \frac{1}{2}k_R(r)(\xi + \widetilde{y^0(\tau(\mathbf{n}))})^2 \\ & \geq -CR^{-2}|\xi + \widetilde{y^0(\tau(\mathbf{n}))}|^3 \geq -CR^{-\frac{13}{8}}. \end{aligned}$$

Thus

$$\begin{aligned} & \mid \sup[f_{r,R}(\xi + \widetilde{y^0}) - \langle y^0, \mathbf{n}\rangle] + \inf\langle y^0, \mathbf{n}\rangle - \frac{1}{2}k_R(r)(\xi + \widetilde{y^0(\tau(\mathbf{n}))})^2 \mid \\ & \leq CR^{-\frac{13}{8}} + \frac{1}{2}R^{-1}\bar{k}_\Lambda \mid (\xi + \widetilde{y^0(\tau_R(\xi))})^2 - (\xi + \widetilde{y^0(\tau(\mathbf{n}))})^2 \mid \end{aligned} \tag{5.105}$$

Substituting this into Eq. (5.101) we get

$$\begin{aligned}|K_7^{A_1}(r+t\mathbf{n},R)| \leq & CR^{-\frac{13}{8}} \int_{\mathcal{X}^0} P^0(dx^0) \int_{-\delta R^\varepsilon}^{\delta R^\varepsilon} d\xi \int_{\mathcal{X}^0} P^0(dy^0) 1_{\sup|\xi+\widetilde{y^0}|\leq \delta R^\varepsilon}(\xi, y^0) \\ & \times \mathcal{W}_{\mathcal{M}(\Pi_r^+)} |\tilde{g}_z|(r+t\mathbf{n}+x^0,(\xi,\eta_0)+y^0) + R^{-1}\bar{k}_\Lambda \\ & \times \int_{\mathcal{X}^0} P^0(dx^0) \int_{-\delta R^\varepsilon}^{\delta R^\varepsilon} d\xi \int_{\mathcal{X}^0} P^0(dy^0) 1_{\sup|\xi+\widetilde{y^0}|\leq \delta R^\varepsilon}(\xi, y^0) \\ & \times \mathcal{W}_{\mathcal{M}(\Pi_r^+)} |\tilde{g}_z|(r+t\mathbf{n}+x^0,(\xi,\eta_0)+y^0) \; \big| \; (\xi + \widetilde{y^0(\tau_R(\xi))})^2 \\ & - (\xi + \widetilde{y^0(\tau(\mathbf{n}))})^2 \; \big| = \hat{K}_7^{A_1}(r+t\mathbf{n},R) + \tilde{K}_7^{A_1}(r+t\mathbf{n},R) \end{aligned} \tag{5.106}$$

Let us fix any $t > 0$. According to Corollary 3.16

$$\begin{aligned}\hat{K}_7^{A_1}(r+t\mathbf{n},R) \leq & CR^{-\frac{13}{8}} \int_{\mathcal{X}^0} P^0(dx^0) \int_{\mathcal{X}^c(B_t(r+t\mathbf{n}))} \rho(dy) \\ & \times \mathcal{W}_{\mathcal{M}} |\tilde{g}_z|(r+t\mathbf{n}+x^0,y) \leq CR^{-\frac{13}{8}}(1+t)^{-l}. \end{aligned} \tag{5.107}$$

Since $\tau_R(\xi) \to \tau(\mathbf{n})$ as $R \to \infty$, by Lebesgue dominated convergence theorem

$$R^{-1}\bar{k}_\Lambda \int_{\partial\Lambda_R} \sigma(dr) \int_0^\infty dt \tilde{K}_7^{A_1}(r+t\mathbf{n},R) \to 0. \tag{5.108}$$

To apply the Lebesgue theorem, we observe that due to Corollary 3.16

$$\begin{aligned}& \int_0^\infty dt \int_{\mathcal{X}^0} P^0(dx^0) \int_{-\infty}^\infty d\xi |\xi|^s \int_{\mathcal{X}^0} P_j^0(dy^0) \mathcal{W}_{\mathcal{M}} |\tilde{g}_z|(r+t\mathbf{n}+x^0,(\xi,\eta_0)+y^0) \\ & \leq \int_0^\infty dt \sum_{k=0}^\infty (k+1)^s \int_{\mathcal{X}^0} P^0(dx^0) \int_{-\infty}^\infty d\xi 1_{k\leq|\xi|<k+1} \int_{\mathcal{X}^0} P_j^0(dy^0) \mathcal{W}_{\mathcal{M}} |\tilde{g}_z|(r+t\mathbf{n} \\ & + x^0,(\xi,\eta_0)+y^0) \leq \int_0^\infty dt \sum_{k=0}^\infty (k+1)^s \int_{\mathcal{X}^0} P^0(dx^0) \int_{\mathcal{X}^c(B_{\max(k,t)}(r+t\mathbf{n}))} \rho_j(dy) \\ & \times \mathcal{W}_{\mathcal{M}} |\tilde{g}_z|(r+t\mathbf{n}+x^0,y) \leq C(\beta,z,l), \quad j,s = 0,1,2. \end{aligned} \tag{5.109}$$

From Eqs. (5.107), (5.108) we get Eq. (5.103) which together with Eqs. (5.95) and (5.99) implies

$$\mathcal{K}^{A_1}(R) = \mathcal{K}_6^{A_1}(R) + o(1) = \int_{\partial\Lambda_R} \sigma(dr) \int_0^\infty dt K_6^{A_1}(r+t\mathbf{n},R) + o(1) \tag{5.110}$$

where $K_6^{A_1}(r+t\mathbf{n},R)$ is given by Eq. (5.100). Thus applying once more Corollary 3.16 to $\mathcal{K}_6^{A_1}(R)$ we get

$$\mathcal{K}^{A_1}(R) = \mathcal{J}_2^c + o(1) \tag{5.111}$$

with the constant term

$$\begin{aligned}\mathcal{J}_2^c = & \frac{1}{2} \int_{\partial\Lambda} \sigma(dr) k(r) \int_0^\infty dt \int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}(\Pi_r^+)}(r+t\mathbf{n}+x^0) \int_{-\infty}^\infty d\xi \\ & \times \int_{\mathcal{X}^0} P^0(dy^0) \mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r+t\mathbf{n}+x^0,(\xi,\eta_0)+y^0)(\xi + \widetilde{y^0(\tau(\mathbf{n}))})^2. \end{aligned} \tag{5.112}$$

Note that due to Eq. (5.109) the integral on the right-hand side of Eq. (5.112) is absolutely convergent in the interval

$$0 < z < (2^l \beta e^{\beta B+1} \lambda ||\phi_l||_1)^{-1}. \tag{5.113}$$

Lemma 5.3 and formulas (5.74) and (5.78) imply that for R large enough

$$\mid \mathcal{W}_{\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})}(\tilde{g}_z)(r + t\mathbf{n} + x^0) \mid \leq C(\Lambda, \beta, z) R^{-\frac{5}{8}} \tag{5.114}$$

On the other hand, using the bound (5.93) with $\xi = 0$, we have

$$\begin{aligned}
&\mid \int_0^{\delta R^\varepsilon} dt(1 - tk_R(r)) \int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(r + t\mathbf{n} + x^0) \mid \\
&\leq 2 \int_{\mathcal{X}^0} P^0(dx^0) 1_{\sup|\widetilde{x}^0| < \delta R^\varepsilon}(x^0) \{\sup[f_{r,R}(\widetilde{x}^0) - \langle x^0, \mathbf{n}\rangle] + \inf\langle x^0, \mathbf{n}\rangle\} \\
&\leq C(\Lambda, \beta) R^{-\frac{3}{4}}
\end{aligned} \tag{5.115}$$

Hence $|\mathcal{K}^{A_2}(R)| \leq C(\Lambda, \beta, z) R^{-\frac{3}{8}}$. Thus we conclude from Eqs. (5.75) and (5.111) that

$$\mathcal{K}^A(R) = \mathcal{J}^c_2(R) + o(1). \tag{5.116}$$

C. Analysis of $\mathcal{K}^B(R)$. The third contribution to the constant term.

The treatment of $\mathcal{K}^B(R)$ in many aspects is similar to that of $\mathcal{K}^A(R)$. Approximating the integral in $K^B(r+t\mathbf{n}, R)$ (see Eqs. (5.67) and (5.117)) over the configurations $(r+t\mathbf{n}+x^0, \omega)$ where the loop $r + t\mathbf{n} + x^0$ leaves Λ_R by the integral over $(r + t\mathbf{n} + x^0, \omega)$ where the loop $r + t\mathbf{n} + x^0$ crosses the parabola tangent to $\partial\Lambda_R$ but not the tangent line, we get the contribution $\mathcal{J}^c_3$. Thus the constant term is the sum $\mathcal{J}^c_1 + \mathcal{J}^c_2 + \mathcal{J}^c_3$. If the potential is rotation invariant each term of this sum is factorized into a potential dependent factor times the integral of the curvature along the boundary $\partial\Lambda_R$, which by the Gauss-Bonnet theorem is the Euler-Poincare characteristic of Λ multiplied by 2π.

From Eqs. (5.67), (5.68) and Corollary 3.16 we get

$$\mathcal{K}^B(R) = \int_{\partial\Lambda_R} \sigma(dr) \int_0^{\delta R^\varepsilon} dt(1 - tk_R(r)) K^B_1(r + t\mathbf{n}, R) + o(1) \tag{5.117}$$

where

$$\begin{aligned}
K^B_1(r + t\mathbf{n}, R) = &\int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(r + t\mathbf{n} + x^0) \\
&\times \mathcal{W}_{\mathcal{M}(\Pi^+_r)}(\tilde{g}_z)(r + t\mathbf{n} + x^0).
\end{aligned} \tag{5.118}$$

Similarly to Eq. (5.87) we have

$$\begin{aligned}
\mathcal{W}_{\mathcal{M}(\Pi^+_r)}(\tilde{g}_z)(r + t\mathbf{n} + x^0) = &\mathcal{W}_{\mathcal{M}(\Pi^+_r)}(\tilde{g}_z)(r + t_0\mathbf{n} + x^0) \\
&+ \frac{\partial}{\partial t}\mathcal{W}_{\mathcal{M}(\Pi^+_r)}(\tilde{g}_z)(r + \bar{t_0}\mathbf{n} + x^0)(t - t_0)
\end{aligned} \tag{5.119}$$

where $t_0 = -\inf\langle x^0, \mathbf{n}\rangle$, $\bar{t_0} = t_0 + \theta(t - t_0)$. Hence

$$K_1^B(r + t\mathbf{n}, R) = K_2^B(r + t\mathbf{n}, R) + K_3^B(r + t\mathbf{n}, R) \tag{5.120}$$

where

$$\begin{aligned} K_2^B(r + t\mathbf{n}, R) = &\int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(r + t\mathbf{n} + x^0) \\ &\times \mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r + t_0\mathbf{n} + x^0) \end{aligned} \tag{5.121}$$

and

$$\begin{aligned} K_3^B(r + t\mathbf{n}, R) = &\int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})}(r + t\mathbf{n} + x^0) \\ &\times \frac{\partial}{\partial t}\mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r + \bar{t_0}\mathbf{n} + x^0)(t - t_0). \end{aligned} \tag{5.122}$$

Using Lemma 5.4, the bound (5.93) with $\xi = 0$ and invoking arguments which were used to derive Eq. (5.94), we can write

$$\begin{aligned} &\left| \int_0^{\delta R^\varepsilon} dt(1 - tk_R(r))K_3^B(r + t\mathbf{n}, R) \right| \leq 2\int_0^\infty dt \int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})} \\ &\times (r + t\mathbf{n} + x^0) \left| \frac{\partial}{\partial t}\mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r + \bar{t_0}\mathbf{n} + x^0) \right| \left| (t - t_0) \right| \leq 2D \int_{\mathcal{X}^0} P^0(dx^0) \\ &\times \{\sup[f_{r,R}(\widetilde{x}^0) - \langle x^0, \mathbf{n}\rangle] + \inf\langle x^0, \mathbf{n}\rangle\}^2 \leq C(\Lambda, \phi, \beta, z)R^{-\frac{3}{2}}. \end{aligned} \tag{5.123}$$

Hence

$$\mathcal{K}_3^B(R) = \int_{\partial\Lambda_R} \sigma(dr) \int_0^{\delta R^\varepsilon} dt(1 - tk_R(r))K_3^B(r + t\mathbf{n}, R) = o(1) \tag{5.124}$$

It remains to separate the last contribution to the constant term coming from $\mathcal{K}_2^B(R)$. We split $\mathcal{K}_2^B(R)$ in two parts

$$\begin{aligned} \mathcal{K}_2^B(R) = &\int_{\partial\Lambda_R} \sigma(dr) \int_0^{\delta R^\varepsilon} dt K_2^B(r + t\mathbf{n}, R) \\ &- \int_{\partial\Lambda_R} \sigma(dr)k_R(r) \int_0^{\delta R^\varepsilon} dt t K_2^B(r + t\mathbf{n}, R) \end{aligned} \tag{5.125}$$

With the help of Lemma 5.3 and the bound (5.93) one can easily show that the second summand on the right-hand side of Eq. (5.125) is $o(1)$. By Corollary 3.16 we can replace the integral $\int_0^{\delta R^\varepsilon} dt$ by $\int_0^\infty dt$ in the first summand of Eq. (5.125) with correction $o(1)$. On the other hand, by Fubini's theorem

$$\begin{aligned} \int_0^\infty dt K_2^B(r + t\mathbf{n}, R) = &\int_{\mathcal{X}^0} P^0(dx^0)\mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r + t_0\mathbf{n} + x^0) \\ &\times \{\sup[f_{r,R}(\widetilde{x}^0) - \langle x^0, \mathbf{n}\rangle] + \inf\langle x^0, \mathbf{n}\rangle\}. \end{aligned} \tag{5.126}$$

Therefore

$$\mathcal{K}_2^B(R) = \int_{\partial\Lambda_R} \sigma(dr) \int_0^\infty dt K_2^B(r + t\mathbf{n}, R) + o(1). \tag{5.127}$$

Now we approximate $\{\sup[f_{r,R}(\widetilde{x}^0) - \langle x^0, \mathbf{n}\rangle] + \inf\langle x^0, \mathbf{n}\rangle\}$ by $\frac{1}{2}k_R(r)$ $\times (\widetilde{x^0(\tau(\mathbf{n}))})^2$ and estimate the correction. We proceed in a similar way as we did in the proof of Eqs. (5.103) and (5.110). It follows from Eq. (5.127) that

$$\begin{aligned}\int_0^\infty dt K_2^B(r + t\mathbf{n}, R) =& \frac{1}{2}k_R(r) \int_{\mathcal{X}^0} P^0(dx^0) \mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r + t_0\mathbf{n} + x^0) \\ & \times (\widetilde{x^0(\tau(\mathbf{n}))})^2 + \int_{\mathcal{X}^0} P^0(dx^0) \mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r + t_0\mathbf{n} \\ & + x^0)\{\sup[f_{r,R}(\widetilde{x}^0) - \langle x^0, \mathbf{n}\rangle] + \inf\langle x^0, \mathbf{n}\rangle \\ & - \frac{1}{2}k_R(r)(\widetilde{x^0(\tau(\mathbf{n}))})^2\}\end{aligned} \tag{5.128}$$

where $\tau(\mathbf{n})$ is defined by $\inf\langle x^0, \mathbf{n}\rangle = \langle x^0(\tau(\mathbf{n})), \mathbf{n}\rangle$. Then applying the inequality (5.105) with $\xi = 0$ we get

$$\begin{aligned}&\int_{\partial\Lambda_R} \sigma(dr) \int_{\mathcal{X}^0} P^0(dx^0) \mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r + t_0\mathbf{n} + x^0)\{\sup[f_{r,R}(\widetilde{x}^0) - \langle x^0, \mathbf{n}\rangle] \\ &+ \inf\langle x^0, \mathbf{n}\rangle - \frac{1}{2}k_R(r)(\widetilde{x^0(\tau(\mathbf{n}))})^2\} = o(1)\end{aligned} \tag{5.129}$$

Thus

$$\int_{\partial\Lambda_R} \sigma(dr) \int_0^\infty dt K_2^B(r + t\mathbf{n}, R) = \mathcal{J}_3^c + o(1) \tag{5.130}$$

where

$$\mathcal{J}_3^c = \frac{1}{2}\int_{\partial\Lambda} \sigma(dr) k(r) \int_{\mathcal{X}^0} P^0(dx^0) \mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r + t_0\mathbf{n} + x^0)(\widetilde{x^0(\tau(\mathbf{n}))})^2. \tag{5.131}$$

Observe that the integral in Eq. (5.131) absolutely converges for z from the interval (5.113). Now combining Eqs. (5.116), (5.117), (5.120), (5.124) and (5.127) we find that

$$\mathcal{K}(R) = \mathcal{J}_2^c + \mathcal{J}_3^c + o(1) \tag{5.132}$$

with $\mathcal{J}_2^c$ and $\mathcal{J}_3^c$ given by Eqs.(5.112) and (5.131) respectively.

The desired expansion (5.58) of the log-partition function follows from formulas (5.25), (5.60), (5.63) and (5.132). We observe that the constant term

$$\begin{aligned}c(\Lambda, \phi, z) =& \mathcal{J}_1^c - \mathcal{J}_2^c - \mathcal{J}_3^c \\ =& \int_{\partial\Lambda} \sigma(dr) k(r) \int_0^\infty dt\ t \int_{\mathcal{X}^0} P^0(dx^0) \mathcal{W}(1_{\mathcal{M}^c(\Pi_r^+)}\tilde{g}_z)(r + t\mathbf{n} + x^0) \\ & - \frac{1}{2}\int_{\partial\Lambda} \sigma(dr) k(r) \int_0^\infty dt \int_{\mathcal{X}^0} P^0(dx^0) 1_{\mathcal{X}(\Pi_r^+)}(r + t\mathbf{n} + x^0) \int_{-\infty}^\infty d\xi \\ & \times \int_{\mathcal{X}^0} P^0(dy^0) \mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r + t\mathbf{n} + x^0, (\xi, \eta_0) + y^0)(\xi + \widetilde{y^0(\tau(\mathbf{n}))})^2 \\ & - \frac{1}{2}\int_{\partial\Lambda} \sigma(dr) k(r) \int_{\mathcal{X}^0} P^0(dx^0) \mathcal{W}_{\mathcal{M}(\Pi_r^+)}(\tilde{g}_z)(r + t_0\mathbf{n} + x^0)(\widetilde{x^0(\tau(\mathbf{n}))})^2.\end{aligned} \tag{5.133}$$

is an analytic function in z for $|z|$ from the interval (5.113).

If ϕ is rotation invariant, the Ursell function and the measure $W_{z\rho}$, also are rotation invariant therefore the integral $\int_0^\infty dt \int_{\mathcal{X}^0} P^0(dx^0)\mathcal{W}(1_{\mathcal{M}^c(\Pi_r^+)}\tilde{g}_z)(r + t\mathbf{n} + x^0)$ does not depend on the orientation of the normal $\mathbf{n}(r)$ and can be evaluated with respect to any fixed unit vector $\mathbf{d}_1$ and the half-plane $\Pi_{\mathbf{d}_1}^+ = \{u \in \mathbf{R}^\nu \mid \langle u, \mathbf{d}_1\rangle \geq 0\}$. In a similar way the corresponding potential dependent factors in the integrals $\mathcal{J}_1^c, \mathcal{J}_2^c, \mathcal{J}_3^c$ do not depend on $r \in \Lambda$ and can be evaluated with the help of an arbitrary fixed pair of orthogonal unit vectors $\mathbf{d}_1, \mathbf{d}_2$. Thus the terms $b(\Lambda, \phi, z)$ and $c(\Lambda, \phi, z)$ of the expansion (5.58) have simpler form:

$$b(\Lambda, \phi, z) = |\partial\Lambda| b(\phi, z) \text{ and } c(\Lambda, \phi, z) = 2\pi\chi(\Lambda) c(\phi, z) \tag{5.134}$$

where

$$b(\phi, z) = -\int_0^\infty dt \int_{\mathcal{X}^0} P^0(dx^0)\mathcal{W}(1_{\mathcal{M}^c(\Pi_{\mathbf{d}_1}^+)}\tilde{g}_z)(t\mathbf{d}_1 + x^0), \tag{5.135}$$

and

$$\begin{aligned} c(\phi, z) = & \int_0^\infty dt\ t \int_{\mathcal{X}^0} P^0(dx^0)\mathcal{W}(1_{\mathcal{M}^c(\Pi_{\mathbf{d}_1}^+)}\tilde{g}_z)(t\mathbf{d}_1 + x^0) - \frac{1}{2}\int_0^\infty dt \int_{\mathcal{X}^0} P^0(dx^0) \\ & \times 1_{\mathcal{X}(\Pi_{\mathbf{d}_1}^+)}(t\mathbf{d}_1 + x^0) \int_{-\infty}^\infty d\xi \int_{\mathcal{X}^0} P^0(dy^0)\mathcal{W}_{\mathcal{M}(\Pi_{\mathbf{d}_1}^+)}(\tilde{g}_z)(t\mathbf{d}_1 + x^0, y^0 \\ & + (\xi, -\inf\langle y^0, \mathbf{d}_1\rangle))[\langle \xi, \mathbf{d}_2\rangle + \langle y^0(\tau(\mathbf{d}_1)), \mathbf{d}_2\rangle]^2 - \frac{1}{2}\int_{\mathcal{X}^0} P^0(dx^0) \\ & \times \mathcal{W}_{\mathcal{M}(\Pi_{\mathbf{d}_1}^+)}(\tilde{g}_z)(r - \inf\langle x^0, \mathbf{d}_1\rangle \mathbf{d}_1 + x^0)\langle x^0(\tau(\mathbf{d}_1)), \mathbf{d}_2\rangle^2. \end{aligned} \tag{5.136}$$

Here $\tau(\mathbf{d}_1)$ is defined by $\inf\langle y^0, \mathbf{d}_1\rangle = \langle y^0(\tau(\mathbf{d}_1)), \mathbf{d}_1\rangle$.

In conclusion we note that the second equality in Eq. (5.134) is a consequence of the Gauss-Bonnet theorem: $\int_{\partial\Lambda} \sigma(dr)k(r) = 2\pi\chi(\Lambda)$.

Theorem 5.2 is proved.

It is worth to note that the expansion of the log-partition function $\ln \Xi_{id}(\Lambda_R, z)$ of the ideal gas is obtained from the expansion (5.58) by setting $\phi \equiv 0$. In this case

$$\ln Z_{id}(\Lambda_R, z) = R^2|\Lambda|\beta z p_{id} + R|\partial\Lambda| z b_{id} + \pi\chi(\Lambda) z c_{id} \tag{5.137}$$

with $p_{id} = \int_{\mathcal{X}^0} P^0(dx^0)$, $b_{id} = \int_{\mathcal{X}^0} P^0(dx^0) \inf\langle x^0, \mathbf{n}\rangle$ and $c_{id} = \int_{\mathcal{X}^0} P^0(dx^0) \times [(\inf\langle x^0, \mathbf{n}\rangle)^2 - (x^0(\tau(\mathbf{n})))^2]$.

Let us show how to get the constant term of the expansion (5.137):

$$c_{id}(\Lambda, z) = \mathcal{J}^c_{id,1} - \mathcal{J}^c_{id,3} = z \int_{\partial\Lambda} \sigma(dr) k(r) \cdot \int_{\mathcal{X}^0} P^0(dx^0) \int_0^{-\inf\langle x^0, \mathbf{n}\rangle} dt\, t$$
$$- z\frac{1}{2} \int_{\partial\Lambda} \sigma(dr) k(r) \int_{\mathcal{X}^0} P^0(dx^0) (\widetilde{x^0(\tau(\mathbf{n}))})^2 = z\frac{1}{2} \int_{\partial\Lambda} \sigma(dr) k(r)$$
$$\times \int_{\mathcal{X}^0} P^0(dx^0) [(\inf\langle x^0, \mathbf{n}\rangle)^2 - (\widetilde{x^0(\tau(\mathbf{n}))})^2] = \pi\chi(\Lambda) c_{id}(z).$$

Thus one recovers the familiar case of large volume asymptotic expansion of Brownian integrals (see Eqs. (2.55) and (2.56) in [62] with $F = 0$).

Notice that the same arguments in the d-dimensional case, $d > 2$ give

$$ln\, Z(\Lambda_R, z) = R^d \beta p(\phi, z)|\Lambda| + R^{d-1}\, b(\phi, z)|\partial\Lambda| + R^{d-2} c_1(\Lambda) c_2(\phi, z) + o(R^{d-2}).$$

Here $p(\phi, z)$ and $b(\phi, z)$ are given by formulas similar to Eqs. (5.23) and (5.134) and $c_1(\Lambda) = (d-1)\int_{\partial\Lambda} \sigma(dr) k_m(r)$, where $k_m(r)$ is the mean curvature of $\partial\Lambda$ at the point $r \in \partial\Lambda$. At the same time, we are not able to get more terms of the expansion. This is a familiar case also for the ideal gas and it is not clear whether the reason is technical or not (cf. [62], section VII).

To prove boundedness property of the derivative of the two-point truncated correlation functions we used the tree identity [16] which involves the function $u(x, y)$ and here for the sake of simplicity we assumed the boundedness of the interaction ϕ. One can release this restrictive condition by developing further the techniques of the Ref. [71] for the bounded function $e^{-u(x,y)} - 1$.

5.3 Asymptotics of Brownian integrals. Bose statistics

This section is devoted to the study of the asymptotics of the Brownian integrals with paths which are constrained to a bounded domain Λ of $\mathbb{R}^d$ when the domain is dilated to infinity. We consider the case of BE statistics where the paths are of random time intervals which are integer multiples of some fixed $\beta > 0$.

The present section consists of two parts. In part one we obtain the three first terms of the asymptotics for the case of small activity. The first two terms are proportional respectively to the volume and the area of the boundary of Λ . We prove that in two dimensional case the third term is purely topological and is proportional to the Euler-Poincare characteristic of the domain.

In part two we consider the Bose gas with repulsive two-body interaction at low activity. We find an explicit expression for the pressure in terms of functional integrals and prove that the correction is of order of the area of the boundary of Λ. The proof is based on the abstract cluster expansion method developed in Chapter 2 and uses the results on the decay of correlations from Chapter 3.

For shortness we use in this section the following notations: we denote by P_z^u the measure $P_{+,z}^u$ given by (3.31), by ρ_z the measure $\rho_{+,z}$ defined by (3.36) and by $W_{z,\Lambda}$ the measure $W_{\rho_{+,z,\Lambda}}$ defined by (3.37).

The main object of our interest is the grand partition function $Z(\Lambda, z)$ of the Bose gas in a bounded domain Λ which is defined by

$$Z(\Lambda, z) = \int\limits_{\mathcal{M}(\Lambda)} \exp\{-U(\omega)\}\,\mathrm{d}W_{z,\Lambda}(\omega), \tag{5.138}$$

We want to study the asymptotics of $\log Z(\Lambda, z)$ for large Λ. Since

$$\int\limits_{\mathcal{X}(\Lambda)} \rho_{z,\Lambda}(X) = \log Z_{id}(\Lambda, z)$$

It is natural to start with the study of the asymptotics of $\log Z_{id}(\Lambda, z)$ for large Λ.

We suppose that ϕ is a non-negative even function which satisfies condition (4) with $l > 0$:

The class of admissible domains Λ consists of open bounded convex subsets of $\mathbb{R}^d$ with n, $n \geq 0$, convex closed holes. We assume that the boundary $\partial\Lambda$ of Λ consists of $n+1$ $(d-1)$-dimensional closed C^3 manifolds. At each point $r \in \partial\Lambda$ we define local coordinates $(\eta, \xi_1, \ldots, \xi_{d-1})$ so that η is along the inward drawn unit normal $\mathbf{n}$ and $\xi_1, \ldots, \xi_{d-1}$ are along the directions of principal curvatures of $\partial\Lambda$ at the point r. In this local coordinates $\partial\Lambda$ is given by a C^3 function f_r:

$$\eta = f_r(\xi_1, \ldots, \xi_{\nu-1}) = f_r(\boldsymbol{\xi}), \quad ||\boldsymbol{\xi}|| < \delta \tag{5.139}$$

for some $\delta > 0$ small enough, $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{d-1})$.

Let $F(X)$, $X \in \mathcal{X}$ be a translation invariant function: $F(X+u) = F(X)$, for all $X \in \mathcal{X}$ and $u \in \mathbb{R}^d$. Hence we can think of F as a function on $\mathcal{X}^0$ and we assume that $F \in L_2(\mathcal{X}^0, P_{\overline{z}}^0)$ for some $\overline{z} > 0$.

Theorem 5.5 . *For any admissible domain Λ and for all z from the interval $0 < z \leq \overline{z}$ the following expansion holds true*

$$\int\limits_{\mathcal{X}(\Lambda_R)} F(X)\mathrm{d}\rho_{z,\Lambda_R}(X) = R^d|\Lambda|a_0(F,z) + R^{d-1}a_1(\Lambda, F, z) + R^{d-2}a_2(\Lambda, F, z) + o(R^{d-2})$$

as $R \to \infty$, the coefficients a_0, a_1 and a_2 are given explicitly in terms of functional integrals by formulas (5.143), (5.169) and (5.170) respectively.

In the case where the function F is in addition rotation invariant the coefficients a_1 and a_2 have simpler form.

Theorem 5.6 . *If under the conditions of Theorem 5.5 the function F is in addition rotation invariant, then*

$$\int_{\mathcal{X}(\Lambda_R)} F(X)\mathrm{d}\rho_{z,\Lambda_R}(X) = R^d|\Lambda|a_0(F,z) + R^{d-1}|\partial\Lambda|\overline{a}_1(F,z) + R^{d-2}\int_{\partial\Lambda} H_\Lambda(r)\mathrm{d}\sigma(r)\overline{a}_2(F,z) + o(R^{d-2})$$

where $\overline{a}_1$ and $\overline{a}_2$ are given by (5.171) and (5.172), $H_\Lambda(r)$ is the mean curvature of $\partial\Lambda$ at the point $r \in \partial\Lambda$ and σ is the $d-1$-dimensional surface measure.

Remark 1 . *In dimension two, $d = 2$, according to Gauss-Bonnet theorem*

$$\int_{\partial\Lambda} H_\Lambda(r)\mathrm{d}\sigma(r) = 2\pi\chi(\Lambda),$$

where $\chi(\Lambda)$ is the Euler-Poincaré characteristic of Λ, $\chi(\Lambda) = 1 - n$, if Λ has n holes. Therefore the corresponding term is purely topological.

Remark 2 . *In particular case where $F \equiv 1$ Theorem 5.6 gives an asymptotic expansion of the log-partition function $\log Z_{id}(\Lambda_R, z)$ of the ideal Bose gas in Λ_R, as $R \to \infty$.*

The next result gives the main term of the asymptotic expansion of the log-partition function of the Bose gas in Λ_R with repulsive interaction ϕ.

Theorem 5.7 . *Let the non-negative potential ϕ satisfy the condition (4) with $l > 1$ and z be from the interval*

$$0 < z < \left[C(d,l)||\phi_l||_1\beta^{1-\frac{d}{2}}\zeta\Big(\frac{d}{2}+1\Big)\right]^{-1} \tag{5.140}$$

then for any admissible domain $\Lambda \subset \mathbb{R}^d$

$$\ln Z(\Lambda_R, z) = R^d \cdot p(\phi, z)|\Lambda| + O(R^{d-1}) \quad as \quad R \to \infty$$

where the so-called pressure $p(\phi, z)$ is given by

$$p(\phi, z) = \int_{\mathcal{X}^0} \mathrm{d}P_z^0(X) \int_{\mathcal{M}(\mathcal{X})} \frac{\varphi(\omega, X)}{|\omega|+1}\mathrm{d}W_{\rho_z}(\omega).$$

5.3.1 Proof of Theorem 5.5

Let

$$I(R,z)=\int_{\mathcal{X}(\Lambda_R)} F(X)\mathrm{d}\rho_{z,\Lambda_R}(X).$$

We decompose this integral as follows

$$I(R,z)=\int\limits_{\Lambda_R}\mathrm{d}u\int\limits_{\mathcal{X}^0}F(X)\mathrm{d}P_z^0(X)-\int\limits_{\Lambda_R}\mathrm{d}u\int\limits_{\mathcal{X}^0}1_{\mathcal{X}^c(\Lambda_R)}(X+u)F(X)\mathrm{d}P_z^0(X)\equiv \tag{5.141}$$
$$\equiv I_0(R,z)-I_1(R,z).$$

This gives the volume term:

$$I_0(R,z)=\mathbb{R}^d\cdot|\Lambda|\cdot a_0(F,z) \tag{5.142}$$

with

$$a_0(F,z)=\int\limits_{\mathcal{X}^0}F(X)\mathrm{d}P_z^0(X). \tag{5.143}$$

Then

$$\begin{aligned}I_1(R,z)= & \int\limits_{\Lambda_{R,\delta}}\mathrm{d}u\int\limits_{\mathcal{X}^0}1_{\mathcal{X}^c(\Lambda_R)}(X+u)F(X)\mathrm{d}P_z^0(X)+\\ & +\int\limits_{\Lambda_R\setminus\Lambda_{R,\delta}}\mathrm{d}u\int\limits_{\mathcal{X}^0}1_{\mathcal{X}^c(\Lambda_R)}(X+u)F(X)\mathrm{d}P_z^0(X)\equiv\\ \equiv & \; I_2(R,z)+I_2'(R,z).\end{aligned} \tag{5.144}$$

Evidently

$$|I_2'(R,z)|\le\int\limits_{\Lambda_R\setminus\Lambda_{R,\delta}}\mathrm{d}u\int\limits_{\mathcal{X}^0}1_{\{\sup||||\ge\delta\sqrt{R}\}}(X)\,|F(X)|\,\mathrm{d}P_z^0(X).$$

By Schwarz inequality and Lemma 3.4

$$\begin{aligned}&\int\limits_{\mathcal{X}^0}1_{\{\sup||X||\ge\delta\sqrt{R}\}}(X)\,|F(X)|\,\mathrm{d}P_z^0(X)\le\\ &\le\|F\|_{L_2}\left[P_{+,z}^0\left(\sup\|X|\ge\delta\sqrt{R}\right)\right]^{1/2}\le\\ &\le C\,(\nu)\,\beta^{-\nu/4}\|F\|_{L_2}\exp[-C(\beta,z)\delta R]\end{aligned} \tag{5.145}$$

for all z, $0<z<1$, where $C(\beta,z)=\left(\frac{|\ln z|}{64\beta}\right)^{1/2}$ and

$$\|F\|_{L_2}=\left(\int\limits_{\mathcal{X}^0}F^2(X)\mathrm{d}P_{+,z}^0(X)\right)^{1/2}.$$

Hence

$$|I_2'(R,z)| \leq |\Lambda| C\,(\nu,\beta,z)\,\|F\|_{L_2}\exp[-C(\beta,z)\delta R] \tag{5.146}$$

Now consider $I_2(R,z)$. We have

$$\begin{aligned} I_2(R,z) = & \int\limits_{\Lambda_{R,\delta}} du \int\limits_{\mathcal{X}^0} 1_{\mathcal{X}^c(\Lambda_R)}\mathcal{X}(\Lambda_R)(X+u) \\ & \cdot 1_{\{\sup\|X\|<\delta\sqrt{R}\}}(X)F(X)\mathrm{d}P_z^0(X)+ \\ & + \int\limits_{\Lambda_{R,\delta}} du \int\limits_{\mathcal{X}^0} 1_{\mathcal{X}^c(\Lambda_R)}\mathcal{X}(\Lambda_R)(X+u) \\ & \cdot 1_{\{\sup\|X\|\geq\delta\sqrt{R}\}}(X)F(X)\mathrm{d}P_z^0(X) \equiv \\ & \equiv I_3(R,z) + I_3'(R,z). \end{aligned} \tag{5.147}$$

According to (5.145)

$$|I_3'(R,z)| \leq |\Lambda| C\,(\nu,\beta,z)\,\|F\|_{L_2}\exp[-C(\beta,z)\delta R] \tag{5.148}$$

To estimate $I_3(R,z)$ we use the local coordinates. Similarly to (5.139) $\partial\Lambda_R$ is given locally by

$$\eta = f_{r,R}(\xi), \quad \|\xi\| < \delta\sqrt{R}$$

We have the following relations between the functions $f_{r,R}$ and $f_r \equiv f_{r,1}$:

$$f_{r,R}(\xi) = Rf_{r,1}(R^{-1}\xi) \tag{5.149}$$

Let $k_i(r|R)$, $i = 1,\ldots,\nu-1$, be the principal curvatures of $\partial\Lambda_R$ at the point $r \in \partial\Lambda_R$. From (5.149) it follows that

$$k_i(r|R) = R^{-1}k_i(r|1), \quad i = 1,\ldots,\nu-1. \tag{5.150}$$

Then similarly to (4.31)

$$\begin{aligned} I_3(R,z) = & \int\limits_{\partial\Lambda_R} \mathrm{d}\sigma_R(r) \int\limits_0^{\delta\sqrt{R}} \prod_{i=1}^{d-1}(1-\tau k_i(r|R))\,\mathrm{d}\tau \cdot \\ & \cdot \int\limits_{\mathcal{X}^0} 1_{\mathcal{X}^c(\Lambda_R)}\mathcal{X}(\Lambda_R)(X+r+\tau\mathbf{n})1_{\sup\|X|<\delta\sqrt{R}}(X)F(X)\mathrm{d}P_z^0(X) \end{aligned}$$

For each $X \in \mathcal{X}^0$ such that $\sup\|X\| < \delta\sqrt{R}$ we put

$$\gamma(X) \equiv \gamma_{r,R}(X) = \inf_t\left[X_{\mathbf{n}}(t) - f_{r,R}(X_T(t)\right].$$

Here

$$X_{\mathbf{n}}(t) = \langle X(t),\mathbf{n}\rangle, \quad X_T(t) = X(t) - \langle X(t),\mathbf{n}\rangle\mathbf{n} \tag{5.151}$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^d$. It is easy to check that, for any $X \in \mathcal{X}^0$ with $\sup \|X\| < \delta\sqrt{R}$, $1_{\mathcal{X}(\Lambda_R)}(X + r + \tau\mathbf{n}) = 0$ iff $\tau + \gamma(X) < 0$. Therefore

$$I_3(R,z) = \int\limits_{\partial\Lambda_R} \mathrm{d}\sigma_R(r) \int\limits_0^{\delta\sqrt{R}} \prod_{i=1}^{d-1} (1 - \tau k_i(r|R))\, d\tau \cdot \cdot \int\limits_{\mathcal{X}^0} 1_{\tau+\gamma(X)<0}(X) 1_{\sup\|X\|<\delta\sqrt{R}}(X) F(X) \mathrm{d}P_z^0(X). \tag{5.152}$$

Using the equality

$$1_{\sup\|X_{\mathbf{n}}\|<\delta\sqrt{R}} = 1_{\sup\|X\|<\delta\sqrt{R}} + 1_{\sup\|X\|\geq\delta\sqrt{R}} 1_{\sup\|X_{\mathbf{n}}\|<\delta\sqrt{R}}$$

we can rewrite (5.152) as

$$\begin{aligned} I_3(R,z) = &\int\limits_{\partial\Lambda_R} \sigma_R(dr) \int\limits_0^{\delta\sqrt{R}} \prod_{i=1}^{d-1} (1 - \tau k_i(r|R))\, d\tau \cdot \\ &\cdot \int\limits_{\mathcal{X}^0} 1_{\tau+\gamma(X)<0}(X) 1_{\sup\|X_{\mathbf{n}}\|<\delta\sqrt{R}}(X) F(X) \mathrm{d}P_z^0(X) - \\ &- \int\limits_{\partial\Lambda_R} \mathrm{d}\sigma_R(r) \int\limits_0^{\delta\sqrt{R}} \prod_{i=1}^{d-1} (1 - \tau k_i(r|R))\, \mathrm{d}\tau \cdot \\ &\cdot \int\limits_{\mathcal{X}^0} 1_{\tau+\gamma(X)<0}(X) 1_{\sup\|X\|\geq\delta\sqrt{R}}(X) 1_{\sup\|X_{\mathbf{n}}\|<\delta\sqrt{R}}(X) F(X) \mathrm{d}P_z^0(X) \equiv \\ &\equiv I^A(R,z) + \tilde{I}^A(R,z). \end{aligned} \tag{5.153}$$

Let us estimate the second term $\tilde{I}^A(R,z)$. It is clear that for each admissible domain Λ

$$\overline{k} = \max_{1\leq i\leq d-1} \sup_{r\in\partial\Lambda} |k_i(r|1)| < \infty.$$

Assuming $\delta < \overline{k}^{-1}$, we have that $\left|\prod_{i=1}^{d-1} (1 - \tau k_i(r|R)\right| < 2^{d-1}$ for all $0 < \tau < \delta R$. Hence using (5.145) we see that

$$\left|\tilde{I}^A(R,z)\right| \leq |\Lambda| C\,(\nu,\beta,z)\, \|F\|_{L_2} \exp[-C(\beta,z)\delta R]. \tag{5.154}$$

The first term $I^A(R,z)$ in (5.153) we decompose as

$$
\begin{aligned}
I^A(R,z) = &\int\limits_{\partial\Lambda_R} \mathrm{d}\sigma_R(r) \int\limits_0^\infty \prod_{i=1}^{d-1} (1-\tau k_i(r|R))\,\mathrm{d}\tau \cdot \\
&\cdot \int\limits_{\mathcal{X}^0} 1_{\tau+\gamma(X)<0}(X) 1_{\sup \|X_{\mathbf{n}}\|<\delta\sqrt{R}}(X) F(X) \mathrm{d}P_z^0(X) - \\
&- \int\limits_{\partial\Lambda_R} \mathrm{d}\sigma_R(r) \int\limits_{\delta\sqrt{R}}^\infty \prod_{i=1}^{d-1} (1-\tau k_i(r|R))\,\mathrm{d}\tau \cdot \\
&\cdot \int\limits_{\mathcal{X}^0} 1_{\tau+\gamma(X)<0}(X) 1_{\sup \|X_{\mathbf{n}}\|<\delta\sqrt{R}}(X) F(X) \mathrm{d}P_z^0(Xb) \equiv \\
&\equiv I_1^A(R,z) + \hat{I}^A(R,z)
\end{aligned}
\tag{5.155}
$$

Let us show that

$$
\left|\hat{I}^A(R,z)\right| \le C \exp\left(-C(\beta,z)\delta\sqrt{R}\right) \tag{5.156}
$$

where $C = C(d,\beta,z,\Lambda,F,\delta)$ does not depend on R.

From (5.150) it follows that

$$
\prod_{i=1}^{d-1} (1-\tau k_i(r|R)) = \sum_{s=0}^{d-1} \tau^s a_s(r|R) = \sum_{s=0}^{d-1} R^{-s}\tau^s a_s(R^{-1}r|1), \tag{5.157}
$$

where $a_0(r|R) = 1$, $a_s(r|R) = (-1)^s \sum\limits_{1\le i_1<\cdots<i_s\le d-1} k_{i_1}(r|R)\cdots k_{i_s}(r|R)$, $s = 1,\cdots,d-1$.

Hence

$$
\begin{aligned}
\left|\hat{I}^A(R,z)\right| \le &\sum_{s=0}^{d-1} \int\limits_{\partial\Lambda_R} |a_s(r|R)| \mathrm{d}\sigma_R(r) \int\limits_{\delta\sqrt{R}}^\infty \tau^s \mathrm{d}\tau \cdot \\
&\cdot \int\limits_{\mathcal{X}^0} 1_{\tau+\gamma(X)<0}(X) 1_{\sup \|X_{\mathbf{n}}\|<\delta\sqrt{R}}(X) |F(X)| \mathrm{d}P_z^0(X).
\end{aligned}
$$

Now with the help of (5.149) and the condition that $f_{r,R}$ is of class C^3 one can easily obtain that

$$
f_{r,R}(\xi) = R^{-1}\frac{1}{2}\sum_{s=0}^{d-1} k_i(R^{-1}r|1)\xi_i^2 + R^{-2}\epsilon_{r,R}(\xi),\ \|\xi\| < \delta\sqrt{R} \tag{5.158}
$$

where

$$
|\epsilon_{r,R}(\xi)| \le C(d)C(\Lambda)\|\xi\|^3 \tag{5.159}
$$

uniformly in $r \in \partial\Lambda_R$ and $R \ge 1$. This implies that for all ξ, $\|\xi\| < \delta R$ and R large enough

$$
|f_{r,R}(\xi)| \le \overline{k}\delta^2.
$$

Using the fact that $\sup_t \|X(t)\| > \tau - \overline{k}\delta^2$ for any loop $\mathbf{X}$ starting at the point $r + \tau\mathbf{n}$ with $\tau > \delta\sqrt{R}$ and such that $\tau + \gamma(X) < 0$, we can write:

$$\int\limits_{\mathcal{X}^0} 1_{\tau+\gamma(X)<0}(X) 1_{\sup\|X_{\mathbf{n}}\|<\sigma\sqrt{R}}(X)|F(X)|\mathrm{d}P_z^0(X) \leq$$
$$\leq \int\limits_{\mathcal{X}^0} 1_{\sup\|X\|>\tau-\overline{k}\delta^2}(X)|F(X)|\mathrm{d}P_z^0(X) \leq$$
$$\leq \|F\|_{L_2}\left[P_{+,z}^0\left(\sup\|X\| > \tau - \overline{k}\delta^2\right)\right]^{1/2} \leq$$
$$\leq C(d)\beta^{-d/4}\|F\|_{L_2}\exp[C(\beta,z)\overline{k}\delta^2]\exp[-C(\beta,z)\tau].$$

Hence

$$|\hat{I}^A(R,z)| \leq C(d,\beta,z,\overline{k},\delta)\|F\|_{L_2}\sum_{s=0}^{d-1}\int\limits_{\partial\Lambda_R} a_s(r|R)\mathrm{d}\sigma_R(r)\cdot$$
$$\cdot\int\limits_{\sigma\sqrt{R}}^{\infty}\tau^s\exp[-C(\beta,z)\tau]\mathrm{d}\tau \leq$$
$$\leq C(d,\beta,z,\Lambda,\delta)\|F\|_{L_2}\exp[-C(\beta,z)\delta\sqrt{R}], \tag{5.160}$$

which proves the formula (5.156).

Hence combining the formulas (5.141)-(5.144), (5.146)-(5.148), (5.153) and (5.156) we find that

$$I_1(R,z) = \int\limits_{\partial\Lambda_R}\sigma_R(dr)\int\limits_0^{\infty}\prod_{i=1}^{d-1}\left(1-\tau k_i(r|R)\right)\mathrm{d}\tau\cdot$$
$$\cdot\int\limits_{\mathcal{X}^0} 1_{\tau+\gamma(X)<0}(X)1_{\sup\|X_{\mathbf{n}}\|<\delta\sqrt{R}}(X)F(X)\mathrm{d}P_z^0(X) + O(e^{-C\sqrt{R}}).$$

Applying Fubini's theorem and formula (5.157) to the last integral we find that

$$I_1(R,z) = \sum_{s=0}^{d-1}\int\limits_{\partial\Lambda_R} a_s(r|R)\mathrm{d}\sigma_R(r)\int\limits_{\mathcal{X}^0} 1_{\sup\|X_{\mathbf{n}}\|<\delta\sqrt{R}}(X)F(X)\mathrm{d}P_z^0(Xb)\cdot$$
$$\cdot\int\limits_0^{-\gamma(X)}\tau^s d\tau + O(e^{-C\sqrt{R}})$$

or

$$I_1(R,z) = \sum_{s=0}^{d-1} L_s(z,R) + O(e^{-C\sqrt{R}}). \tag{5.161}$$

with

$$L_s(z,R) = \frac{1}{s+1}\int\limits_{\partial\Lambda_R} a_s(r|R)\sigma_R(dr)\cdot \tag{5.162}$$
$$\cdot\int\limits_{\mathcal{X}^0} 1_{\sup\|X_{\mathbf{n}}\|<\delta\sqrt{R}}(X)F(X)\left(-\gamma(X)\right)^{s+1}\mathrm{d}P_z^0(X).$$

Let $e_1, \ldots, e_{d-1}$ be unit vectors drawn along the directions of the principal curvatures of $\partial\Lambda_R$ at the point $r \in \partial\Lambda_R$. For each $X \in \mathcal{X}^0$, with $\sup \|X\| < \delta R$, we choose $t_{\mathbf{n}} = t_{\mathbf{n}}(X)$ and $t_R = t_R(X)$ from the interval $[0, |X|\beta]$ so that $X_{\mathbf{n}}(t_{\mathbf{n}}) = \inf_t X_{\mathbf{n}}(t)$ and $X_{\mathbf{n}}(t_R) - f_{r,R}(X_T(t_R)) = \inf_t (X_{\mathbf{n}}(t) - f_{r,R}(X_T(t)))$. By Proposition 5.11 from Appendix A.3 $t_{\mathbf{n}}$ is P_z^0-almost surely unique and by Proposition 5.12 from Appendix A.4 $t_{\mathbf{n}} \to t_R$, as $R \to \infty$, P_z^0-almost surely for all $0 < z \leq 1$.

Let us show that the following representation of $\gamma(X)$ is valid:

$$-\gamma(X) = -X_{\mathbf{n}}(t_{\mathbf{n}}) + \frac{R^{-1}}{2} \sum_{i=1}^{d-1} k_i(R^{-1}r|1) \langle X_T(t_{\mathbf{n}}), e_i \rangle^2 + R^{-1} \tilde{\epsilon}_{r,R}(X), \tag{5.163}$$

where

$$|\tilde{\epsilon}_{r,R}(X)| \leq C(d,\beta) \left\{ \sum_{i=1}^{d-1} \left(\langle X_T(t_R), e_i \rangle^2 - \langle X_T(t_{\mathbf{n}}), e_i \rangle^2 \right) + R^{-1} \|X\|^3 \right\}. \tag{5.164}$$

Note that from Proposition 5.12 and the Lebesgue dominant convergence theorem it follows that

$$\int\limits_{\partial\Lambda_R} \mathrm{d}\sigma_R(r) \int\limits_{\mathcal{X}^0} \tilde{\epsilon}_{r,R}(X) \mathrm{d}P_z^0(X) = o\left(R^{d-1}\right), \quad \text{as } R \to \infty. \tag{5.165}$$

Let us prove (5.163). We have that

$$-\gamma(X) = f_{r,R}(X_T(t_{\mathbf{n}})) - X_{\mathbf{n}}(t_{\mathbf{n}}) + \Delta\left(X|r,R\right),$$

where

$$0 \leq \Delta\left(X|r,R\right) = f_{r,R}(X_T(t_R)) - X_{\mathbf{n}}(t_R) - f_{r,R}(X_T(t_{\mathbf{n}})) + X_{\mathbf{n}}(t_{\mathbf{n}}).$$

Using 5.158 we find that

$$\Delta\left(X|r,R\right) \leq f_{r,R}(X_T(t_R)) - f_{r,R}(X_T(t_{\mathbf{n}})) = \frac{R^{-1}}{2} \sum_{i=1}^{d-1} k_i\left(R^{-1}r|1\right) \cdot$$
$$\cdot \left[\langle X_T(t_R), e_i \rangle^2 - \langle X_T(t_{\mathbf{n}}), e_i \rangle^2\right] + R^{-2}\left[\epsilon_{r,R}(X_T(t_R)) - \epsilon_{r,R}(X_T(t_{\mathbf{n}}))\right].$$

This according to 5.159 and Proposition 5.12, Appendix A.4, implies (5.163) and (5.164).

With the help of (29) we can treat the terms $L_s(z,R)$ from (5.162). Consider $L_0(z,R)$. We have that

$$\begin{aligned} L_0(z,R) &= R^{d-1} \int\limits_{\partial\Lambda} \mathrm{d}\sigma(r) \int\limits_{\mathcal{X}^0} F(X) \left(-X_{\mathbf{n}}(t_{\mathbf{n}}) + \frac{R^{-1}}{2} \sum_{i=1}^{d-1} k_i(r|1) \right) \cdot \\ &\cdot \langle X_T(t_{\mathbf{n}}), e_i \rangle^2 + R^{-1} \tilde{\epsilon}_{r,R}(X) \Big) \mathrm{d}P_z^0(X) = \\ &= -R^{d-1} \int\limits_{\partial\Lambda} \mathrm{d}\sigma(r) \int\limits_{\mathcal{X}^0} F(X) \inf X_{\mathbf{n}} \mathrm{d}P_z^0(X) + \\ &+ \frac{R^{d-2}}{2} \int\limits_{\partial\Lambda} \mathrm{d}\sigma(r) \int\limits_{\mathcal{X}^0} \sum_{i=1}^{d-1} k_i(r|1) F(X) \langle X_T(t_{\mathbf{n}}), e_i \rangle^2 \mathrm{d}P_z^0(X) + o(R^{d-2}) \end{aligned} \tag{5.166}$$

In a similar way, according to (5.157),

$$L_1(z,R) = -\frac{1}{2}R^{d-2}\int\limits_{\partial\Lambda}\sum_{i=1}^{d-1}k_i(r|1)\mathrm{d}\sigma(r)\cdot \qquad (5.167)$$

$$\cdot\int\limits_{\mathcal{X}^0}F(X)X_{\mathbf{n}}^2(t_{\mathbf{n}})\mathrm{d}P_z^0(X) + O(R^{d-3}).$$

It is easy to check that

$$\sum_{s=2}^{d-1}L_s(z,R) = O(R^{d-3}). \qquad (5.168)$$

Indeed for R large enough $|\gamma(X)|^s \leq C\sup\|X\|^s$.

With the help of Lemma 3.4 it is easy to check that $\sup\|X\|^s \in L_1(\mathcal{X}^0, P_z^0)$. Therefore

$$\sum_{s=2}^{d-1}|L_s(z,R)| \leq \sum_{s=2}^{d-1}\frac{R^{d-1}}{s+1}\int\limits_{\partial\Lambda}R^{-s}a_s(r|1)\mathrm{d}\sigma(r)\cdot$$

$$\cdot\int\limits_{\mathcal{X}^0}F(X)\sup\|X\|^{s+1}\mathrm{d}P_z^0(X) = O(R^{d-3}).$$

Now from (5.161), (5.166)-(5.168) it follows that

$$I_1(R,z) = R^{\nu-1}a_1(\Lambda,F,z) + R^{\nu-2}a_2(\Lambda,F,z) + o(R^{\nu-2})$$

where

$$a_1 = -\int\limits_{\partial\Lambda}\mathrm{d}\sigma(r)\int\limits_{\mathcal{X}^0}F(X)\inf X_{\mathbf{n}}\mathrm{d}P_z^0(X) \qquad (5.169)$$

$$a_2 = \frac{1}{2}\int\limits_{\partial\Lambda}\mathrm{d}\sigma(r)\int\limits_{\mathcal{X}^0}F(X)\sum_{i=1}^{d-1}k_i(r|1)\cdot \qquad (5.170)$$

$$\cdot\left[\langle X_T(t_{\mathbf{n}}),e_i\rangle^2 - X_{\mathbf{n}}^2(t_{\mathbf{n}})\right]\mathrm{d}P_z^0(X)$$

This together with (5.141) complete the proof of Theorem (5.5).

Now suppose that the function $F(X)$ is in addition rotation invariant. Then the integral

$$\int\limits_{X^0}F(X)\inf X_{\mathbf{n}}\mathrm{d}P_z^0(X)$$

does not depend on the orientation of the unit normal $\mathbf{n}$ in $\mathbb{R}^d$, because the measure P_z^0 also is rotation invariant. Hence a_1 takes a simple form:

$$a_1 = |\partial\Lambda|\overline{a}_1(F,z)$$

with

$$\overline{a}_1(F,z) = -\int\limits_{X^0}F(X)\inf\langle X,\mathbf{d}_1\rangle\mathrm{d}P_z^0(X) \qquad (5.171)$$

where d_1 is any fixed unit vector in $\mathbb{R}^d$. In the same way

$$a_2 = \frac{1}{2} \int\limits_{\partial\Lambda} \sum_{i=1}^{d-1} k_i(r|1) \mathrm{d}\sigma(r) \int\limits_{\mathcal{X}^0} F(X) \cdot$$
$$\cdot \left[\left\langle X_T(\bar{t}), \mathbf{d}_2 \right\rangle^2 - \left\langle X_{\mathbf{n}}(\bar{t}), \mathbf{d}_1 \right\rangle^2 \right] \mathrm{d}P_z^0(X)$$

or

$$a_2 = \int\limits_{\partial\Lambda} H_\Lambda(r) \mathrm{d}\sigma(r) \overline{a}_2(F, z),$$

where

$$H_\Lambda(r) = \frac{1}{d-1} \sum_{i=1}^{d-1} k_i(r|1)$$

is the mean curvature of $\partial\Lambda$ at the point r and

$$\overline{a}_2(F, z) = \frac{d-1}{2} \int\limits_{\mathcal{X}^0} F(X) \left[\left\langle X_T(\bar{t}), d_2 \right\rangle^2 - \left\langle X(\bar{t}), d_1 \right\rangle^2 \right] \mathrm{d}P_z^0(X)$$

Here $\mathbf{d}_1, \mathbf{d}_2$ is an arbitrary fixed pair of orthogonal unit vectors in $\mathbb{R}^d$ and $\bar{t}$ is defined by $\left\langle X_{\mathbf{n}}(\bar{t}), \mathbf{d}_1 \right\rangle = \inf \langle X_{\mathbf{n}}(t), \mathbf{d}_1 \rangle$.

Theorem 5.6 is proved.

5.3.2 Proof of Theorem 5.7

To develop the large volume asymptotics of the log-partition function $\ln Z(\Lambda_R, z)$ of the Bose gas with interaction we start as before with the cluster representation of $\ln Z(\Lambda_R, z)$:

$$\ln Z(\Lambda_R, z) = \int\limits_{\mathcal{M}(\Lambda_R)} \varphi(\omega) \mathrm{d}W_{z,\Lambda_R}(\omega)$$

where φ is the Ursell function. Then

$$\begin{aligned} \ln Z(\Lambda_R, z) &= \int_{\mathcal{X}(\Lambda_R)} G_z(X) \mathrm{d}\rho_z(X) - \int_{\mathcal{X}(\Lambda_R)} \mathrm{d}\rho_z(X) \int_{\mathcal{M}^c(\Lambda_R)} \frac{\varphi(X, \omega)}{|\omega| + 1} \mathrm{d}W_{\rho_z}(\omega) \\ &\equiv A_0(R, z) - A_1(R, z), \end{aligned} \tag{5.172}$$

where

$$G_z(X) = \int\limits_{\mathcal{M}} \frac{\varphi(X, \omega)}{|\omega| + 1} \mathrm{d}W_{\rho_z}(\omega).$$

Note that G_z is translation invariant function: $G_z(X + u) = G_z(X)$, for any $u \in \mathbb{R}^d$ and $X \in \mathcal{M}$. This follows from the translation invariance of the Ursell function and the measure W_{ρ_z}. By Theorem 2.1, Eq. (2.8), $G_z \in L^2(\mathcal{X}^0, P_z^0)$ for all z from the interval (5.140). According to Theorem 5.5

$$A_0(R, z) = R^d |\Lambda| a_0(G_z) + R^{d-1} a_1(\Lambda, G_z) + R^{\nu-2} a_2(\Lambda, G_z) + o(R^{d-2}).$$

Now consider $A_1(R,z)$. We will show below that $A_1(R,z) = O(R^{d-1})$. Similarly to (5.144) we decompose A_1 as:

$$\begin{aligned} A_1(R,z) = &\int\limits_{\Lambda_{R,\delta}} \mathrm{d}u \int\limits_{\mathcal{X}^u} 1_{\mathcal{X}(\Lambda_R)}(X) \mathrm{d}P_z^u(X) \int\limits_{\mathcal{M}^c(\Lambda_R)} \frac{\varphi(X,\omega)}{|\omega|+1} \mathrm{d}W_{\rho_z}(\omega) + \\ + &\int\limits_{\Lambda_R \setminus \Lambda_{R,\delta}} \mathrm{d}u \int\limits_{\mathcal{X}^u} 1_{\mathcal{X}(\Lambda_R)}(\mathbf{X}) P_z^u d(\mathbf{X}) \int\limits_{\mathcal{M}^c(\Lambda_R)} \frac{\varphi(X,\omega)}{|\omega|+1} \mathrm{d}W_{\rho_z}(\omega) \equiv \\ \equiv &A_2(R,z) + A_2'(R,z). \end{aligned} \tag{5.173}$$

Applying Lemma 3.4 we find that

$$\begin{aligned} |A_2'(R,z)| \leq &\int\limits_{\Lambda_R \setminus \Lambda_{R,\delta}} du \int\limits_{\mathcal{X}^u} P_z^u d(\mathbf{X}) \int\limits_{\mathcal{M}^c(B_u(\delta R)} \left| \frac{\varphi(X,\omega)}{|\omega|+1} \right| W_{\rho_z}(d\omega) \leq \\ \leq &C\,(1+\delta R)^{-l} R^{\nu} |\Lambda| = O(R^{\nu-1}), \end{aligned} \tag{5.174}$$

where $C = C\,(\Phi, \beta, \nu, z, l) > 0$.

Consider $A_2(R,z)$. Using the local coordinate system we can write

$$A_2(R,z) = \int\limits_{\partial\Lambda_R} \sigma_R(dr) \int\limits_0^{\delta R} \prod_{i=1}^{d-1} (1 - tk_i(r|R))\, dt \cdot \tag{5.175}$$

$$\cdot \int\limits_{\mathcal{X}^0} 1_{\mathcal{X}(\Lambda_R)}(\mathbf{X}^0 + r + t\mathbf{n}) \mathrm{d}P_z^0(X^0) \int\limits_{\mathcal{M}^c(\Lambda_R)} \frac{\varphi(X^0 + r + t\mathbf{n}, \omega)}{|\omega|+1} \mathrm{d}W_{\rho_z}(\omega).$$

Again applying Lemma 3.4 we have that

$$\begin{aligned} |A_2(R,z)| \leq 2^{\nu-1} &\int\limits_{\partial\Lambda_R} \mathrm{d}\sigma_R(r) \int\limits_0^{\delta R} \mathrm{d}t \int\limits_{\mathcal{X}^0} \mathrm{d}P_z^0(X^0) \cdot \\ \cdot &\int\limits_{\mathcal{M}^c(B_{r+t\mathbf{n}}(t))} \left| \frac{\varphi(X^0 + r + t\mathbf{n}, \omega)}{|\omega|+1} \right| \mathrm{d}W_{\rho_z}(\omega) \leq \\ \leq C\,(\Phi, \beta, \nu, z, l) &\int\limits_{\partial\Lambda_R} \mathrm{d}\sigma_R(r) \int\limits_0^{\infty} (1+t)^{-l}\, \mathrm{d}t = O(R^{\nu-1}). \end{aligned} \tag{5.176}$$

This completes the proof of Theorem 5.7.

5.4 Polygonal regions

The previous section was devoted to the study of the asymptotic behavior of the the logarithm of the grand partition function $\ln Z(\Lambda, z)$ of ideal Bose gases in a bounded domain with smooth boundary. The three first terms of the expansion were obtained. In this final section applying different method we find all the nondecreasing terms of the large volume asymptotics of $\ln Z(\Lambda, z)$ for interacting Bose gas in a bounded Λ with polygonal boundaries.

For simplicity, only the two-dimensional case is analyzed.

Let Λ be a polygonal domain in $\mathbf{R}^2$, and for simplicity we consider only the case of two-dimensional convex polygonal domains Λ with m obtuse angles $\theta_1, \dots, \theta_m$ with vertices $A_1, \dots, A_m$. We fix the orientation of Λ by inner unit normals n_j so that n_j is a normal to the side $A_{j-1}A_j$ and $\langle n_j, n_{j+1}\rangle = |cos\theta_j|$, $j = 1, \dots, m$, with $m+1$ identified to 1, where $\langle \cdot\,, \cdot\rangle$ stands for the scalar product in $\mathbf{R}^2$.

Theorem 5.8 (Polygonal regions) . *If the interaction potential ϕ satisfies the conditions (a)– (c) from Section 5.3, with $l > 16$ in (c), then for all z from the interval*

$$0 < z < [C(l)||\phi_l||_1\zeta\,(2)]^{-1} \tag{5.177}$$

where ζ is the Riemann zeta function, the following expansion holds true:

$$\begin{aligned} \ln Z(\Lambda_R, \beta, z) = {} & R^2|\Lambda|p(\beta, z) + R\sum_{j=1}^{m} |A_{j-1}A_j| b_{n_j}(\beta, z) \\ & + \sum_{j=1}^{m} c_{n_j,n_{j+1}}(\beta, z) + o(1) \quad as \quad R \to \infty, \end{aligned} \tag{5.178}$$

where the contribution of the angle θ_j has the form

$$\begin{aligned} c_{n_j,n_{j+1}}(\beta, z) = {} & \int_0^{\theta_j} d\varphi \hat{c}_{n_j,n_{j+1}}(\varphi, \beta, z) + \int_{\pi-\theta_j}^{\pi/2} d\varphi c_{n_j}(\varphi, \beta, z) \\ & + \int_{\pi/2}^{\theta_j} d\varphi c_{n_{j+1}}(\varphi, \beta, z), \quad j = 1, \dots, m, \end{aligned} \tag{5.179}$$

with $m+1$ identified to 1. Here $\beta p(\beta, z)$ is the pressure given by formula (5.185) *below, $|A_{j-1}A_j|$ is the length of the side $A_{j-1}A_j$ and the quantities $b_{n_j}(\beta, z)$, $c_{n_j}(\varphi, \beta, z)$ and $\hat{c}_{n_j,n_{j+1}}(\varphi, \beta, z)$ are explicitly given in terms of the Brownian integrals by formulas* (5.186), (5.189) *and* (5.193) *respectively.*

If the interaction ϕ is rotation-invariant, then the expansion (5.178) takes a simpler form. Namely, the terms $b_{n_j}(\beta, z)$and $c_{n_j}(\varphi, \beta, z)$ do not depend on the orientation of the normals n_j and can be evaluated for a fixed unit vector e. Similarly, the term $\hat{c}_{n_j,n_{j+1}}(\varphi, \beta, z)$ becomes independent of the orientation of the corresponding angle θ_j. Therefore, this angle can be defined by a pair of unit vectors $e_1(\theta_j)$ and $e_2(\theta_j)$ that are orthogonal to the sides of the angle and satisfy the equality $\langle e_1(\theta_j), e_2(\theta_j)\rangle = |cos\theta_j|$. Thus, the following corollary of Theorem 5.8 is true.

Corollary 5.9 . *If, in addition to the conditions of Theorem, the potential ϕ is rotation-invariant, then*

$$\Xi(\Lambda_R, \beta, z) = R^2|\Lambda|p(\beta, z) + R|\partial\Lambda|b_e(\beta, z) + \sum_{j=1}^{m} c(\theta_j, \beta, z) + o(1)$$

as $R \to \infty$, where

$$c(\theta_j, \beta, z) = \int_0^{\theta_j} d\varphi \hat{c}_{e1(\theta_j),e2(\theta_j)}(\varphi, \beta, z) + \int_{\pi-\theta_j}^{\theta_j} d\varphi c_e(\varphi, \beta, z). \tag{5.180}$$

Here $b_e(\beta, z)$, $c_e(\varphi, \beta, z)$ and $\hat{c}_{\theta_j}$ are given by the below formulas (5.194), (5.195) *and* (5.196) *respectively.*

A similar result is true for the case $d > 2$. We formulate it without proof.

Theorem 5.10 . *Let the potential ϕ be the same as in Corollary 4.9 then for all z from the interval*

$$0 < z < \left[C(d, l)||\phi_l||_1 \beta^{1-\frac{d}{2}} \zeta\left(\frac{d}{2} + 1\right)\right]^{-1}$$

and any convex polyhedron $\Lambda \subset \mathbb{R}^d$

$$\ln Z(\Lambda_R, \beta, z) = R^d|\Lambda|p(\phi, \beta, z) + \sum_{i=1}^{d} R^{d-i} c_i(\phi, \beta, z) \sum_{\xi \in \Lambda^{(d-i)}} |\xi| + o(1). \tag{5.181}$$

Here $\Lambda^{(i)}$, $i = 0, \cdots, d-1$ is the family of all i-faces of the polyhedron Λ and $|\xi|$, $\xi \in \Lambda^{(i)}$, is the i- dimensional volume of ξ.

Proof of Theorem 5.8. We write the grand partition function in terms of the composite (or winding) loops:

$$Z(\Lambda_R, \beta, z) = \int_{\mathcal{M}(\Lambda)} \mathrm{d}W_{\rho_z,\Lambda}(\omega) e^{U(\omega)}$$

where the energy $U(\omega)$ of the configuration ω is given by (3.38) and the measure $W_{\rho_z,\Lambda} = W_{\rho_{+,z,\Lambda}}$ is given by formula (3.37). Then write the cluster representation of $\ln Z(\Lambda_R, \beta, z)$:

$$\ln Z(\Lambda_R, \beta, z) = \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega)\varphi(\omega) 1_{\mathcal{M}(\Lambda_R)}(\omega), \tag{5.182}$$

where φ is the Ursell function with $q(X, Y) = \exp(-U_2(X, Y)) - 1$ where $U_2(X, Y)$ is that of (3.40).

This implies

$$\begin{aligned} \ln Z(\Lambda_R, \beta, z) &= \int_{\mathcal{X}} \mathrm{d}\rho_z(X) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) g\left(X, \omega\right) 1_{\mathcal{M}(\Lambda_R)}\left(X, \omega\right) \\ &= \int_{\Lambda_R} \mathrm{d}u \int_{\mathcal{X}^0} \mathrm{d}P_z^0\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) g\left(X^0 + u, \omega\right) 1_{\mathcal{M}(\Lambda_R)}\left(X^0 + u, \omega\right) \end{aligned} \tag{5.183}$$

where $g(\omega) = \frac{\varphi(\omega)}{|\omega|}$. Let $A_{R,1}, \dots, A_{R,m}$ be the vertices of the convex m-gone Λ_R. Then, defining $H(t) = 1_{[0,+\infty)}(t)$, $-\infty < t < \infty$, we get

$$1_{\mathcal{M}(\Lambda_R)}(\omega) = \prod_{j=1}^{m} H\left(\inf \langle \omega, n_j \rangle\right),$$

where

$$\inf\langle \omega; n_j \rangle = \min_{X \in \omega} \inf_t \langle X(t); n_j \rangle .$$

Hence,

$$1_{\mathcal{M}(\Lambda_R)}\left(X^0 + u, \omega\right) = \sum_{J \subset \{1,\dots,m\}} (-1)^{|J|} \prod_{j \in J} H\left(-\inf \left\langle X^0 + u, \omega, n_j \right\rangle\right)$$

in virtue of the identity $H(t) = 1 - H(-t)$. Substituting the above equality into (5.183), we get

$$\begin{aligned}
\ln Z(\Lambda_R, \beta, z) = & \int_{\Lambda_R} \mathrm{d}u \int_{\mathcal{X}^0} \mathrm{d}P_z^0(X^0) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) g\left(X^0 + u, \omega\right) \\
& - \sum_{j=1}^{m} \int_{\Lambda_R} \mathrm{d}u \int_{\mathcal{X}^0} \mathrm{d}P_z^0\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) g\left(X^0 + u, \omega\right) \\
& \cdot H\left(-\inf \left\langle X^0 + u, \omega, n_j \right\rangle\right) \\
& + \sum_{j=1}^{m} \int_{\Lambda_R} \mathrm{d}u \int_{\mathcal{X}^0} \mathrm{d}P_z^0\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) \\
& \cdot g\left(X^0 + u, \omega\right) H\left(-\inf \left\langle X^0 + u, \omega, n_j \right\rangle\right) \\
& \cdot H\left(-\inf \left\langle X^0 + u, \omega, n_{j+1} \right\rangle\right) + I'(R, z) \\
\equiv & I^A(R, z) + \sum_{j=1}^{m} I_j^B(R, z) + \sum_{j=1}^{m} I_j^C(R, z) + I'(R, z),
\end{aligned} \tag{5.184}$$

where

$$\begin{aligned}
I'(R, z) = & \sum_{J \subset \{1,\dots,m\}, |J| \geq 3} (-1)^{|J|} \int_{\Lambda_R} \mathrm{d}u \int_{\mathcal{X}^0} \mathrm{d}P_z^0\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) \\
& \cdot g\left(X^0 + u, \omega\right) \prod_{j \in J} H\left(-\inf \left\langle X^0 + u, \omega, n_j \right\rangle\right).
\end{aligned}$$

Note that the configurations which contribute to $I'(R, z)$ contain loops that cross at least two non-adjacent faces. Hence, we can estimate this term with the help of Theorem 3.15, Corollary 3.16 and get $I'(R, z) = o(1)$ as $R \to \infty$.

Now, by the translation-invariance of g, W_{ρ_z} and $\mathcal{M}$ we obtain

$$I^A(R, z) = R^d |\Lambda| \beta p(\beta, z),$$

where the pressure $p(\beta, z)$ has the cluster representation

$$p(\beta, z) = \beta^{-1} \int_{\mathcal{X}^0} \mathrm{d}P_z^0\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) g\left(X^0 + u, \omega\right). \tag{5.185}$$

Let us fix any j, $1 \le j \le m$, and consider the term $I_j^b(R,z)$ defined by (5.184). Consider the corresponding side $A_{R,j-1}A_{R,j}$ between the vertices of the angles θ_{j-1} and θ_j. Here and below we identify 0 with m, so that $A_{R,0} = A_{R,m}$, $\theta_0 = \theta_m$, etc. Further, we choose the coordinates u_1, u_2, with u_1 along the side $A_{R,j-1}A_{R,j}$ and u_2 along the normal n_j, to provide that the coordinates of the vertices $A_{R,j-1}$ and $A_{R,j}$ be $(Ra_{j-1},0)$ and $(Ra_j,0)$ respectively. To this side, we associate three disjoint regions in Λ_R: a rectangle D_j of the height δR (with a fixed $\delta = \delta(\Lambda)$) and the length $R(a_j - a_{j-1})$ and two sectors $S_1(j-1)$ and $S_2(j)$ of the radius δR and angles $\theta_{j-1} - \pi/2$ and $\theta_j - \pi/2$ respectively. Then we have the following decomposition of the term $I_j^B(R,z)$:

$$\begin{aligned}
I_j^B(R,z) = &\int_{S_1(j-1)} \mathrm{d}u \int_{\mathcal{X}^0} \mathrm{d}P_z^0(X^0) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(d\omega) g\left(X^0+u,\omega\right) H\left(-\inf\left\langle X^0+u,\omega,n_j\right\rangle\right) \\
&- \int_{D_j} \mathrm{d}u \int_{\mathcal{X}^0} \mathrm{d}P_z^0\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(d\omega) g\left(X^0+u,\omega\right) H\left(-\inf\left\langle X^0+u,\omega,n_j\right\rangle\right) \\
&- \int_{S_2(j)} \mathrm{d}u \int_{\mathcal{X}^0} \mathrm{d}P_z^0(X^0) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(d\omega) g\left(X^0+u,\omega\right) H(-\inf\langle X^0+u,\omega,n_j\rangle) \\
&+ o(1) = \; I_{S_1(j-1)}^B(R,z) + I_{D_j}^B(R,z) + I_{S_2(j)}^B(R,z) + o(1) \quad \text{as} \quad R \to \infty.
\end{aligned}$$

Note that in this decomposition the term $I_{D_j}^B(R,z)$ gives the contribution to the boundary term, while $I_{S_1(j-1)}^B(R,z)$ and $I_{S_2(j)}^B(R,z)$ contribute to the constant term.

Observing that $D_j = \left\{(u_1,u_2) \in \Lambda \;\middle|\; Ra_{j-1} \le u_1 \le Ra_j,\ 0 \le u_2 \le \delta R\right\}$, we get

$$\begin{aligned}
I_{D_j}^B(R,z) = &-\int_{Ra_{j-1}}^{Ra_j} \mathrm{d}u_1 \int_0^{\delta R} \mathrm{d}u_2 \int_{\mathcal{X}^0} \mathrm{d}P_z^0\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) \\
&\cdot g\left(X^0+(u_1,u_2),\omega\right) H\left(-\inf\left\langle X^0+(u_1,u_2),\omega,n_j\right\rangle\right) \\
= &\int_{Ra_{j-1}}^{Ra_j} \mathrm{d}u_1 b_{n_j}(u_1,\beta,z) + o(1),
\end{aligned}$$

where

$$\begin{aligned}
b_{n_j}(u_1,\beta,z) = &-\int_0^\infty \mathrm{d}u_2 \int_{\mathcal{X}^0} \mathrm{d}P_z^0\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) \\
&\cdot g\left(X^0+(u_1,u_2),\omega\right) H\left(-\inf\left\langle X^0+(u_1,u_2),\omega,n_j\right\rangle\right). \qquad (5.186)
\end{aligned}$$

Now, note that

$$\inf\left\langle X^0+(u_1,u_2),\omega,n_j\right\rangle = \inf\left\langle X^0+(0,u_2),\omega,n_j\right\rangle.$$

Hence, the functions g_1 and H do not depend on u_1, and therefore also the quantity $b_{n_j}(u_1,\beta,z)$ is independent of u_1 due to the translation-invariance of W_{ρ_z}. Thus, we have

$$I_{D_j}^B(R,z) = R(a_j - a_{j-1}) b_{n_j}(\beta,z) + o(1) \qquad (5.187)$$

with $b_{n_j}(\beta,z) \equiv b_{n_j}(0,\beta,z)$ defined by (5.186).

Proceeding to the sector $S_1(j-1)$ and using polar coordinates centered at $A_{R,j-1}$, we get

$$S_1(j-1) = \left\{(r,\varphi) \;\middle|\; 0 \le r \le \delta R,\; \frac{\pi}{2} \le \varphi \le \theta_{j-1}\right\},$$

where the angle φ is measured from the side $A_{R,j-1}A_{R,j}$. Therefore, by (**??**)

$$\begin{aligned} I^B_{S_1(j)}(R,z) = &- \int_{\pi/2}^{\theta_{j-1}} \mathrm{d}\varphi \int_0^{\delta R} \mathrm{d}r \int_{\mathcal{X}^0} \mathrm{d}P^0_z\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) \\ &\cdot g\left(X^0 + (r,\varphi), \omega\right) H\left(-\inf\left\langle X^0 + (r,\phi), \omega, n_j\right\rangle\right) \\ = &\int_{\pi/2}^{\theta_{j-1}} \mathrm{d}\varphi c_{n_j}(\varphi,\beta,z) + o(1), \end{aligned} \tag{5.188}$$

where

$$\begin{aligned} c_{n_j}(\varphi,\beta,z) = &- \int_0^\infty \mathrm{d}r \int_{\mathcal{X}^0} \mathrm{d}P^0_z\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) \\ &\cdot g_1\left(X^0 + (r,\varphi), \omega\right) H\left(-\inf\left\langle X^0 + (r,\varphi), \omega, n_j\right\rangle\right). \end{aligned} \tag{5.189}$$

Observe that by translation-invariance $c_{n_j}(\varphi,\beta,z)$ is invariant with respect to the shift of the center of polar coordinates by any vector $a = (a_1, 0)$.

Similarly, we obtain that the contribution of the sector $S_2(j)$ is equal to

$$I^B_{S_2(j)}(R,z) = \int_{\pi-\theta_j}^{\pi/2} \mathrm{d}\varphi c_{n_j}(\varphi,\beta,z) + o(1), \tag{5.190}$$

where $c_{n_j}(\varphi,\beta,z)$ is that of (5.189). Combining formulas (5.187)-(5.190), we get

$$\begin{aligned} \sum_{j=1}^m I^B_j(R,z) = &R \sum_{j=1}^m (a_j - a_{j-1}) b_{n_j}(u_1,\beta,z) \\ &+ \sum_{j=1}^m \left\{ \int_{\pi-\theta_j}^{\pi/2} \mathrm{d}\varphi c_{n_j}(\varphi,\beta,z) + \int_{\pi/2}^{\theta_j} \mathrm{d}\varphi c_{n_{j+1}}(\varphi,\beta,z) \right\} + o(1). \end{aligned} \tag{5.191}$$

For analyzing the term $I^C_j(R,z)$, $1 \le j \le m$, in (5.184), consider the angle θ_j with vertex $A_{R,j}$, which is measured from the face $A_{R,j}A_{R,j+1}$ with the inner normal n_{j+1}. Let S_j be a sector in Λ_R of the radius δR, angle θ_j and vertex $A_{R,j}$. Choosing u_1-axis along $A_{R,j}A_{R,j+1}$ and u_2-axis along the normal n_{j+1} we get

$$\begin{aligned} I^C_j(R,z) = &\int_{S_j} \mathrm{d}u \int_{\mathcal{X}^0} \mathrm{d}P^0_z\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) g\left(X^0 + u, \omega\right) \\ &\cdot H\left(-\inf\left\langle X^0 + u, \omega, n_j\right\rangle\right) H\left(-\inf\left\langle X^0 + u, \omega, n_{j+1}\right\rangle\right) + o(1). \end{aligned}$$

Using polar coordinates (r,φ) centered at $A_{R,j}$, we can rewrite this equality in the form

$$\begin{aligned} I^C_j(R,z) = &\int_0^{\theta_j} \mathrm{d}\varphi \int_0^\infty \mathrm{d}r \int_{\mathcal{X}^0} \mathrm{d}P^0_z\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) g\left(X^0 + (r,\varphi), \omega\right) \\ &\cdot H\left(-\inf\left\langle X^0 + (r,\varphi), \omega, n_j\right\rangle\right) \\ &\cdot H\left(-\inf\left\langle X^0 + (r,\varphi), \omega, n_{j+1}\right\rangle\right) + o(1) \\ = &\int_0^{\theta_j} \mathrm{d}\varphi \hat{c}_{n_j,n_{j+1}}(\varphi,\beta,z) + o(1), \end{aligned} \tag{5.192}$$

where

$$\hat{c}_{n_j,n_{j+1}}(\varphi,\beta,z) = \int_0^\infty \mathrm{d}r \int_{\mathcal{X}^0} \mathrm{d}P_z^0\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) g\left(\{X^0+(r,\varphi)\}\cup\omega\right) \cdot H\left(-\inf\left\langle X^0+(r,\varphi),\omega,n_j\right\rangle\right) \cdot H\left(-\inf\left\langle X^0+(r,\varphi),\omega,n_{j+1}\right\rangle\right). \tag{5.193}$$

To get the constant term of the expansion (5.180), we collect all contributions of angles given in formulas (5.191) and (5.192):

$$c_{n_j,n_{j+1}}(\beta,z) = \int_0^{\theta_j} \mathrm{d}\varphi \hat{c}_{n_j,n_{j+1}}(\varphi,\beta,z) + \int_{\pi-\theta_j}^{\pi/2} \mathrm{d}\varphi c_{n_j}(\varphi,\beta,z) + \int_{\pi/2}^{\theta_j} \mathrm{d}\varphi c_{n_{j+1}}(\varphi,\beta,z),$$

where c_{n_j} and $\hat{c}_{n_j,n_{j+1}}$ are defined by (5.189) and (5.193) respectively. This completes the proof of Theorem.

If, in addition to translation-invariance, the interaction ϕ is rotation-invariant, then $b_{n_j}(\beta,z)$ and $c_{n_j}(\varphi,\beta,z)$ do not depend on n_j, due to the rotation-invariance of P_z^0 and W_{ρ_z}. Assuming that $e\in\mathbb{R}^2$ is an arbitrarily fixed unit vector, we choose the Cartesian coordinates (u_1,u_2) with u_2 along e and set

$$b_e(\beta,z) = -\int_0^\infty \mathrm{d}u_2 \int_{\mathcal{X}^0} \mathrm{d}P_z^0\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) \cdot g\left(X^0+(0,u_2),\omega\right) H\left(-\inf\left\langle X^0+(0,u_2),\omega,e\right\rangle\right). \tag{5.194}$$

Similarly, evaluating the terms c_{n_j} given by (5.189) for a fixed vector e, we get

$$c_e(\varphi,\beta,z) = -\int_0^\infty \mathrm{d}r \int_{\mathcal{X}^0} \mathrm{d}P_z^0\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) \cdot g\left(X^0+(r,\varphi),\omega\right) H\left(-\inf\left\langle X^0+(r,\varphi),\omega,e\right\rangle\right). \tag{5.195}$$

Finally, the terms $c_{n_j,n_{j+1}}$ in (5.193) are independent of the orientation of n_j, n_{j+1} and depend only on the opening θ_j of the corresponding angle, and therefore we can evaluate them by any fixed pair of unit vectors $e_1(\theta_j), e_2(\theta_j)\in\mathbb{R}^2$ such that $\langle e_1(\theta_j), e_2(\theta_j)\rangle = |cos\theta_j|$, $j=1,\ldots,m$.

We assume that

$$\hat{c}_{\theta_j}(\varphi,\beta,z) = \int_0^\infty \mathrm{d}r \int_{\mathcal{X}^0} \mathrm{d}P_z^0\left(X^0\right) \int_{\mathcal{M}} \mathrm{d}W_{\rho_z}(\omega) g\left(X^0+(r,\varphi),\omega\right) \cdot H\left(-\inf\left\langle X^0+(r,\varphi),\omega,e_1(\theta_j)\right\rangle\right) \cdot H\left(-\inf\left\langle X^0+(r,\varphi),\omega,e_2(\theta_j)\right\rangle\right). \tag{5.196}$$

Then, for obtaining (5.180) it remains to replace in (5.179) the terms c_{n_j} and $\hat{c}_{n_j,n_{j+1}}$ by $c_e(\varphi,\beta,z)$ and $\hat{c}_{\theta_j}(\varphi,\beta,z)$ respectively, and replacing the terms b_{n_j} by b_e in (5.178) we complete the proof of Corollary.

Appendix

A.1 Proof of Lemma 4.3

Let $\omega = \{x_1, \cdots, x_m\}$. Then, with the help of Proposition 6.1 (b) from Ref. 14 and Stirling's formula we can write

$$\mathcal{W}(|\tilde{g}_z|)(\omega) \leq \sum_{n=0}^{\infty} \frac{(ze^{2\beta B})^{n+m}}{n!} \sum_{T\in\mathcal{T}(1,\cdots,m,m+1,\cdots,m+n)} \int_{\mathcal{X}^n} \prod_{\{i,j\}\in T} |u(x_i,x_j)|$$
$$\cdot\, dx_{m+1}\cdots dx_{m+n} \leq \frac{1}{\bar{u}} \sum_{n=0}^{\infty} \frac{(z\bar{u}e^{2\beta B})^{n+m}}{n!}(m+n)^{m+n-2}$$
$$\leq \frac{(m-1)!}{e\bar{u}} \left[\frac{z\bar{u}e^{2\beta B+1}}{1 - z\bar{u}e^{2\beta B+1}}\right]^m .$$

Here, we used the well-known fact that the number of trees with $n+m$ vertices is $(n+m)^{n+m-2}$, and the last line is a consequence of the formula:

$$\sum_{n=0}^{\infty} \frac{(m+n-1)!}{n!} t^n = \frac{(m-1)!}{(1-t)^m}, \quad |t| < 1.$$

from [105], section 4.4 . This completes the proof of Lemma 4.3.

A.2 Proof of Lemma 4.4

To prove Lemma 4.4 we use the tree identity from [15].

$$\sum_{\gamma\in\mathcal{C}_n} \prod_{\{i,j\}\in\gamma} (e^{-u_{i,j}} - 1) = \sum_{T\in\mathcal{T}_n} \prod_{\{i,j\}\in T} (-u_{i,j}) \int d\lambda_T(\{s_{ij}\}) \exp\left(-\sum_{i<j} s_{ij}u_{i,j}\right)$$

Here $\mathcal{C}_n$ is the set of connected graphs with n vertices, the real numbers $u_{i,j}$, $1 \leq i < j \leq n$, satisfy the stability condition $\sum_{1\leq i<j\leq n} u_{i,j} \geq -bn$, $s_{ij} = s_{ji}$, $0 \leq s_{ij} \leq 1$, λ_T depends on the tree T and is a probability measure supported on the set $s_{i,j}, 1 \leq i < j \leq n$, such that

$$\sum_{i<j} s_{ij}u_{i,j} \geq -bn. \tag{A.6}$$

The details can be found in [15]. We will apply this identity with $u_{i,j} = u(x_i, x_j)$ and $b = \beta B$, where B is the stability constant for ϕ.

Setting $x_1 = x_1^0 + (\xi_1, \eta_1)$, due to the tree identity we have

$$\frac{\partial}{\partial\eta_1} g(x_1,\cdots,x_n) = \sum_{T\in\mathcal{T}_n} \int d\lambda_T(\{s_{ij}\}) \exp\left(-\sum_{i<j} s_{ij}u(x_i,x_j)\right) \frac{\partial}{\partial\eta_1} \prod_{\{i,j\}\in T} (-u$$
$$\times (x_i,x_j)) + \sum_{T\in\mathcal{T}_n} \int d\lambda_T(\{s_{ij}\}) \exp\left(-\sum_{i<j} s_{ij}u(x_i,x_j)\right)$$
$$\times \frac{\partial}{\partial\eta_1}\left(-\sum_{i<j} s_{ij}u(x_i,x_j)\right) \prod_{\{i,j\}\in T} (-u(x_i,x_j)).$$

Interchanging integration and differentiation (see [59], Lemma 2.2), we get

$$\frac{\partial}{\partial \eta_1}\mathcal{W}(g_z)(x_1^0+(\xi_1,\eta_1),x_2)=G_1(x_1,x_2)+G_2(x_1,x_2). \tag{A.7}$$

with

$$\begin{aligned}G_1(x_1,x_2)=&z^2\sum_{n=2}^{\infty}\frac{z^{n-2}}{(n-2)!}\int_{\mathcal{X}^{n-2}}\rho(dx_3)\cdots\rho(dx_n)\sum_{T\in\mathcal{T}_n}\int d\lambda_T(\{s_{ij}\})\\&\times\exp\left(-\sum_{i<j}s_{ij}u(x_i,x_j)\right)\left[\frac{\partial}{\partial\eta_1}\prod_{\{i,j\}\in T}(-u(x_i,x_j))\right]\end{aligned}$$

and

$$\begin{aligned}G_2(x_1,x_2)=&z^2\sum_{n=2}^{\infty}\frac{z^{n-2}}{(n-2)!}\int_{\mathcal{X}^{n-2}}\rho(dx_3)\cdots\rho(dx_n)\sum_{T\in\mathcal{T}_n}\int d\lambda_T(\{s_{ij}\})\prod_{\{i,j\}\in T}\\&\times(-u(x_i,x_j))\exp\left(-\sum_{i<j}s_{ij}u(x_i,x_j)\right)\frac{\partial}{\partial\eta_1}\left(-\sum_{i<j}s_{ij}u(x_i,x_j)\right)\end{aligned}$$

Let us estimate $G_1(x_1,x_2)$. Evidently

$$\frac{\partial}{\partial\eta_1}\prod_{\{i,j\}\in T}(-u(x_i,x_j))=\sum_{j:\{1,j\}\in T}\left(-\frac{\partial}{\partial\eta_1}u(x_1,x_j)\right)\prod_{\{k,l\}\in T;\{k,l\}\neq\{1,j\}}(-u(x_k,x_l)).$$

Hence by the stability inequality (A.6) we have

$$\begin{aligned}\mid G_1(x_1,x_2)\mid\leq&\sum_{n=2}^{\infty}\frac{(ze^{\beta B})^n}{(n-2)!}\sum_{T\in\mathcal{T}_n}\sum_{j:\{1,j\}\in T}\int_{\mathcal{X}^{n-2}}\rho(dx_3)\cdots\rho(dx_n)\mid\frac{\partial}{\partial\eta_1}u(x_1,x_j)\mid\\&\times\prod_{\{k,l\}\in T;\{k,l\}\neq\{1,j\}}|u(x_k,x_l)|\leq e\beta\max(M,M')(ze^{\beta B})^2\sum_{n=2}^{\infty}n\\&\times(n-1)[z\beta e^{\beta B+1}P^0(\mathcal{X}^0)\max(||\phi||_1,||\nabla\phi||_1)]^{n-2}.\end{aligned} \tag{A.8}$$

Now we consider $G_2(x_1,x_2)$. Since ϕ has bounded derivative,

$$\left|\frac{\partial}{\partial\eta_1}\left(-\sum_{i<j}s_{ij}u(x_i,x_j)\right)\right|\leq\sum_{j=2}^{n}s_{1j}\mid\frac{\partial}{\partial\eta_1}u(x_1,x_j)\mid\leq\beta M'n.$$

Therefore

$$\mid G_2(x_1,x_2)\mid\leq e\beta^2MM'(ze^{\beta B})^2\sum_{n=2}^{\infty}n(n-1)[z\beta e^{\beta B+1}P^0(\mathcal{X}^0)||\phi||_1]^{n-2}. \tag{A.9}$$

Now observe that if $z\beta e^{\beta B+1}P^0(\mathcal{X}^0)\max(||\phi||_1,||\nabla\phi||_1)<1$, both series in the last lines of Eqs. (A.8) and (A.9) converge, hence Lemma 4.4 is proved.

A.3 Proposition 4.11

Proposition 5.11 *[42]. The time $\underline{t}$ at which one dimensional composite Brownian loop attains its infimum is P_z^0-almost surely unique for all $z:\ 0 < z \le 1$*

Proof: Let $\mathcal{X}^0$ be the space of all one dimensional composite Brownian loops. Let $T(X) = \{t \in [0, |X|\beta | X(t) = \inf_s X(s)\}$. We need to show that $P_z^0\{X \in \mathcal{X} | \text{card } T(X) > 1\} = 0$.

Let $\overline{h}(X) = \sup T(X)$ and $\underline{h}(X) = \inf T(X)$, $X \in \mathcal{X}^0$.

For each $X \in \mathcal{X}^0$ let $\hat{X} \in \mathcal{X}^0$ be defined by $\hat{X}(t) = X(j\beta - t)$ if $X \in \mathcal{X}^0_{j\beta}$. Evidently $\hat{}:\ \mathcal{X}^0 \to \mathcal{X}^0$ is one to one mapping which preserves the measure $P^0_{j\beta}$, $j = 1, 2, \ldots$, on each $\mathcal{X}_{j\beta}$. Therefore $\hat{}$ preserves the measure P_z^0. Taking into account that $\overline{h}(\hat{X}) = \underline{h}(X)$ we have that

$$\int_{\mathcal{X}^0} \overline{h}(X) \mathrm{d}P_z^0(X) = \int_{\mathcal{X}^0} \overline{h}(\hat{X}) \mathrm{d}P_z^0(\hat{X}) = \int_{\mathcal{X}^0} \underline{h}(X) \mathrm{d}P_z^0(X)$$

Thus $\overline{h} - \underline{h} \ge 0$ with

$$\int_{\mathcal{X}^0} (\overline{h}(X) - \underline{h}(X)) \mathrm{d}P_z^0(X) = 0$$

which implies that $P_z^0\{X \in \mathcal{X}^0 | cardT(X) > 1\} = 0$.

A.4 Proposition 4.12

Proposition 5.12 *[42]. For each $X \in \mathcal{X}^0$, and all z, $0 < z \le 1$, $t_R(X) \to t_{\mathbf{n}}(X)$, as $R \to \infty$, P_z^0-almost surely.*

Proof: It is sufficient to show that

$$|X_{\mathbf{n}}(t_R) - X_{\mathbf{n}}(t_{\mathbf{n}})| \to 0, \quad as\ R \to \infty, \tag{5.197}$$

for each $X \in \mathcal{X}^0$. Indeed, if $\overline{\tau}(X)$ is a limiting point for the set $\{t_R(X),\ R \ge 1\}$ then (5.197) implies that $\langle X \cdot \mathbf{n} \rangle(\overline{\tau}) = \langle X \cdot \mathbf{n} \rangle(t_{\mathbf{n}}) = \inf \langle X \cdot \mathbf{n} \rangle$ and by Proposition 1 $\overline{\tau}(X) = t_{\mathbf{n}}(X)$ P_z^0-almost surely.

Let us prove (5.197). By definitions of t_R and $t_{\mathbf{n}}$

$$\inf_t (X(t_{\mathbf{n}}) - f_{r,R}(X_T(t))) - \inf_t X_{\mathbf{n}}(t) \le X_{\mathbf{n}}(t_{\mathbf{n}}) - f_{r,R}(X_T(t_{\mathbf{n}})) - X_{\mathbf{n}}(t_{\mathbf{n}})$$

which implies

$$X_{\mathbf{n}}(t_R) - f_{r,R}(X_T(t_R)) - X_{\mathbf{n}}(t_{\mathbf{n}}) \le -f_{r,R}(X_T(t_{\mathbf{n}})),$$

which together with the bound

$$|f_{r,R}(\xi)| \le CR^{-1} \|\xi\|^2,$$

(see (5.158)) gives

$$0 \le X_{\mathbf{n}}(t_R) - X_{\mathbf{n}}(t_{\mathbf{n}}) \le 2CR^{-1} \|X\|^2.$$

Formula (5.197) is proved.

6 Conclusion

The thesis presents a new general approach to the cluster expansion method, one of the most powerful method for the study of Gibbs random fields. This approach can be applied to classical and quantum systems, both continuous and discrete

The method of cluster expansions is presented in Chapter 2.

With the help of this approach efficient bounds for the two-point semiinvariants of the Gibbs random fields in a bounded domaines are obtained.

The main bound for an abstract two-point semiinvariant is proved in Section 3.1. We are interested mainly in applications to quantum systems. To apply the techniques, developed in Chapter 2 and Section 3.1, to the study of partition functions of quantum gases, the quantum mechanical problem involving unbounded noncommuting operators is reduced to a classical-like problem involving only scalar functions. The key to this reduction is the remarkable *Feynman - Kac representation* [40, 46, 86] of a quantum gas as a model of interacting Brownian loops which is called the *interacting loop gas*. One can think about the loop gas model as a classical system where Euclidean points are replaced by Brownian loops, the Lebesgue integrals by Wiener integrals and classical interaction between points by an interaction between trajectories.

For a function of two Brownian loops an integral type decay property is introduced and it is proved that if the classical pair potential has a power decay then the two-point semiinvariant of the corresponding loop gas has the same type of decay.

In Sections 3.4 and 3.5 the cases of repulsive integrable potentials, general stable integrable potentials and potentials with hard core are considered and the corresponding bounds for the two-point semiinvariants are derived.

Chapter 4 is devoted to classical gases. The cases of continuous gas with pair potential and lattice spin gas with many body potential are considered.

A new approach for the derivation of the large volume asymptotics of the log-partition function is presented. This approach, in contrast to the existing ones (see [92], [19]), uses bounds only for the two-point semiinvariants. By a modification of this method a similar problem for the model of interacting Brownian loops is solved in Chapter 5. In Chapter 4 we prove the central local limit theorem [7, 9], give a bound for the convergence rate [84] and prove the local limit theorem for the probabilities of large deviations of the particle number in a grand canonical ensemble [91, 90].

Chapter 5 is devoted to the asymptotic expansion of the log-partition function of the Gibbs distribution of a quantum gas in a bounded domain.

The following expansion for an interacting Boltzmann gas is the main result obtained in

Chapter 5:

$$\ln\, Z(\Lambda_R, z) = R^2|\Lambda|\beta p(\phi, z) + R\,|\partial\,\Lambda|b(\phi, z) + 2\pi\chi(\Lambda)c(\phi, z) + o(1). \tag{6.1}$$

Here β is the inverse temperature, $|\Lambda|$ is the area, $|\partial\,\Lambda|$ the length of the boundary of Λ and $\chi(\Lambda)$ is the Euler-Poincare characteristic of the domain Λ. The coefficients $p(\phi, z)$, $b(\phi, z)$ and $c(\phi, z)$ are explicitly expressed as functional integrals and are analytic functions of the activity z in a neighborhood of the origin; $p(\phi, z)$ is the pressure and $b(\phi, z)$ can be interpreted as the surface tension.

For the case of ideal Bose gas a similar expansion is proved.

Using different method the asymptotic expansion of the log-partition function of the interacting Bose gas in polygonal domains is proved.

References

[1] A. Abdesselam, V. Rivasseau, *Trees, forests, and jungles: a botanical garden for cluster expansions*, Lect. Notes Phys. 446, 7–36; hep-th/9409094 (1994).

[2] R. V. Ambartzumian, *On condensable point processes* New Trends in Probability and Statistics, V. Sazonov and T. Shervashidze (Eds.) VSP/Mokslas, 655–667 (1991).

[3] R. V. Ambartzumian, *Random graph approach to Gibbs processes with pair interaction* Acta Applicandae Mathematicae 22 , 3–14 (1991).

[4] R. V. Ambartzumian, H. S. Sukiasian *Inclusion - Exclusion and point processes* Acta Applicandae Mathematicae 22 , 15–31 (1991).

[5] V. Arzumanian, B. Nahapetian, S. Poghosyan, *Cluster properties of classical lattice spin systems*, Teor. Mat. Fiz. 67, 21–31 (1986).

[6] V. Arzumanian, B. Nahapetian, S. Poghosyan, *Asymptotic expansion of the logarithm of the partition function for lattice spin system with vacuum*, Teor. Mat. Fiz. 81, 175–184 (1989).

[7] V. Arzumanian, B. Nahapetian, S. Poghosyan, *Classical spin lattice systems with vacuum*, Acta Applicandae Mathematicae 22 , 33–53 (1991).

[8] V. Arzumanian, B. Nahapetian, S. Poghosyan, *Cluster properties of lattice systems with continuous spin*, Dokl. Acad. Nauk Arm. SSR 80: 5 , 195–199 (1985).

[9] V. Arzumanian, B. Nahapetian, S. Poghosyan, *Local limit theorem for a number of particles in lattice spin systems*, Teor. Mat. Fiz.89 , 178–189 (1991).

[10] G. Battle, P. Federbush, *A phase cell cluster expansion for Euclidean field theories*, Ann. Phys. 142, 95–139 (1982).

[11] G. Benfatto, M. Cassandro, I. Merola and E. Presutti, *Limit theorems for statistics of combinatorial partitions with applications to mean field Bose gas*, J. Math. Phys. 46, 033303–033341 (2005).

[12] V. Betz, D. Ueltschi, *Spatial random permutations and infinite cycles*, Comm. Math. Phys. 285 (2), 469–501 (2009).

[13] A. Borovkov, *Theory of Probability*, Nauka, Moscow (1976) In Russian.

[14] A. Bovier, M. Zahradník, *A simple approach to the problem of convergence of cluster expansions of polymer models*, J. Statist. Phys. 100, 765–778 (2000).

[15] O. Bratteli, D. Robinson, *Operator Algebras and Quantum Statistical Mechanics II* Springer, Berlin (1981).

[16] D. C. Brydges, *A short course on cluster expansions*, in "Phénomènes critiques, systèmes aléatoires, théories de jauge", Les Houches 1984, 129–183 (1986).

[17] D. C. Brydges, P. Federbush, *A new form of the Mayer expansion in classical statistical mechanics*, J. Math. Phys. 19, 2064–2067 (1978).

[18] D. C. Brydges, T. Kennedy, *Mayer expansions and the Hamilton-Jacobi equation*, J. Statist. Phys. 48, 19–49 (1987).

[19] P. Collet, F. Dunlop, *Geometric expansion of the boundary free energy of a dilute gas*, Commun. Math. Phys. 108, 1 – 12 (1987).

[20] H. Cramer, *Sur on nouveal theoreme limite de la theorie des probabilites* Actuar Sci. et ind 736 Paris (1938) (Uspekhi Mat. Nauk 10, 166–178 (1944)).

[21] S. Dashian, B. Nahapetian, *Description of random fields by means of one-point conditional distributions and some applications* Markov Processes Relat. Fields 7, 193–214, (2001).

[22] S. Dashian, B. Nahapetian, *Description of specifications by means of probability distributions in small volumes under condition of very weak positivity*, Journal of Statistical Physics, 117 (1-2), 281–300, (2004).

[23] S. Dashian, B. Nahapetian, *On Gibbsianness of Random Fields*, Markov Processes Relat. Fields, 5, 81–104, (2009).

[24] G. Del Grosso, *On the local central limit theorem for Gibbs random processes*, Commun. Math. Phys. 37, 141–160 (1973).

[25] R. L. Dobrushin, *Estimates of semi-invariants for the Ising model at low temperatures*, in "Topics in statistical and theoretical physics", AMS Transl. Ser. 2, 177, 59–81 (1996).

[26] R. L. Dobrushin, *Perturbation methods of the theory of Gibbsian fields*, Lecture Notes in Math. 1648, 1–66, Springer, Berlin (1996).

[27] R. L. Dobrushin, *Description of a random field by means of conditional probabilities and the conditions governing its regularity*, Theory of Prob. and its Appl. 13, 197 – 224 (1968).

[28] R. L. Dobrushin, *Gibbsian random fields for lattice systems with pair-wise interactions*, Funkt. Anal. and its Appl., 2, 292 – 301 (1968).

[29] R. L. Dobrushin, *The problem of uniqueness of a Gibbsian random field and the problem of phase transition*, Funkt. Anal. and its Appl., 2, 302 – 312 (1968).

[30] R. L. Dobrushin, *Gibbsian random fields. The general case*, Funkt. Anal. and its Appl., 3, 22 – 28 (1969).

[31] R. L. Dobrushin, R. A. Minlos, *Existence and continuity of pressure in classic statistical physics*, Teor. Veroyatnost. i Primenen., 12:4 , 595-618 (1967).

[32] R. L. Dobrushin, B. S. Nahapetian, *Strong convexity of pressure for lattice systems of classical statistical physics*, Teor. Mat. Fiz. 20, 223–234 (1986).

[33] T. Dorlas, Ph. Martin, J. Pule, *Long cycles in a perturbed mean field model of a Boson Gas*, J. Stat. Phys. 121, 433–461 (2005).

[34] M. Duneau, D. Iagolnitzer, B. Souillard, *Decrease properties of truncated correlation functions and analyticity properties of classical lattice and continuous systems* Commun. math. Phys. 31, 191 (1973).

[35] M. Duneau, D. Iagolnitzer, B. Souillard, *Decay of correlations for infinite range interactions* J. Math. Phys. 16, 1662–1666 (1975).

[36] M. Duneau, B. Souillard, *Cluster properties of lattice and continuous systems* Commun. math. Phys. 47, 155–166 (1976).

[37] W. G. Faris, *A connected graph identity and convergence of cluster expansions*, J. Math. Phys. 49, 113302 (2008).

[38] W. Faris, *Combinatorics and cluster expansions*, Probab. Surveys 7, 157–206 (2010).

[39] R. Fernández, A. Procacci, *Cluster expansion for abstract polymer models. New bounds from an old approach*, Commun. Math. Phys. 274, 123–140 (2007).

[40] R. Feynman, *Space-time approach to non-relativistic quantum mechanics*, Rev. Mod. Phys. 20:2, 367–387 (1948).

[41] M. Fisher, G. Caginalp, *Wall and boundary free energies, I Ferromagnetic scalar spin systems*, Commun. Math. Phys. 56, 11–56 (1977).

[42] S. FrigioS, S. Poghosyan, *Asymptotics of Brownian integrals and pressure. Bose-Einstein statistics*, J.Contemp. Mathem. Analysis 42, 49 – 68 (2007).

[43] I. Gelfand, A. Yaglom, *Integration in functional spaces in quantum physics*, Uspekhi Mat. Nauk 11, 77–114 (1956).

[44] J. Ginibre, *Reduced density matrices of quantum gases. III. Hard-core potentials*, J. Math. Phys. 6, 1432–1446 (1965).

[45] J. Ginibre, *Reduced density matrices of quantum gases. I. Limit of infinite volume*, J. Math. Phys. 6, 238–251 (1965).

[46] J. Ginibre, *Some applications of functional integration in statistical mechanics*, in "Mécanique statistique et théorie quantique des champs", Les Houches 1970, 327–427 (1971).

[47] J. Glimm, A. Jaffe, *Quantum physics. A functional integral point of view*, Springer-Verlag (1981).

[48] J. Glimm, A. Jaffe, T. Spencer, *The Wightman Axioms and Particle Structure in the $\mathcal{P}(\phi)_2$ Quantum Field Model*, Ann. Math. 100, 585 (1974).

[49] C. Gruber, H. Kunz, *General properties of polymer systems*, Comm. Math. Phys. 22, 133–161 (1971).

[50] A. Halfina, *The limiting equivalence of the canonical and grand canonical ensembles (low density case)*, Math. Sbornuk v.80 (122) N 1 (9), 3– 51 (1969).

[51] D. Iagolnitzer, B. Souillard, *On the Analyticity in the Potential in Classical Statistical Mechanics* Comm. Math. Phys. 60, 131-152 (1978).

[52] D. Iagolnitzer, B. Souillard, *Random fields and limit theorems* Colloq. Math. Soc. Janos Bolyai 27, Random Fields, Estergom Hungary, (1979).

[53] B. Jackson, A. Procacci, A. D. Sokal, *Complex zero-free regions at large $|q|$ for multivariate Tutte polynomials (alias Potts-model partition functions) with general complex edge weights*, preprint, arXiv:0810.4703 (2008).

[54] M. Kac, *Can one hear the shape of a drum*, Amer. Math. Monthly 73, 1–23 (1966).

[55] I. Karatzas, S. Shreve, *Brownian Motion and Stochastic Calculus*, Springer (1988).

[56] L. Koralov, *The Existence of Pair Potential Corresponding to Specified Density and Pair Correlation* arXiv:0903.0432v1 [math-ph] 3 Mar 2009.

[57] R. Kotecký, D. Preiss, *Cluster expansion for abstract polymer models*, Comm. Math. Phys. 103, 491–498 (1986).

[58] T. Kuna, J. L. Lebowitz, E. R. Speer, *Realizability of point processes* arXiv:math-ph/0612075, 2, 20 Mar 2007.

[59] S. Lang, *Real Analysis*, Addison-Wesley, London (1983).

[60] O. E. Lanford, D. Ruelle *Observables at infinity and states with short range correlations in statistical mechanics*, Comm. Math. Phys. 13, 194–215 (1969).

[61] T. D. Lee, C. N. Yang *Statistical theory of equations of state and phase transitions, I. Theory of condensation*, Phys. Rev. 87, 404–409 (1952).

[62] N. Macris, P. A. Martin, J. V. Pule, *Large volume asymptotics of Brownian integrals and orbital magnetism*, Ann. Inst. Henry Poincare 66:2, 147–183 (1997).

[63] N. Macris, P. A. Martin, J. V. Pule, *Large volume asymptotics of Brownian integrals and orbital magnetism*, Comm. Math. Phys. 117, 215–241 (1988).

[64] V. A. Malyshev, *Cluster expansions in lattice models of statistical physics and quantum field theory*, Russ. Math. Surveys 35, 1–62 (1980).

[65] V. Malyshev, R. Minlos, *Gibbs random fields: Cluster expansions*, Kluwer, Dodrecht (1991).

[66] *Bose-Einstein condensation: A mathematically unsolved problem*, Intern. Journal of Theoret. Phys. 44, 11, 2021–2027 (2005).

[67] S. Mase, *Mean characteristics of gibbsian point field*, Ann. of the Inst. of Stat. Math. 42, 2, 203–220 (1990).

[68] K. Mecke, *Integral geometry in statistical physics*, Intern. Journal of Modern Phys. 12, 9, 861–899 (1998).

[69] R. A. Minlos, *Gibbs' limit distribution* , Funk. Anal. i Pril. 2, 60–73 (1967).

[70] R. A. Minlos, *Regularity of Gibbs' limit distribution* , Funk. Anal. i Pril. 1, 40–53 (1967).

[71] R. A. Minlos, S. Poghosyan, *Estimates of Ursell functions, group functions, and their derivatives*, Theor. Math. Phys. 31, 408–418 (1977).

[72] R. A. Minlos, Ya. G. Sinai, *Some new results on first order phase transitions in lattice gas models*, Trans. Moscow Math. Soc. 17, 237–267 (1967).

[73] R. A. Minlos, Ya. G. Sinai, *The phenomenon of "phase separation" at low temperatures in some lattice models of gas. I*, Math. USSR-Sb. (3) 2, 335–395 (1967).

[74] R. A. Minlos, Ya. G. Sinai, *The phenomenon of "phase separation" at low temperatures in some lattice models of gas. II*, Trans. Moscow Math. Soc. 19, 121–196 (1968).

[75] D. Martirosian, Uspechi Mat. Nauk 30, N6, p. 181 (1975)

[76] D. Martirosian, Teor. Mat. Fiz. 22, 3, p. 335 (1975)

[77] S. Miracle-Solé, *On the convergence of cluster expansions*, Physica A 279, 244–249 (2000).

[78] B. Nahapetian, *Limit theorems and some applications in statistical physics*, Teubner - Verlag, Leipzig (1990).

[79] B. Nahapetian, *Strong mixing conditions for Gibbs random fields with descrete parameter and some applications*, Izvestia Acad. Nauk Arm. SSR 10, 242 –254 (1975).

[80] B. Nahapetian, *Limit theorems and statistical mechanics*, Random Fields Colloquia mathematica societatis Janos Bolyai 27 (1979).

[81] B. Nahapetian, *On a criterion of weak dependence*, Theory Probab. Appl. SSR 25, 374 – 381 (1980).

[82] B. Nahapetian, *An approach to limit theorems for dependent random variables*, Theory Probab. Appl. SSR 32, 589 – 595 (1987).

[83] B. Nahapetian, S. Poghosyan, *Decay of correlations in classical lattice spin systems with vacuum*, J. Contemp. Mathem. Analysis 30, 6, 31 – 46 (1995).

[84] B. Nahapetian, S. Poghosyan, *Estimate of convergence rate in local limit theorem for the particle number in spin systems*, Teor. Mat. Fiz. 95 , 497–512 (1993).

[85] B. Nehring, H. Zessin, *A path integral representation of the moment measures of the general ideal Bose gas*, Mathem. Nachr. 1-11 (2012).

[86] E.Nelson, *Feynman integrals and the Schrö*dinger equation, J. Math. Phys. 5, 332–343 (1964).

[87] B. O'Neil, *Elementary Differential Geometry*, Academic Press, New York (1966).

[88] O. Penrose, *Convergence of fugacity expansions for classical systems*, in "Statistical Mechanics, Foundations and Applications", Benjamin (1967).

[89] C.-E. Pfister, *Large deviations and phase separation in the two-dimensional Ising model*, Helv. Phys. Acta 64, 953–1054 (1991).

[90] S. Poghosyan, *Probabilities of large deviations for Gibbs random fields*, Izv. Akad. Nauk Arm SSR 25, 432–447, (1990).

[91] S. Poghosyan, *Large deviations for Gibbs random fields*, Uspekhi Mat. Nauk 36:2, 201-202 (1981).

[92] S. Poghosyan, *Asymptotic expansion of the logarithm of the partition function*, Commun. Math. Phys. 95, 227–245 (1984).

[93] S. Poghosyan, *Strong cluster properties of Ginibre gas. Quantum statistics*, J. Contemp. Mathem. Analysis 40, 4, 57–79 (2005).

[94] S. Poghosyan, *Asymptotic expansion of the log-partition function for a gas of interacting Brownian loops II*, J. Math. Phys. 51, 073302:1–22 (2010).

[95] S. Poghosyan, *Decay of correlations in a quantum gas with hard core potential*, Markov Processes Relat. Fields 18, 457-472 (2012)

[96] S. Poghosyan, H. Zessin, *Decay of correlations of the Ginibre gas obeying Maxwell-Boltzmann statistics*, Markov Processes Relat. Fields 7, 561 – 580 (2001).

[97] S. Poghosyan, H. Zessin, *Geometric expansion of the log-partition function for the Ginibre gas obeying Maxwell-Boltzmann statistics*, Markov Processes Relat. Fields 7, 581 – 593 (2001).

[98] S. Poghosyan, H. Zessin, *Existence of pressure for the Ginibre gas in n-connected domains*, J. Contemp. Mathem. Analysis 38, 63–73 (2003).

[99] S. Poghosyan, H. Zessin, *Asymptotic expansion of the log-partition function for a gas of interacting Brownian loops*, J. Math. Phys. 48, 093301:1–13 (2007).

[100] S. Poghosyan, H. Zessin, *An integral characterization of random permutations. A point process approach*, J.Contemp. Math. Analysis 45, 5, 67 – 76 (2010).

[101] S. Poghosyan, D. Ueltschi, *Abstract cluster expansion with applications to statistical mechanical systems*, J. Math. Phys. 50, 053509: 1–17 (2009).

[102] A. Procacci, *Abstract polymer models with general pair interactions*, arxiv.org/0707.0016 (version 2 of 26 Nov. 2008).

[103] M. Rafler, *Gaussian Loop- and Polya processes: A point process approach* Thesis, Potsdam (2009).

[104] M. Reed, B. Simon, *Methods of modern mathematical physics II*, Academic Press (1975).

[105] D. Ruelle, *Statistical mechanics: Rigorous results*, World Scientific (1999).

[106] D. Ruelle, *States of classical statistical mechanics*, J. Math. Phys. 8, 1657–1668 (1967).

[107] E. Seiler, *Gauge theories as a problem of constructive quantum field theory and statistical mechanics*, In Lecture Notes in Physics, Springer (1982).

[108] R. Seiringer, D. Ueltschi, *Rigorous upper bounds on the critical temperature of dilute Bose gases*, preprint (2009); arxiv.org:0904.0050

[109] B. Simon, *Functional integration and quantum physics*, Academic Press (1979)

[110] A. D. Sokal, *Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions*, Comb. Probab. Comput. 10, 41–77 (2001)

[111] D. Ueltschi, *Cluster expansions and correlation functions*, Moscow Math. J. 4, 511–522 (2004); math-ph/0304003

[112] D. Ueltschi, *Feynman cycles in the Bose gas* J. Math. Phys. 47, 123303:1–15 (2006)

[113] L. Van Hove, *Quelques proprietes generales de l'integrale de configuration d'un systeme de particules avec interaction*, Physica, 15, 951 – 961 (1949)