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On the Links between Miura Transformations of Bogoyavlensky Lattices and Inverse Spectral Problems for Band Operators

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Abstract. We consider semi-infinite and finite Bogoyavlensky lattices

$$\dot{a}_i = a_i \left(\prod_{j=1}^{p} a_{i+j} - \prod_{j=1}^{p} a_{i-j} \right),$$

$$\dot{b}_i = b_i \left(\sum_{j=1}^{p} b_{i+j} - \sum_{j=1}^{p} b_{i-j} \right),$$

for some $p \geq 1$, and Miura-like transformations between these systems, defined for $p \geq 2$. Both lattices are integrable (via Lax pair formalism) by the inverse spectral problem method for band operators, i.e., operators generated by band matrices. The key role in this method is played by the moments of the Weyl matrix of the corresponding band operator and their evolution in time. We find a description of the above-mentioned transformations in terms of these moments and apply this result to study finite Bogoyavlensky lattices and, in particular, their first integrals.

Key Words: Difference Operators, Inverse Spectral Problems, Nonlinear Lattices, Miura Transformations
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Introduction

Since the pioneering work of Gardner, Greene, Kruskal and Miura [12], the integration of nonlinear equations by using various inverse problems methods is among the main topics in modern mathematical physics. This integration task has inspired the development of many aspects of the theory of differential and difference operators (the inverse problems for the latter can be considered as a part of operator theory) as well as the areas of mathematics related to inverse problems. For almost half a century, see [16–18], the inverse spectral problems for difference operators have been applied for integration of certain nonlinear dynamical systems called nonlinear lattices. As an example of such an application, we mention the work by Berezanski [7], where the initial boundary value problem for the semi-infinite Toda lattice was solved by using the classical inverse spectral problem for Jacobi operators. Further on, the inverse spectral problem method was developed aiming to cover wider classes of nonlinear lattices, see e.g. [13, 21, 30]. Such activity has also inspired the development of the areas of the function theory connected with the study of difference operators [2, 3, 5, 28].

Turning to nonlinear integrable equations, note that an important role in their study is played by various Miura-like transformations which relate the equations and their solutions. For example, a discrete Miura transformation between Kac-van Moerbeke (Volterra) and Toda lattices allows one to derive the N–soliton solutions for both such systems starting from the trivial ones [15]. Also, as noted in [9, 10], such transformation maps onto each other the first integrals, Hamiltonians, Poisson brackets, master symmetries of these two systems (both of them are rich in these objects of interest of the integrable systems theory). In [22–24], an easy description of this transformation in terms of the inverse spectral data for Jacobi operators, which appear in the Lax representation for both lattices (including the matrix Volterra and Toda systems), was obtained. Note that it was recently found in [23, 25] that such transformations can be applied to the study of self-adjointness of Jacobi operators. The latter result illustrates that the links between the theory of nonlinear integrable equations and the operator theory are not based entirely upon the above-mentioned inverse problems.

Here we study similar transformations between Bogoyavlensky lattices [8, 29, 32] and obtain their description via the inverse spectral data for band operators (the latter may also be regarded as a high order difference ones [30]) which arise in the Lax pairs of such systems. We also apply this result to the case of finite Bogoyavlensky lattices. In particular, we show how our findings can be helpful for obtaining some "nonstandard" first integrals of these systems.

The paper is organized as follows. In the next section, the semi-infinite Bogoyavlensky lattices and their integration by means of the inverse spectral problem for band operators is considered. In Section 3, the above-mentioned description of Miura-like transformations between these lattices is obtained. In the final two sections, we consider the finite lattices and show how the previous result can be applied to study their first integrals.

1 Bogoyavlensky lattices. Inverse problem method

Consider the Cauchy problem for two nonlinear dynamical systems in the class of bounded solutions in the semi-infinite case:

$$\dot{a}_i = a_i \left(\prod_{j=1}^{r} a_{i+j} - \prod_{j=1}^{r} a_{i-j} \right) \quad \text{for a fixed} \quad r \in \mathbb{N}; \tag{1}$$

$$\dot{b}_i = b_i \left(\sum_{j=1}^{q} b_{i+j} - \sum_{j=1}^{q} b_{i-j} \right) \quad \text{for a fixed} \quad q \in \mathbb{N}; \tag{2}$$

for $i \in \mathbb{Z}_+$, where $a_i = a_i(t)$, $b_i = b_i(t)$, $a_i, b_i \in \mathbb{C}$, $t \in [0, T)$, $0 < T \leq \infty$; $a_i, b_i \neq 0$, $(a_i(t))_{i=0}^{\infty}$, $(b_i(t))_{i=0}^{\infty} \in l_\infty$; $b_l = a_l = 0$ for $l < 0$.

Both systems (1) and (2) (when $i \in \mathbb{Z}$) were introduced by Bogoyavlensky, see [8] and references thereafter, where it was shown that they can be regarded as discrete versions of the Korteweg-de Vries equation.

The system (1) admits the Lax representation $\dot{L} = [L, A]$ with matrices

$$\begin{aligned} L &= L1 = L1(t) = (L1_{ij})_{i,j=0}^{\infty} = \\ &= \begin{pmatrix} 0 & 0 & \dots & 0_{0,\,r-1} & 1 & 0 & 0 & 0 & \dots \\ a_0 & 0 & 0 & \dots & 0_{1,r} & 1 & 0 & 0 & \dots \\ 0 & a_1 & 0 & 0 & \dots & 0_{2,\,r+1} & 1 & 0 & \dots \\ 0 & 0 & a_2 & 0 & 0 & \dots & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \end{aligned} \tag{3}$$

$$\begin{aligned} A &= A1 = A1(t) = (A1_{ij})_{i,j=0}^{\infty} = \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0_{r,0} & 0 & 0 & 0 & 0 & 0 & \dots \\ a_0 \cdots a_r & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_1 \cdots a_{r+1} & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots \end{pmatrix}, \end{aligned} \tag{4}$$

and the Lax pair for the system (2) can be defined as follows

$$\begin{aligned} L &= L2 = L2(t) = (L2_{ij})_{i,j=0}^{\infty} = \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \dots & \ddots & \dots & \dots & \dots & \dots & \dots \\ 0_{q-1,\,0} & \dots & 0 & 1_{q-1,\,q} & 0 & 0 & 0 & \dots \\ b_0 & 0_{q,1} & \dots & 0 & 1_{q,\,q+1} & 0 & 0 & \dots \\ 0 & b_1 & 0 & \dots & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \dots \end{pmatrix}, \end{aligned} \tag{5}$$

$$
\begin{aligned}
A &= A2 = A2(t) = (A2_{ij})_{i,j=0}^{\infty} = \\
&= -\begin{pmatrix}
b_0 & 0 & \dots & 0_{0,\,q} & 1 & 0 & 0 & 0 & 0 & \dots \\
0 & b_0+b_1 & 0 & 0 & 0_{1,\,q+1} & 1 & 0 & 0 & 0 & \dots \\
\vdots & \dots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0_{q,\,0} & \dots & 0_{q,\,q-1} & \sum\limits_{i=0}^{q} b_i & 0 & \dots & 0_{q,\,2q} & 1 & 0 & \dots \\
0 & 0 & \dots & 0_{q+1,\,q} & \sum\limits_{i=1}^{q+1} b_i & 0 & \dots & 0 & 1 & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\end{aligned}
\tag{6}
$$

Both $L1$ and $L2$ are special cases of the matrix $M = (a_{i,k})_{i,k=0}^{\infty}$ with the following elements:

$$
\begin{gathered}
a_{i,k} \in \mathbb{C}, \quad a_{i,k} = 0, \quad k > i + r, \quad i > k + q, \\
a_{i,\,i+r} = 1, \quad a_{i+q,\,i} \neq 0, \quad i \geq 0;
\end{gathered}
\tag{7}
$$

i.e.,

$$
M = \begin{pmatrix}
a_{0,0} & \dots & 1_{0,\,r} & 0 & 0 & \dots & \dots & \dots \\
a_{1,0} & a_{1,1} & \dots & 1 & 0 & \dots & \dots & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots \\
a_{q,\,0} & a_{q\,,1} & a_{q,\,2} & \dots & \dots & 1_{q,\,q+r} & 0 & \dots \\
0 & a_{q+1,\,1} & a_{q+1,\,2} & a_{q+1,\,3} & \dots & \dots & 1 & \dots \\
0 & 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\tag{8}
$$

thus M is an infinite nonsymmetric band matrix which consists of $q + r + 1$ (possibly) nonzero diagonals. Denote as M any matrix of this structure. As we see, the matrix $L1$ fits into the case of M with $q = 1$, whereas for $L2$ the corresponding case is $r = 1$.

Denote by $l^2[0, \infty)$ the Hilbert space of the complex sequences $y = (y_n)_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} |y_n|^2 < \infty$, with inner product $(y, z) = \sum_{n=0}^{\infty} y_n \bar{z}_n$. Also, denote by $\{e_n\}_{n=0}^{\infty}$ its standard orthonormal basis. We identify the matrix M with the operator defined as the closure of the operator acting on the dense set of finite vectors from $l^2[0, \infty)$, where its action is described via matrix calculus (and keep the same notation M for this operator).

Now, consider in brief the inverse spectral problem for the operators M; for its full description, see e.g. [26]. First, for $\lambda \in \Omega(M)$, where $\Omega(M)$ is the resolvent set of M, we define the following functions named the Weyl solutions of M [26, 30, 31]:

$$
\begin{gathered}
\Phi(\lambda) = (\Phi_i(\lambda))_{i=0}^{\infty}, \quad \Phi_i(\lambda) = (\Phi_i^1(\lambda), \dots, \Phi_i^q(\lambda)), \\
\Phi_i^n(\lambda) := (R_\lambda e_{n-1})_i, \quad n = 1, \dots, q,
\end{gathered}
\tag{9}
$$

where $R_\lambda = (\lambda E - M)^{-1}$ is the resolvent of M and E is the identity operator.

In other words, $\Phi(\lambda) = (R_\lambda e_0, \ldots, R_\lambda e_{q-1})$. Also define the Weyl matrix for M as follows:

$$\mathfrak{M}(\lambda, M) = (\mathfrak{M}_{m,n}(\lambda, M))_{m=1,\ldots,r}^{n=1,\ldots,q}, \qquad \lambda \in \Omega(A),$$
$$\mathfrak{M}_{m,n}(\lambda, M) := \Phi_{m-1}^n(\lambda) = (R_\lambda e_{n-1})_{m-1} = (R_\lambda e_{n-1}, e_{m-1}). \tag{10}$$

For $q = 1$, $r = 1$, the matrix $\mathfrak{M}(\lambda, M)$ coincides with the Weyl function for the corresponding tridiagonal matrix of Jacobi type, and if J is the classical Jacobi operator, then $\mathfrak{M}(\lambda, J)$ is the Stieltjes transform of its spectral measure, see [1, 6, 7, 22, 23]. We also introduce the following system of formal power series with the parameter $\lambda \in \mathbb{C}\backslash\{0\}$:

$$W(\lambda) = (W_{m,n}(\lambda))_{m=1\ldots r}^{n=1\ldots,q},$$
$$W_{m,n}(\lambda) = \sum_{k=0}^{\infty} \frac{S_k^{m,n}}{\lambda^{k+1}}, \quad S_k^{m,n} = (M^k)_{m-1,n-1}, \tag{11}$$

where (M^k) is the k-th power of the matrix M. If the operator M is bounded (when $\sup_{i,j} \mid a_{i,i+j} \mid < \infty$, which holds for $L1$ and $L2$), the Neumann formula for its resolvent is valid for $\mid \lambda \mid > \|M\|$, and, as follows from (9),

$$\Phi_i^n(\lambda) = \sum_{k=0}^{\infty} \frac{(M^k)_{i,n-1}}{\lambda^{k+1}} \tag{12}$$

for all i and n. Therefore, for the bounded operators M, the functions $\mathfrak{M}_{m,n}(\lambda) = \Phi_{m-1}^n(\lambda)$ are holomorphic at infinity and $\mathfrak{M}_{m,n}(\lambda) = W_{m,n}(\lambda)$ in its neighborhood. This allows us to define, in the general case, the asymptotic expansion of the Weyl matrix of M at infinity, as the matrix (11):

$$\mathfrak{M}_\infty(\lambda, M) \stackrel{def}{=} W(\lambda).$$

Now we introduce the object which plays the key role in the considered inverse spectral problem method. Namely, the moment sequence of the Weyl matrix of M is defined by

$$S(\mathfrak{M}(\lambda, M)) = (S_k)_{k=0}^{\infty}, \quad S_k = \begin{pmatrix} S_k^{1,1} & \ldots & S_k^{1,q} \\ \vdots & & \vdots \\ S_k^{r,1} & \ldots & S_k^{r,q} \end{pmatrix},$$

where the elements $S_k^{m,n}$ for $m = 1, \ldots, r$, $n = 1, \ldots, q$ are defined by (11), and called the moments of $\mathfrak{M}(\lambda, M)$.

As in the tridiagonal matrix case [22], our inverse spectral problem admits the following formalization: given $S(\mathfrak{M}(\lambda, M))$, find M.

The elements of M can be recovered from $S(\mathfrak{M}(\lambda, M))$ in the recurrent manner starting from $a_{i+q,i}$, see [26], and the latter are obtained from the following formulas:

$$a_{i+q,i} = \frac{\Delta_{i+q}\Delta_{i-1}}{\Delta_{i+q-1}\Delta_i}, \quad i \geq 0, \tag{13}$$

where

$$\Delta_k = \det(H_k) \quad \text{for } H_k = (\alpha_{i,j})_{i,j=0}^k,\ \Delta_{-1} = 1, \quad \text{and } \alpha_{i,j} = S_{k1+k2}^{m1,n1}, \tag{14}$$

$$k1 = \left\lfloor \frac{i}{r} \right\rfloor, \quad k2 = \left\lfloor \frac{j}{q} \right\rfloor, \quad m1 = (i \bmod r) + 1, \quad n1 = (j \bmod q) + 1;$$

(we use the common notation: $r = a \bmod b$ for the remainder from integer division). Also, set

$$H = (\alpha_{i,j})_{i,j=0}^{\infty}. \tag{15}$$

Another important issue is a solvability criterion for the considered inverse problem method for M in terms of the moment sequence of its Weyl matrix. It can be formulated as follows.

Theorem 1 *The sequence*

$$S(\mathfrak{M}(\lambda, M)) = (S_k)_{k=0}^{\infty};\ S_k = \begin{pmatrix} S_k^{1,1} & \dots & S_k^{1,q} \\ \vdots & & \vdots \\ S_k^{r,1} & \dots & S_k^{r,q} \end{pmatrix},$$

$$S_k^{m,n} \in \mathbb{C}, \quad m = 1, \dots, r,\ n = 1, \dots, q,$$

is the moment sequence of the Weyl matrix of the operator M, *if and only if the following conditions hold:*

(i) normalization condition:

$$S_l^{m,n} = \delta_{m+lr,n}, \qquad n > lr, \qquad l = 0, \dots, \left[\frac{q}{r}\right];$$

(ii) for every $k \geq 0$, $\Delta_k \neq 0$, *where* Δ_k *are the determinants defined according to* (14).

For $r = 1$, $q = 2$, the proof is given in [19]; the case of arbitrary r and q can be proved similarly (see also [20, 30, 31] where a similar criterion was established for another classes of band operators). Note that from the proof it follows that there is a one-to-one correspondence between the matrices M and the sequences satisfying the above conditions $(i) - (ii)$. Obviously, the condition (ii) implies that

$$\operatorname{rank} H = \infty,$$

where H is defined in (15).

Using Theorem 1 (condition (i)) and (13) and applying induction on k, we establish the following result.

Lemma 1 *The determinants* Δ_k (14) *are calculated by the formulas*

$$\begin{aligned} \Delta_0 &= \cdots = \Delta_{q-1} = 1, \\ \Delta_{i+q} &= a_{i+q,\,i} \cdots a_{i+1,\,i-q+1} (a_{i,\,i-q} \cdots a_{i-q+1,\,i-2q+1})^2 \times \cdots \\ &\times\ (a_{i-(h-3)q,\,i-(h-2)q} \cdots a_{i-(h-2)q+1,\,i-(h-1)q+1})^{h-1} \times \\ &\times\ (a_{i-(h-2)q,\,i-(h-1)q} \cdots a_{q,\,0})^h; \end{aligned} \tag{16}$$

$i \geq 0,\ h = \left\lfloor \dfrac{i+q}{q} \right\rfloor$; $a_{i,k} = 1$ *for* $k < 0$

We now turn back to the systems (1)–(2) and, respectively, to the matrices $L1$ and $L2$. As follows from the above, for the eponymous operators the moment sequences are

$$S(\mathfrak{M}(\lambda, L1))(t) = \begin{pmatrix} S_k^{1,1} \\ \vdots \\ S_k^{r,1} \end{pmatrix}_{k=0}^{\infty} \qquad \text{where} \quad S_k^{m,1} = (L1^k)_{m-1,0}; \tag{17}$$

and

$$S(\mathfrak{M}(\lambda, L2))(t) = (S_k^{1,1}, \ldots, S_k^{1,q})_{k=0}^{\infty} \qquad \text{where} \quad S_k^{1,n} = (L2^k)_{0,\,n-1}. \tag{18}$$

Both $L1$ and $L2$ correspond to the case of "sparse" matrices M, i.e., such ones that in addition to (7) the conditions

$$a_{i,k} = 0 \quad \text{for} \quad k = i-q+1, \ldots, i+r-1$$

are fulfilled. In terms of $S(\mathfrak{M}(\lambda, L1))(t)$ and $S(\mathfrak{M}(\lambda, L2))(t)$, this property implies that for $k' \in \mathbb{Z}_+$,

$$S_k^{m,1} = 0, \quad m = 1, \ldots, r, \qquad k \neq m - 1 + (r+1)k'; \tag{19}$$

$$S_k^{1,n} = 0, \quad n = 1, \ldots, q, \qquad k \neq n - 1 + (q+1)k'. \tag{20}$$

For the complete proof of sparsity criterion for the matrices M in terms of $S(\mathfrak{M}(\lambda, M))$, see [26], Theorem 2. Another property of the moments of $S(\mathfrak{M}(\lambda, L1))(t)$ and $S(\mathfrak{M}(\lambda, L2))(t)$ can be established directly using (3), (5) and (17)–(18), namely,

Lemma 2 *For the elements of* $S(\mathfrak{M}(\lambda, L1))(t)$ *and* $S(\mathfrak{M}(\lambda, L2))(t)$ *defined according to* (17)–(18)

$$\begin{gathered} S_{m-1}^{m,1} = a_0 \ldots a_{m-2},\ m \geq 2; \qquad S_{r+1}^{1,1} = a_0 \ldots a_{r-1}; \\ S_{n-1}^{1,n} = 1, \qquad S_{n+q}^{1,n} = \sum_{l=0}^{n-1} b_l. \end{gathered} \tag{21}$$

Note that $S_0^{1,1} = 1$ according to the condition (i) of the Theorem 1.

Our next aim is to find the evolution equations for the moments. First we check that since $\dot{R}_\lambda(t) = -R_\lambda(t)(\lambda E - \dot{L}(t))R_\lambda(t)$, it follows from the Lax equation that

$$\begin{aligned}\dot{R}_\lambda(t) &= R_\lambda(t)\left(A(t)(\lambda I - L(t)) - (\lambda I - L(t))A(t)\right)R_\lambda(t) = \\ &= [R_\lambda(t), A(t)].\end{aligned} \tag{22}$$

Let R^1_λ and R^2_λ be the resolvents of $L1$ and $L2$ respectively. Denote by $(R^1_{i,j})_{i,j=0}^\infty$ and $(R^2_{i,j})_{i,j=0}^\infty$ their matrix representations in the basis $\{e_n\}$. As follows from (10), $\mathfrak{M}_{m,1}(\lambda, L1) = \mathfrak{M}_{m,1}(\lambda, L1)(t) = R^1_{m-1,0}$, $\mathfrak{M}_{1,n}(\lambda, L2) = \mathfrak{M}_{1,n}(\lambda, L2)(t) = R^2_{0,n-1}$. Then, using (22) and (3)–(4), we find that

$$\dot{R}^1_{i,0} = a_0 \dots a_r R^1_{i,r+1}; \quad i = 0, \dots, r-1. \tag{23}$$

From the identity

$$R^1_\lambda(\lambda E - L1) = E, \tag{24}$$

written in the matrix form, we get the following chain of relations

$$\begin{aligned} -R^1_{0,0} + \lambda R^1_{0,r} - a_r R^1_{0,r+1} &= 0, \\ \lambda R^1_{0,r-1} - a_{r-1} R^1_{0,r} &= 0, \\ &\vdots \\ \lambda R^1_{0,1} - a_1 R^1_{0,2} &= 0, \\ \lambda R^1_{0,0} - a_0 R^1_{0,1} &= 1; \end{aligned}$$

from which we obtain

$$a_r R^1_{0,r+1} = \frac{\lambda^{q+1} R^1_{0,0} - \lambda^q}{a_0 \dots a_{r-1}} - R^1_{0,0}.$$

Substituting the latter into (23) for $i = 0$, we find the equation for $\mathfrak{M}_{1,1}(\lambda, L1)$:

$$\dot{\mathfrak{M}}_{1,1}(\lambda, L1) = \lambda^{r+1}\mathfrak{M}_{1,1}(\lambda, L1) - \lambda^r - a_0 \dots a_{r-1}\mathfrak{M}_{1,1}(\lambda, L1).$$

Using (11)–(12) and (17), (21), we get the corresponding equation for the moments

$$\dot{S}^{1,1}_k = S^{1,1}_{k+r+1} - S^{1,1}_{r+1} S^{1,1}_k.$$

For $i = 1, \dots, r-1$, we derive from (24) the following relations

$$\begin{aligned} -R^1_{i,0} + \lambda R^1_{i,r} - a_r R^1_{i,r+1} &= 0, \\ \lambda R^1_{i,r-1} - a_{r-1} R^1_{i,r} &= 0, \\ &\vdots \\ \lambda R^1_{i,0} - a_0 R^1_{i,1} &= 0; \end{aligned}$$

from which we have

$$a_r R^1_{i,r+1} = \frac{\lambda^{q+1} R^1_{i,0}}{a_0 \dots a_{r-1}} - R^1_{i,0};$$

and the substitution of the latter into (23) leads to

$$\begin{aligned} \dot{R}^1_{m-1,0} = & \quad \dot{\mathfrak{M}}_{m,1}(\lambda, L1) = \\ = & \quad \lambda^{r+1}\mathfrak{M}_{m,1}(\lambda, L1) - a_0 \dots a_{r-1}\mathfrak{M}_{m,1}(\lambda, L1); \quad m = 2, \dots, r. \end{aligned}$$

Thus we arrive at the following equations for the elements of $S(\mathfrak{M}(\lambda, L1))(t)$

$$\dot{S}^{m,1}_k = S^{m,1}_{k+r+1} - S^{1,1}_{r+1} S^{m,1}_k; \quad m = 1, \dots, r; \tag{25}$$

which can be written in the equivalent form

$$S^{m,1}_k(t) = X(t)(S^{m,1}_k(0) + \int_0^t X(t_1)^{-1} S^{m,1}_{k+r+1}(t_1) dt_1); \tag{26}$$

where $X(t)$ is the solution of:

$$\dot{X}(t) = -S^{1,1}_{r+1}(t) X(t); \quad X(0) = 1,$$

and we find for the latter

$$X(t) = e^{-\int_0^t S^{1,1}_r(\tau)} = e^{-\int_0^t a_0(\tau) \dots a_{r-1}(\tau) d\tau}. \tag{27}$$

In order to get the equations for the elements of $S(\mathfrak{M}(\lambda, L2))(t)$, first, using the formula $\dot{R}^1_\lambda = [R^2_\lambda, A2]$, we establish the relations similar to (23), namely,

$$\dot{R}^2_{0,0} = R^2_{q+1,0}; \; \dot{R}^2_{0,j} = R^2_{q+1,j} - (\sum_{l=1}^{j} b_l) R^2_{0,j}; \quad j = 1, \dots, q-1. \tag{28}$$

From the matrix equation $(\lambda E - L2) R^2_\lambda = E$ and (28), acting similarly as above, we derive:

$$\begin{aligned} \dot{R}^2_{0,n-1} \quad = & \quad \dot{\mathfrak{M}}_{1,n}(\lambda, L2) = \\ = & \quad \lambda^{q+1}\mathfrak{M}_{1,n}(\lambda, L2) - \lambda^{q+1-n} - (\sum_{l=0}^{n-1} b_l)\mathfrak{M}_{1,n}(\lambda, L2); \quad n = 1, \dots, q; \end{aligned}$$

and using (21), we finally get

$$\dot{S}^{1,n}_k = S^{1,n}_{k+q+1} - S^{1,n}_{q+n} S^{1,n}_k; \quad n = 1, \dots, q. \tag{29}$$

Similarly to (26), we find that

$$S_k^{1,n}(t) = Y_n(t)(S_k^{1,n}(0) + \int_0^t Y_n(t_1)^{-1} S_{k+q+1}^{1,n}(t_1)dt_1); \tag{30}$$

and each Y_n is found from

$$\dot{Y}_n(t) = -S_{q+n}^{1,n}(t)Y_n(t); \quad Y_n(0) = 1.$$

In view of the above, we have obtained that if (1) and (2) have a solution, then the elements of $S(\mathfrak{M}(\lambda, L1))(t)$ and $S(\mathfrak{M}(\lambda, L2))(t)$ satisfy (25) and (29).

Assuming that $(a_i(t))_{i=0}^{\infty}$, $(b_i(t))_{i=0}^{\infty} \in l_\infty$ (which, in turn, implies that the operator norms $\|L1(t)\|$ and $\|L2(t)\|$ are bounded) and solving the integral equations (26) and (30) by iteration, one can get the following formulas for the moments:

$$S_k^{m,1}(t) = \frac{\sum_{l=0}^{\infty} \frac{t^l}{l!} S_{k+(r+1)l}^{m,1}(0)}{\sum_{l=0}^{\infty} \frac{t^l}{l!} S_{(r+1)l}^{1,1}(0)}; \; S_k^{1,n}(t) = \frac{\sum_{l=0}^{\infty} \frac{t^l}{l!} S_{k+(q+1)l}^{1,n}(0)}{\sum_{l=0}^{\infty} \frac{t^l}{l!} S_{n-1+(q+1)l}^{1,n}(0)}; \; k \in \mathbb{Z}_+. \tag{31}$$

Indeed, take $m = 1$ and $k = r + 1$. Applying (26) N times, we obtain the following equation for $S_{r+1}^{1,1}$:

$$S_{r+1}^{1,1} = X(t) \left(\sum_{l=0}^{N} \frac{t^l}{l!} S_{r+1+l(r+1)}^{1,1}(0) + \right.$$

$$\left. + \int_0^t \int_0^{t_1} \cdots \int_0^{t_N} X^{-1}(t_{N+1}) S_{(N+1)(r+1)}^{1,1}(t_{N+1}) dt_{N+1} \ldots dt_1 \right).$$

As follows from (17),

$$|S_{(N+1)(r+1)}^{1,1}(t_{N+1})| = |(L1^{(N+1)(r+1)}(t_{N+1}))_{0,0}| \leq \|(L1(t_{N+1})\|^{(N+1)(r+1)}.$$

Hence, due to our assumption and (27), we find that the integral in the above formula converges uniformly to 0 as $N \to \infty$ on $[0, T)$. Thus,

$$S_{r+1}^{1,1}(t) = X(t) \sum_{l=0}^{\infty} \frac{S_{r+1+l(r+1)}^{1,1}(0)t^l}{l!}. \tag{32}$$

From (26) we obtain that $S_{r+1}^{1,1}(t) = -\dot{X}(t)X^{-1}(t)$. On the other hand,

$$\frac{dX^{-1}(t)}{dt} = -\dot{X}(t)X^{-2}(t) = \left(\sum_{l=0}^{\infty} \frac{S_{r+1+l(r+1)}^{1,1}(0)t^l}{l!} \right).$$

Using this relation, we find that

$$X(t)^{-1} = \left(\sum_{l=0}^{\infty} \frac{S^{1,1}_{l(r+1)}(0) t^l}{l!} \right)$$

(here we used the sparsity condition (19)). Substituting the latter relation into (32), we obtain (31) for $m = 1$ and $k = r + 1$. Taking other values of k and using the above arguments we find that

$$S^{1,1}_k(t) = X(t) \sum_{l=0}^{\infty} \frac{S^{1,1}_{k+l(r+1)}(0) t^l}{l!},$$

from which (31) follows for $m = 1$ and all k. The cases for other values of m and n are considered similarly.

Note: Comparing (25) and (29), we see that all equations (25) contain the same multiplier $S^{1,1}_{r+1}$, and this is not the case for the equations (29), where each of the q equations has its own multiplier $S^{1,n}_{q+n}$. Further on, we will use this while studying the finite Bogoyavlensky lattices.

For arbitrary initial data $(a_i(0))_{i=0}^{\infty}, (b_i(0))_{i=0}^{\infty} \in l_\infty$, $a_i(0)$, $b_i(0) \neq 0$, the local existence and uniqueness theorem for (2) (in the class of bounded solutions) was established in [13, 21] (in fact, the lattices with matrix/operator coefficients were considered there); for the system (1), it can be proved in a similar way.

In view of the above, we obtain the following integration method for (1)–(2).

Theorem 2 *For each set of initial complex data* $(a_i(0))_{i=0}^{\infty}$, $(b_i(0))_{i=0}^{\infty} \in l_\infty$; $a_i(0)$, $b_i(0) \neq 0$, *there exists* $\delta > 0$ *such that the Cauchy problem for* (1)–(2) *has a unique solution for* $t \in [0, \delta)$ *which can be found in the following way:*

1. *Construct the matrices* $L1(0)$, $L2(0)$ *defined by* (3) *and* (5), *respectively, at* $t = 0$ *out of the initial data and find the moments* $S(\mathfrak{M}(\lambda, L1))(0)$ *and* $S(\mathfrak{M}(\lambda, L2))(0)$.

2. *Calculate the moments* $S(\mathfrak{M}(\lambda, L1))(t)$ *and* $S(\mathfrak{M}(\lambda, L2))(t)$ *according to* (31).

3. *Using* (13) *find the elements* $a_i(t)$, $b_i(t)$ *for* $i \in \mathbb{Z}_+$ *and* $t \in [0, \delta)$, *which give the required solution.*

If $a_i(0)$ and $b_i(0)$ are all real and positive (or all negative) then the global existence and uniqueness property for the solutions of (1) was established in [28]; a similar result for the system (2) was obtained in [3, 19]. As follows from the above considerations, in this case, our integration method can be applied to find the global solution. To our knowledge, it remains an open question for which (complex or real) initial data there exists the bounded solution of (1) or (2) for all $t \in [0, T)$.

2 Miura-like transformations

From now on, we will assume that in (1)–(2), $r = q = p$ for some $p \geq 2$. In this case, the Weyl matrices $\mathfrak{M}(\lambda, L1)$ and $\mathfrak{M}(\lambda, L2)$ are of the sizes $p \times 1$ and $1 \times p$, respectively, and formulas (13) are written as follows:

$$a_i = \frac{\Delta_{i+1}\Delta_{i-1}}{\Delta_i^2}, \qquad b_i = \frac{\Delta_{i+p}\Delta_{i-1}}{\Delta_{i+p-1}\Delta_i}, \qquad i \geq 0, \tag{33}$$

where Δ_i's are defined in (14).

As mentioned in [8], the systems

$$\dot{a}_i = a_i \left(\prod_{j=1}^{p} a_{i+j} - \prod_{j=1}^{p} a_{i-j} \right), \quad i \in \mathbb{Z};$$

after denoting

$$b_i = a_i \cdots a_{i+p-1}, \tag{34}$$

take the form

$$\dot{a}_i = a_i(b_{i+1} - b_{i-p}),$$

and differentiating both sides of (34), one gets

$$\dot{b}_i = b_i(\sum_{j=1}^{p} b_{i+j} - \sum_{j=1}^{p} b_{i-j}), \quad i \in \mathbb{Z}. \tag{35}$$

Conversely, fix an arbitrary $i = I$ in (35). Then, for $p \geq 2$, let $a_I, \ldots, a_{I+p-2}$ be solutions of the equations

$$\begin{aligned} \dot{a}_I &= a_I(b_{I+1} - b_{I-p}); \\ &\vdots \\ \dot{a}_{I+p-2} &= a_{I+p-2}(b_{I+p-1} - b_{I-2}). \end{aligned} \tag{36}$$

Then, by setting

$$a_{I+p-1} = \frac{b_I}{a_I \cdots a_{I+p-2}}, \tag{37}$$

from (35)–(36), one gets

$$\begin{aligned} \dot{a}_{I+p-1} &= \frac{b_I \left(\sum_{j=1}^{p} b_{I+j} - \sum_{j=1}^{p} b_{I-j} \right)}{a_I \cdots a_{I+p-2}} - \\ &- \frac{b_I \left((b_{I+1} - b_{I-p}) + (b_{I+2} - b_{I-p+1}) + \cdots + (b_{I+p-1} - b_{I-2}) \right)}{a_I \cdots a_{I+p-2}} \\ &= a_{I+p-1}(b_{I+p} - b_{I-1}). \end{aligned}$$

In the same manner, defining successively

$$a_i := \frac{b_{i-p+1}}{a_{i-p+1}\cdots a_{i-1}}, \quad \text{for} \quad i \geq I+p, \tag{38}$$

and

$$a_i := \frac{b_i}{a_{i+1}\cdots a_{i+p-1}}, \quad \text{for} \quad i \leq I-1; \tag{39}$$

for these values of i, one finds that

$$\dot{a_i} = a_i(b_{i+1} - b_{i-p}).$$

The later implies that an inverse to the transformation (34) can be defined according to (36)–(39).

We now turn back to the Cauchy problem for semi-infinite systems (1)–(2), considered in the previous section. The Miura transformation, as we call it (in [32] it was called a Bäcklund transformation; since Miura mappings may be regarded as a special cases of Bäcklund transforms, this name is also justified), defined by (34), maps the system (1) to the system (2) with initial conditions

$$b_i(0) = a_i(0)\cdots a_{i+p-1}(0).$$

The inverse Miura transformation for $p \geq 2$ from (2) to (1) is defined as follows:

$$a_i(t) = a_i(0)e^{\int_0^t b_{i+1}(\tau)d\tau} \text{ for some } 0 \neq a_i(0) \in \mathbb{C}, i = 0,\ldots,p-2; \tag{40}$$

and for $i \geq p-1$, it is defined recurrently as

$$a_i(t) = \frac{b_{i-p+1}(t)}{a_{i-p+1}(t)\cdots a_{i-1}(t)}. \tag{41}$$

Note that one can define the inverse Miura transformation starting from (36) and set $a_i(t) = a_i(0)\exp(\int_0^t (b_{i+1}(\tau) - b_{i-p}(\tau))d\tau)$ for certain complex $a_i(0) \neq 0$, $i = I,\ldots,I-p+2$. Then, using (39), one arrives at the elements $a_0(t),\ldots,a_{p-2}(t)$. However, if in (40) we set for $a_i(0)$ the values of the latter at $t = 0$, then, applying (40)–(41), we get the same semi-infinite system (1) with the same initial data as after using the transformations defined according to (36)–(39); in this sense, the two procedures are equivalent.

Our next aim here is to find the expression for these Miura transformations in terms of the moment sequences $S(\mathfrak{M}(\lambda, L1))(t)$ and $S(\mathfrak{M}(\lambda, L2))(t)$ introduced in the previous section. For convenience, denote their elements defined in (17)–(18) as

$$\begin{pmatrix} S_k^1 \\ \vdots \\ S_k^p \end{pmatrix} \quad \text{and} \quad (\tilde{S}_k^1,\ldots,\tilde{S}_k^p), \qquad k \in \mathbb{Z}_+,$$

respectively. Note that as follows from (19)–(20),

$$S_k^l = \tilde{S}_k^l = 0 \quad \text{for} \quad l = 1, \dots, p, \quad k \neq l - 1 + (p+1)k',\ k' \in \mathbb{Z}_+. \tag{42}$$

Theorem 3 *The Miura transformation* (34) *between the systems* (1) *and* (2) *for* $r = q = p \geq 2$ *can be described as the transformation* $S \to \tilde{S}$ *between* $S := S(\mathfrak{M}(\lambda, L1))(t)$ *and* $\tilde{S} := S(\mathfrak{M}(\lambda, L2))(t)$ *as follows:*

$$\frac{S_k^l(t)}{S_{l-1}^l(t)} = \tilde{S}_k^l(t), \quad l = 1, \dots, p; \quad k \in \mathbb{Z}_+. \tag{43}$$

Conversely, the transformation (40)–(41) *can be expressed as* $\tilde{S} \to S$ *in the following way:*

$$\begin{aligned} S_k^1(t) &= \tilde{S}_k^1(t), \\ S_k^l(t) &= a_0(0) \cdots a_{l-2}(0) e^{\displaystyle\int_0^t (\tilde{S}_{l+p}^l(\tau) - \tilde{S}_{p+1}^1(\tau)) d\tau} \tilde{S}_k^l(t), \end{aligned} \tag{44}$$

$l = 2, \dots, p$.

Proof. First consider (1) and the corresponding moment sequence S. In this case we have $\alpha_{i,j} = S_{k_1+j}^{m1}(t)$, $r = p$ in (14), and formula (16) reads

$$\Delta_k = a_{k-1} a_{k-2}^2 \cdots a_0^k, \qquad k \geq 1. \tag{45}$$

For $k \geq 0$, set

$$\hat{S}_k^l(t) = \frac{S_k^l(t)}{S_{l-1}^l(t)}, \quad l = 1, \dots, p; \tag{46}$$

and consider the determinants $\tilde{\Delta}_k = \tilde{\Delta}_k(t) = \det(\hat{\alpha}_{i,j})_{i,j=0}^k$ defined in (14) (with $\hat{\alpha}_{i,j} = \hat{S}_{k_1+j}^{m1}(t)$). Then, for $k = 0, \dots, p-1$, applying Lemma 2, we get

$$\tilde{\Delta}_k = \frac{\Delta_k}{S_k^{k+1}(t) \cdots S_0^1(t)} = \frac{a_{k-1} a_{k-2}^2 \cdots a_0^k}{S_k^{k+1}(t) \cdots S_0^1(t)} = 1. \tag{47}$$

For $k \geq p$, $k = hp + h_1$ and $h = \left\lfloor \dfrac{k}{p} \right\rfloor$, we have

$$\begin{aligned} \tilde{\Delta}_k &= \frac{\Delta_k}{(S_{p-1}^p(t) \cdots S_0^1(t))^h (S_{h_1}^{h_1+1}(t) \cdots S_0^1(t))} = \\ &= \frac{a_{k-1} a_{k-2}^2 \cdots a_0^k}{(a_{p-2} a_{p-3}^2 \cdots a_0^{p-1})^h (a_{h_1}^0 \cdots a_0^{h_1})} = \\ &= \frac{a_{k-1} a_{k-2}^2 \cdots a_{h_1+1}^{ph-1} (a_{h_1} \cdots a_0)^{ph}}{(a_{p-2} a_{p-3}^2 \cdots a_0^{p-1})^h} := \frac{\mathrm{I}}{\mathrm{II}} \end{aligned} \tag{48}$$

As in (34), set $b_k = a_k \dots a_{k+p-1}$. Then it can be checked that

$$\begin{aligned} & b_{k-vp} \cdots b_{k-(v+1)p+1} = \\ = \quad & a_{k-(v-1)p-1} a^2_{k-(v-1)p-2} \cdots a^p_{k-vp} a^{p-1}_{k-vp-1} \cdots a^2_{k-(v+1)p} a_{k-(v+1)p+1}, \end{aligned} \tag{49}$$

for $v = 1, \dots, h-1$. Applying (49), we rearrange the numerator in the right-hand side of (48) as follows:

$$\mathrm{I} = (b_{k-p} \cdots b_{k-2p+1})(b_{k-2p} \cdots b_{k-3p+1})^2 \cdots (b_{k-(h-1)p} \cdots b_{k-hp+1})^{h-1} \times \mathrm{III},$$

where

$$\mathrm{III} = a^h_{p+h_1-1} a^{2h}_{p+h_1-2} \cdots a^{(p-1)h}_{h_1+1} (a_{h_1} \cdots a_0)^{ph}. \tag{50}$$

Then from (50) and (48), we have

$$\begin{aligned} \frac{\mathrm{III}}{\mathrm{II}} &= \frac{a^h_{p+h_1-1} a^{2h}_{p+h_1-2} \cdots a^{ph}_{h_1} a^{ph}_{h_1-1} a^{ph}_0}{a^h_{p-2} a^{2h}_{p-3} a^{(p-h_1-1)h}_{h_1} \cdots a^{(p-1)h}_0} = \\ &= a^h_{p+h_1-1} a^{2h}_{p+h_1-2} \cdots a^{(h_1+1)h}_{p-1} a^{(h_1+1)h}_{p-2} \cdots a^{(h_1+1)h}_{h_1} a^{h_1 h}_{h_1-1} \cdots a^h_0 = (b_{h_1} \cdots b_0)^h. \end{aligned}$$

Substituting the latter formulas into (48), we finally get

$$\begin{aligned} \tilde{\Delta}_k \quad = \quad & (b_{k-p} \cdots b_{k-2p+1})(b_{k-2p} \cdots b_{k-3p+1})^2 \cdots (b_{k-(h-1)p} \cdots b_{k-hp+1})^{h-1} \times \\ \times \quad & (b_{h_1} \cdots b_0)^h. \end{aligned} \tag{51}$$

Comparing (47) and (51) with formulas (16) corresponding to the special case of matrices $L2$ (when $q = p$), we find that they coincide with each other. Due to (33), (42) and Theorem 1 (its condition (i) follows from (42) and (46); the condition (ii) follows from the fact that (ii) is fulfilled for the moment sequence S), this coincidence implies that $(\hat{S}_1(t), \dots, \hat{S}_p(t))$ is the moment sequence $\tilde{S} = S(\mathfrak{M}(\lambda, L2))(t)$ of the matrix $L2$ with the coefficients b_i defined by (34). Thus, the "direct" part of the theorem is proved.

To prove the converse part, we consider the system (2), the moment sequence $\tilde{S}$ and the corresponding determinants $\tilde{\Delta}_k$ defined according to (14) and satisfying (51). Then we consider the sequence $S = S(t)$ with the elements defined from (44). Using the latter, we set

$$a_i(t) = \frac{S^{i+2}_{i+1}(t)}{S^{i+1}_{i+1}(t)} = a_i(0) e^{\displaystyle\int_0^t (\tilde{S}^{i+2}_{i+p+2}(\tau) - \tilde{S}^{i+1}_{i+p+1}(\tau)) d\tau}, \quad i = 0, \dots p-2.$$

As follows from Lemma 2, $\tilde{S}^{i+2}_{i+p+2}(\tau) - \tilde{S}^{i+1}_{i+p+1}(\tau) = b_{i+1}(\tau)$, and we arrive at formula (40). Then we consider the determimants Δ_k defined according to (14), where $\alpha_{i,j} = S^{m1}_{k_1+j}(t)$, and show, reversing the arguments used to prove the "direct" part and applying (41), that they satisfy (45) (in the latter, the elements a_k for $k \geq p-1$ are found from (41)). Using this fact and Theorem 1, we find that S is the moment sequence of the matrix $L1(t)$ which appears in the Lax representation (3)–(4) for the system (1) with the elements satisfying (40)–(41). □

3 Finite case. First integrals

Now consider the finite systems (1)–(2), namely,

$$\dot{a}_i = a_i \left(\prod_{j=1}^{p} a_{i+j} - \prod_{j=1}^{p} a_{i-j} \right); \tag{52}$$

$$\dot{b}_i = b_i \left(\sum_{j=1}^{p} b_{i+j} - \sum_{j=1}^{p} b_{i-j} \right); \tag{53}$$

where $i = 0, \ldots, N$; $a_i, b_i \in \mathbb{C}$, $t \in [0, T)$, $0 < T \leq \infty$; $a_i, b_i \neq 0$, $b_l = a_l = 0$ for $l < 0$ and $l > N$; for a certain $p \geq 1$ and $N \geq 2p - 1$. They can be called the Bogoyavlensky lattices with open-end boundary conditions [29]. As in the semi-infinite case, the system (52) admits the Lax representation with the matrix

$$L = L1_N = \begin{pmatrix} 0 & 0 & \ldots & 0_{0,\,p-1} & 1 & 0 & 0 & \ldots & 0 \\ a_0 & 0 & 0 & \ldots & 0_{1,p} & 1 & 0 & \ldots & 0 \\ 0 & a_1 & 0 & 0 & \ldots & 0_{2,\,p+1} & 1 & \ldots & 0 \\ \vdots & \ldots & \ddots & \ldots & \ldots & \vdots & \ldots & \ddots & \vdots \\ 0 & \ldots & 0 & a_{N-1} & 0 & 0 & 0 & \ldots & 1 \\ 0 & \ldots & 0 & 0 & a_N & 0 & 0 & \ldots & 0 \end{pmatrix}$$

of order $N+2 \times N+2$ while the L matrix for the system (53) takes the form

$$L = L2_N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \ldots & \ddots & \ldots & \vdots & \vdots & \ldots & \vdots \\ 0_{p-1,\,0} & \ldots & 0 & 1_{p-1,\,p} & 0 & 0 & \ldots & 0 \\ b_0 & 0_{p,\,1} & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & b_1 & 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \ldots & \ddots & \ldots & \vdots & \ldots & \ddots & \vdots \\ 0 & \ldots & 0 & b_{N-1} & 0 & 0 & \ldots & 1 \\ 0 & \ldots & 0 & 0 & b_N & 0 & \ldots & 0 \end{pmatrix},$$

and its order is $N+p+1 \times N+p+1$. Both of them are special cases of the matrices M_N; the latter may be regarded as leading principal submatrices of order $N+q+1$ of the above considered matrices M defined by (8):

$$M_N = \begin{pmatrix} a_{0,0} & \dots & 1_{0,\,r} & 0 & 0 & \dots & 0_{0,\,q+N} \\ \vdots & \dots & \vdots & \ddots & \dots & \vdots & \vdots \\ a_{q,\,0} & a_{q,\,1} & a_{q,2} & \dots & 1_{q,\,q+r} & \dots & 0 \\ \vdots & \ddots & \vdots & \dots & \dots & \ddots & \vdots \\ 0 & \dots & a_{q+N-r,\,N-r} & \dots & \dots & \dots & 1_{q+N-r,\,q+N} \\ \vdots & \vdots & \ddots & \dots & \dots & \dots & \vdots \\ \\ \vdots & \vdots & \dots & \ddots & \dots & \dots & \vdots \\ 0 & 0 & \dots & 0 & a_{q+N,\,N} & \dots & a_{q+N,\,q+N} \end{pmatrix},$$

The inverse spectral problem method considered in Section 2, including the reconstruction algorithm, is applicable to M_N as well. The condition (i) of Theorem 1 remains the same, while the condition (ii) is replaced by

(ii_N) *For* $k = 0, \dots, q+N$, $\Delta_k \neq 0$ and

$$\operatorname{rank} H = N+q+1, \tag{54}$$

where H *is defined in* (15).

Due to the Hankel type structure of matrix H (for $r = q = 1$, this is an infinite Hankel matrix. As known, its subsequent rows/columns are the "shortened" versions of the preceding ones, see [11], Chapter XV, Theorem 7, and this property is retained in the general case of matrix H) the condition (54) is equivalent to

$$\alpha_{i,j} = \sum_{v=0}^{N+q} C_v \alpha_{i,j-v-1}, \quad C_v \in \mathbb{C}, \quad j \geq N+q+1, \tag{55}$$

or

$$\alpha_{i,j} = \sum_{v=0}^{N+q} D_v \alpha_{i-v-1,j}, \quad D_v \in \mathbb{C}, \quad i \geq N+q+1, \tag{56}$$

for certain sets $C_0, \dots, C_{N+q}$ or $D_0, \dots, D_{N+q}$, and $N+q$ is the least number for which (55)–(56) are fulfilled. Also, if one of these conditions is hold, then the second one is hold as well (the row rank of a matrix is equal to its column rank). We call $C_0, \dots, C_{N+q}$ and $D_0, \dots, D_{N+q}$ the finite rank coefficients (FRC) of the matrix H. Here, we will not give the proof of analogue of Theorem 1 for the matrices M_N, instead we refer to [20] where a similar result was established for another class of finite band matrices (see also [13]).

As in the previous section, we denote as

$$S(\mathfrak{M}(\lambda, L1_N))(t) := S_N(t) = S_N = \begin{pmatrix} S_k^1 \\ \vdots \\ S_k^p \end{pmatrix}_{k=0}^{\infty}$$

and

$$S(\mathfrak{M}(\lambda, L2_N))(t) := \tilde{S}_N(t) = \tilde{S}_N = (\tilde{S}_k^1, \ldots, \tilde{S}_k^p)_{k=0}^{\infty}$$

the moment sequences of the Weyl matrices corresponding to $L1_N$ and $L2_N$ respectively. The relations (55) and (56) for the elements of S_N and $\tilde{S}_N$ can be written as

$$S_k^l = \sum_{v=0}^{N+1} C_v S_{k-v-1}^l, \qquad k \geq N+2; \tag{57}$$

$$S_{k_l+k}^{m_l} = \sum_{v=0}^{N+1} D_v S_{\hat{k}_{l,v}+k}^{\hat{m}_{l,v}}, \quad k \geq 0; \qquad l = 1, \ldots, p, \tag{58}$$

where

$$k_l = \left\lfloor \frac{N+1+l}{p} \right\rfloor, \qquad \hat{k}_{l,v} = \left\lfloor \frac{N+l-v}{p} \right\rfloor,$$

$$m_l = (N+1+l \bmod p) + 1, \quad \hat{m}_{l,v} = (N+l-v \bmod p) + 1$$

(as mentioned above, the order of $L1_N$ and, consequently, the rank of H equals to $N+2$);

$$\tilde{S}_k^l = \sum_{v=0}^{N+p} \tilde{C}_v \tilde{S}_{k-v-1}^l, \qquad k \geq N+p+1; \tag{59}$$

$$\tilde{S}_{\tilde{k}_l+k}^{\tilde{n}_l} = \sum_{v=0}^{N+p} \tilde{D}_v \tilde{S}_{\tilde{k}_{l,v}+k}^{\tilde{n}_{l,v}}, \quad k \geq 0; \qquad l = 1, \ldots, p, \tag{60}$$

where

$$\tilde{k}_l = \left\lfloor \frac{N+p+l}{p} \right\rfloor, \quad \tilde{k}_{l,v} = \left\lfloor \frac{N_v}{p} \right\rfloor, \; N_v = N+p+l-(v+1),$$

$$\tilde{n}_l = (N+p+l \bmod p) + 1, \quad \tilde{n}_{l,v} = (N_v \bmod p) + 1$$

(in the $L2_N$ case, the rank of H is $N+p+1$).

The integration procedure for the semi-infinite Bogoyavlensky lattices considered in Section 2 can be applied for integration of their finite counterparts as well. In particular, formulas (25) and (29) are valid for the elements of $S_N(t)$ and $\tilde{S}_N(t)$. An additional question here is how the finite rank coefficients defined in (57)–(60) evolve in time. It turns out that all of them except $\tilde{D}_v$ are time independent. Namely, the following result holds for the systems (52)–(53).

Theorem 4 *The FRC* $C := \{C_0, \dots, C_{N+1}\}$ *and* $D := \{D_0, \dots, D_{N+1}\}$ *introduced in* (57)–(58) *are the integrals of motion (first integrals of) the system* (52)*, while the FRC* $\tilde{C} := \{\tilde{C}_0, \dots, \tilde{C}_{N+p}\}$ *defined in* (59) *are the first integrals of the system* (53)*.*

Proof. First, consider the sequence $S_N(t)$. In accordance with (14), for a fixed $j \in \mathbb{Z}_+$, the j-th column of the corresponding matrix $H = H(t)$ consists of the moments $\alpha_{k,j} = S^{(k \bmod p)+1}_{\lfloor \frac{k}{p} \rfloor + j}$, $k = 0, 1, \dots$. We apply (57) and (25) (here, $r = p$) to the first $N+2$ elements of the $(N+2)$-th column of H. In particular, for $\alpha_{0,N+2} = S^1_{N+2}$, we have

$$\dot{S}^1_{N+2} = \sum_{v=0}^{N+1} C_v \dot{S}^1_{N+1-v} \ + \ \sum_{v=0}^{N+1} \dot{C}_v S^1_{N+1-v} =$$

$$= \sum_{v=0}^{N+1} C_v (S^1_{N+1-v+p+1} - S^1_{p+1} S^1_{N+1-v}) \ + \ \sum_{v=0}^{N+1} \dot{C}_v S^1_{N+1-v} =$$

$$= S^1_{N+p+3} - S^1_{p+1} S^1_{N+2} + \sum_{v=0}^{N+1} \dot{C}_v S^1_{N+1-v} \ = \ \dot{S}^1_{N+2} + \sum_{v=0}^{N+1} \dot{C}_v S^1_{N+1-v}.$$

Thus,

$$\sum_{v=0}^{N+1} \dot{C}_v S^1_{N+1-v} = 0.$$

After that, applying successively this procedure $N+1$ times to $\alpha_{1,N+2}, \dots$, $\alpha_{N+1,N+2}$, where

$$\alpha_{N+1,N+2} = S^{(N+1 \bmod p)+1}_{\lfloor \frac{N+1}{p} \rfloor + N+2} = S^{\hat{m}_{1,0}}_{\hat{k}_{1,0}+N+2},$$

we arrive at the system of equations

$$\begin{array}{ccccccc} \dot{C}_0 S^1_{N+1} & + & \dots & + & \dot{C}_{N+1} S^1_0 & = & 0, \\ \vdots & & & & \vdots & & \\ \dot{C}_0 S^{(N+1 \bmod p)+1}_{\lfloor \frac{N+1}{p} \rfloor + N+1} & + & \dots & + & \dot{C}_{N+1} S^{(N+1 \bmod p)+1}_{\lfloor \frac{N+1}{p} \rfloor} & = & 0. \end{array} \tag{61}$$

By the above-stated condition (ii_N) (here, $q = 1$), its determinant $\Delta_{N+1} \neq 0$, therefore, $\dot{C}_0 = \dots = \dot{C}_{N+1} = 0$.

Note that for a fixed $i \in \mathbb{Z}_+$, the i-th row of the matrix H consists of the moments $\alpha_{i,k} = S^{(i \bmod p)+1}_{\lfloor \frac{i}{p} \rfloor + k}$, $k = 0, 1, \dots$ We set $i = N+2$ and apply (58) and (25) to the first $N+2$ elements of the $(N+2)$-th row. First, take

$\alpha_{N+2,0} = S_{k_1}^{m_1}$. Differentiating with respect to t both sides of the relation $S_{k_1}^{m_1} = \sum_{v=0}^{N+1} D_v S_{\hat{k}_{1,v}}^{\hat{m}_{1,v}}$ and applying (25), we get

$$\dot{S}_{k_1}^{m_1} = \sum_{v=0}^{N+1} D_v(S_{\hat{k}_{1,v}+p+1}^{\hat{m}_{1,v}} - S_{p+1}^1 S_{\hat{k}_{1,v}}^{\hat{m}_{1,v}}) + \sum_{v=0}^{N+1} \dot{D}_v S_{\hat{k}_{1,v}}^{\hat{m}_{1,v}} =$$

$$= S_{k_1+p+1}^{m_1} - S_{p+1}^1 S_{k_1}^{m_1} + \sum_{v=0}^{N+1} \dot{D}_v S_{\hat{k}_v}^{\hat{m}_v} = \dot{S}_{k_1}^{m_1} + \sum_{v=0}^{N+1} \dot{D}_v S_{\hat{k}_{1,v}}^{\hat{m}_{1,v}},$$

and we find that $\sum_{v=0}^{N+1} \dot{D}_v S_{\hat{k}_{1,v}}^{\hat{m}_{1,v}} = 0$ (here, $S_{\hat{k}_{1,N+1}}^{\hat{m}_{1,N+1}} = S_0^1$). Then applying this procedure to the remaining $N+1$ elements $(\alpha_{N+2,1}, \dots, \alpha_{N+2,N+1}) = (S_{k_1+1}^{m_1}, \dots S_{k_1+N+1}^{m_1})$, we obtain the system

$$\begin{array}{ccc} \dot{D}_0 S_{\hat{k}_{1,0}}^{\hat{m}_{1,0}} + & \dots & + \dot{D}_{N+1} S_0^1 = 0 \\ & \vdots & \vdots \\ \dot{D}_0 S_{\hat{k}_{1,0}+N+1}^{\hat{m}_{1,0}} + & \dots & + \dot{D}_{N+1} S_{N+1}^1 = 0. \end{array}$$

Again, its determinant $\Delta_{N+1} \neq 0$, and $\dot{D}_0 = \dots = \dot{D}_{N+1} = 0$ as well.

Further on, consider the sequence $\tilde{S}_N(t)$ and the corresponding matrix $H = H(t)$. By (14), for a fixed i the i-th row of H consists of the moments $\alpha_{i,k} = \tilde{S}_{\lfloor \frac{k}{p} \rfloor + i}^{(k \bmod p)+1}$, $k = 0, 1, \dots$ Set i to be $N+p+1$ and take its first p elements: $\tilde{S}_{N+p+1}^l$, $l = 1, \dots, p$. Acting as above and using (59) and (29), we obtain

$$\dot{\tilde{S}}_{N+p+1}^l = \sum_{v=0}^{N+p} \tilde{C}_v \dot{\tilde{S}}_{N+p-v}^l + \sum_{v=0}^{N+p} \dot{\tilde{C}}_v \tilde{S}_{N+p-v}^l =$$

$$= \sum_{v=0}^{N+p} \tilde{C}_v (\tilde{S}_{N+p-v+p+1}^l - \tilde{S}_{p+1}^l \tilde{S}_{N+p-v}^l) + \sum_{v=0}^{N+p} \dot{\tilde{C}}_v \tilde{S}_{N+p-v}^l =$$

$$= \tilde{S}_{N+2p+2}^l - \tilde{S}_{p+1}^l \tilde{S}_{N+p+1}^l + \sum_{v=0}^{N+p} \dot{\tilde{C}}_v \tilde{S}_{N+1-v}^l = \dot{\tilde{S}}^l{}_{N+p+1} + \sum_{v=0}^{N+p} \dot{\tilde{C}}_v \tilde{S}_{N+p-v}^l.$$

Therefore, $\sum_{v=0}^{N+p} \dot{\tilde{C}}_v \tilde{S}_{N+p-v}^l = 0$ for $l = 1, \dots, p$. After repeating this procedure $N+1$ times for the elements $\alpha_{N+p+1,p}, \dots, \alpha_{N+p+1,N+p}$, we obtain the linear homogenous system similar to (61) for the unknowns $\dot{\tilde{C}}_0, \dots, \dot{\tilde{C}}_{N+p}$ with the determinant $\tilde{\Delta}_{N+p} \neq 0$, and we immediately find that

$$\dot{\tilde{C}}_0 = \dots = \dot{\tilde{C}}_{N+p} = 0.$$

□

Note that an analogues result was obtained in [14] and [22], where a similar approach based upon an inverse problem method was applied to integration of finite Volterra and Toda lattices and discrete modified Korteweg–de Vries equation in the finite case. In these works H is a Hankel matrix of a finite rank.

Remark. Unlike the case of D, the above arguments are not applicable to $\tilde{D} := \{\tilde{D}_0, \ldots, \tilde{D}_{N+p}\}$ due to the different structure of equations (25) and (29) (see the note before Theorem 2). As we will see below, the latter, generally speaking, are not the first integrals of (53).

It can be easily verified (see e.g. [4]) that if a finite dynamical system satisfies the Lax equation, then the coefficients of characteristic polynomial of the corresponding L matrix (and, therefore, its eigenvalues) are the first integrals of the system. Below we establish the relations between the above introduced FRC and these coefficients.

Proposition 1 *The FRC C and $\tilde{C}$ defined in Theorem 4 coincide up to the sign with the coefficients (except for the leading ones) of characteristic polynomials of the matrices $L1_N$ and $L2_N$, respectively.*

Proof. First, consider the matrix $L1_N$ and its characteristic polynomial

$$P_{L1_N}(\lambda) := \det(\lambda I - L1_N) = \lambda^{N+2} + c_0\lambda^{N+1} + \cdots + c_{N+1},$$

where I is identity matrix. As follows from the Cayley-Hamilton theorem,

$$L1_N^{N+2+k'} + c_0 L1_N^{N+1+k'} + \cdots + c_{N+1} L1_N^{k'} = O, \quad k' \in \mathbb{Z}_+ \tag{62}$$

(here O is a zero $N+2 \times N+2$ matrix). According to the definition, $S_k^l = (L1_N^k)_{l-1,0}$, $l = 0, \ldots, p$, $k \in \mathbb{Z}_+$. Using equation (62) for $k' = 0, \ldots, \lfloor \frac{N+1}{p} \rfloor = \hat{k}_{1,0}$, we construct the system of $N+2$ equations

$$\begin{array}{rcl} S_{N+2}^1 &=& -c_0 S_{N+1}^1 - \ldots - c_{N+1} S_0^1, \\ S_{N+2}^2 &=& -c_0 S_{N+1}^2 - \ldots - c_{N+1} S_0^2, \\ \vdots & & \vdots \\ S_{\hat{k}_{1,0}+N+2}^{\hat{m}_{1,0}} &=& -c_0 S_{\hat{k}_{1,0}+N+1}^{\hat{m}_{1,0}} - \cdots - c_{N+1} S_{\hat{k}_{1,0}}^{\hat{m}_{1,0}}; \end{array} \tag{63}$$

its determinant $\Delta_{N+1} \neq 0$. Comparing (63) with (57), one can find that $C_0 = -c_0, \ldots, C_{N+1} = -c_{N+1}$. The case of $\tilde{C}_0, \ldots, \tilde{C}_{N+p}$ and $L2_N$ is considered similarly. □

In view of the above, $C_0, \ldots, C_{N+1}$ and $\tilde{C}_0, \ldots, \tilde{C}_{N+p}$ may be regarded as "standard" first integrals of the finite systems (52) and (53) respectively, whereas $D_0, \ldots, D_{N+1}$ as nonstandard first integrals of (52). Note that for non-Abelian finite Bogoyavlensky lattices (e.g. lattices with matrix elements), it can be also shown that the corresponding (matrix) finite rank coefficients $C_0, \dots, C_{N+1}$ and $\tilde{C}_0, \dots, \tilde{C}_{N+p}$ are their first integrals (in [14], a similar result was established for non-Abelian discrete modified KdV equation in the finite case) and they are not directly linked with the characteristic polynomials of the corresponding Lax matrices (the existence of possible links between these objects is, to our knowledge, an open issue).

At the end of this section, consider the Miura transformation between (52) and (53). To define it correctly, see (34), we set N to be equal to n_0 for system (53) and $N = n_0 + p - 1$ for certain $n_0 \geq 2p - 1$ for (52). All findings of the previous section, including the analog of Theorem 3 for the sequences S_N and $\tilde{S}_N$, hold unaltered. To prove the latter, one should take the determinants Δ_k with k ranging from 0 to $n_0 + p$, rather then $k \in \mathbb{Z}_+$ as in the semi-infinite case. It is of interest to study the behavior of the first integrals of these systems in relation to the Miura transformation. For the finite Volterra and Toda lattices this issue was considered in [22]. As already mentioned, for both such lattices the corresponding moment matrix H is a Hankel matrix of a finite rank. It was shown that the finite rank coefficients of both moment matrices, as in the above Proposition, are coincide with the coefficients of characteristic polynomials of their (finite) Lax matrices L. These coefficients are transformable into each other by Miura-like mapping between Volterra and Toda lattices considered in [22], and that is all the information about the first integrals we can get from studying such mapping. In what follows, we will show that the situation is more promising in the case of finite Bogoyavlensky lattices.

4 Example

Consider, as an example, for $p = 2$ system (52) with $N = 4$:

$$\begin{cases} \dot{a}_0 = a_0 a_1 a_2, \\ \dot{a}_1 = a_1 a_2 a_3, \\ \dot{a}_2 = a_2 a_3 a_4 \; - \; a_2 a_1 a_0, \\ \dot{a}_3 = -a_3 a_2 a_1, \\ \dot{a}_4 = -a_4 a_3 a_2; \end{cases} \tag{64}$$

and, respectively, system (53) with $N = 3$:

$$\begin{cases} \dot{b}_0 = b_0(b_1 + b_2), \\ \dot{b}_1 = b_1(b_2 + b_3 - b_1), \\ \dot{b}_2 = b_2(b_3 - b_1 - b_0), \\ \dot{b}_3 = b_3(-b_2 - b_1). \end{cases} \tag{65}$$

Let S_N and $\tilde{S}_N$ be the corresponding moment sequences which consist of the elements defined by (17)–(18) with the matrices $L1_4$ and $L2_3$, respectively. According to (42), $S_k^1 = \tilde{S}_k^1 = 0$, $(k \bmod 3) \neq 0$, $S_k^2 = \tilde{S}_k^2 = 0$, $(k \bmod 3) \neq 1$. Then, the rank of matrix H defined by (15) equals to 6 for the both systems (64) and (65) as well as the order of the matrices $L1_4$ and $L2_3$. For the elements of S_N, it can be checked (e.g. by solving two 6×6 systems composed of the equations (57) and (58), respectively, in the unknowns C_v and D_v) that in accordance with (57)–(58),

$$\begin{aligned} S_k^l &= (a_0a_1 + a_1a_2 + a_2a_3 + a_3a_4)S_{k-3}^l - (a_0a_1a_3a_4)S_{k-6}^l, \quad k \geq 6,\\ S_{k_l+k}^{m_l} &= (a_1 + a_3)S_{\hat{k}_{l,2}+k}^{\hat{m}_{l,2}} - (a_0a_3)S_{\hat{k}_{l,5}+k}^{\hat{m}_{l,5}}, \qquad l = 1, 2; \end{aligned}$$

where, as found from (58) $k_1 = 3$, $m_1 = 1$, $\hat{k}_{1,2} = 1$, $\hat{m}_{1,2} = 2$, $\hat{k}_{1,5} = 0$, $\hat{m}_{1,5} = 1$, $k_2 = 3$, $m_2 = 2$, $\hat{k}_{2,2} = 2$, $\hat{m}_{2,2} = 1$, $\hat{k}_{2,5} = 0$, $\hat{m}_{2,5} = 2$. In particular,

$$S_3^1 = (a_1 + a_3)S_1^2 - (a_0a_3)S_0^1. \tag{66}$$

Thus, for the matrix $L1_4$, the nonzero elements in the FRC sets C and D defined in Theorem 4 are $\{a_0a_1 + a_1a_2 + a_2a_3 + a_3a_4, -a_0a_1a_3a_4\}$ and $\{a_1 + a_3, -a_0a_3\}$, respectively, and all of them are the first integrals of (64). As follows from Proposition 1, the coefficients of characteristic polynomial of the matrix $L1_4$ coincide with the set $-C$, and the polynomial itself is written as

$$P_{L1_4}(\lambda) = \lambda^6 - (a_0a_1 + a_1a_2 + a_2a_3 + a_3a_4)\lambda^3 + a_0a_1a_3a_4.$$

Note that $\sum_{i=0}^{3} a_ia_{i+1} = \mathrm{Tr}(L^3)/3$ for $L = L1_4$.

We now turn to the system (65) and to the Miura transformations between (64) and (65). Since $S_0^1 = 1$, we find from (43) that $\tilde{S}_k^1 = S_k^1$. Then, using the first of the equations (59) (for $l = 1$), we find that for the matrix $L2_3$ with the elements b_i defined from Miura mapping (34), the set $\tilde{C}$ (Theorem 4) coincides with C, namely,

$$\tilde{C} = \{a_0a_1+a_1a_2+a_2a_3+a_3a_4,\ -a_0a_1a_3a_4\} = \{b_0+b_1+b_2+b_3, -b_0b_3\} \tag{67}$$

and gives the "standard" (see previous section) first integrals of the corresponding system (65); we denote them as $\{I_1, I_2\}$. Moreover, since the elements of $\tilde{C}$ defined in (67) can be expressed entirely via $\{b_i\}$, these are the first integrals for the general case of (65). Using (67) and Proposition 1, we also find the characteristic polynomial of $L2_3$ without its direct calculation:

$$P_{L2_3}(\lambda) = \lambda^6 - (b_0 + b_1 + b_2 + b_3)\lambda^3 + b_0b_3.$$

Now consider the pair $\{J_1, J_2\} := \{a_1 + a_3, -a_0a_3\}$ of the first integrals of (64) from the set D. Obviously, they are also the first integrals of the system

(65) with the elements $b_0, \dots, b_3$ defined according to (34). As follows from the latter, $J_2 = -a_0a_3 = -b_0b_2/b_1$, and $J_2 = J_2(b_0, b_1, b_2)$ is the first integral of (65) in the general case of this system. To find the expression of J_1 via $\{b_i\}$, we first consider the mapping $S_4 \to \tilde{S}_3$, which is equivalent to (34), and the resulting sequence $\tilde{S}_3$. As follows from (60) and (42), for the element $\tilde{S}^{\tilde{n}_1}_{\tilde{k}_1} = \tilde{S}^1_3$ of the latter, we have

$$\tilde{S}^1_3 = \tilde{D}_2\tilde{S}^2_1 + \tilde{D}_5\tilde{S}^1_0,$$

and, as in the general case of (65), $\tilde{D}_2$ and $\tilde{D}_5$ can be expressed in terms of $\{b_i\}$. Comparing the latter expression with (66) and using Theorem 3 (formula (43)) we find that $\tilde{D}_5 = J_2 = -b_0b_2/b_1$ and

$$\tilde{D}_2 = a_0J_1 = a_0a_1 + a_0a_3 = b_0 + \frac{b_0b_2}{b_1} = b_0 - J_2.$$

In particular, from the latter formula follows that $\tilde{D}_2$ is not the first integral of (65), see the remark after Theorem 4. Thus, $J_1 = b_0(b_1 + b_2)/b_1a_0$, and applying formula (40) for the inverse Miura transformation, we finally get the expression for J_1 via $\{b_i\}$:

$$J_1 = \frac{b_0(b_1 + b_2)}{b_1a_0(0)e^{\int_0^t b_1(\tau)d\tau}} \quad \text{for certain} \quad a_0(0) \in \mathbb{C}\backslash 0.$$

Thus, we have found that system (65) in the general case has four integrals of motion $\{I_1, I_2, J_1, J_2\}$ such that first two of them are the coefficients of characteristic polynomial of the Lax matrix corresponding to (65), whereas J_1 and $J2$ are the "nonstandard" integrals related to the system (64).

5 Concluding remarks and open issues

In view of the above, we can conclude that the description of Miura transformation between Volterra and Toda lattices via the inverse spectral data of the corresponding Lax operators, obtained in [22]- [23], can be extended to the case of Bogoyavlensky lattices (1)–(2). The latter, like the former, are systems with a rich Hamiltonian structure, see [29], Chapter 17 and [32]. For example, the finite system (52) can be written as

$$\dot{a}_i = \{H^a_2, a_i\}^a_2, \qquad i = 0, \dots, N,$$

with the Hamiltonian $H^a_2 := \sum_i a_ia_{i+1}$ and the quadratic Poisson bracket $\{\cdot, \cdot\}^a_2$ defined in the coordinates $\{a_i\}$ as follows:

$$\{a_i, a_j\}^a_2 = -\{a_j, a_i\}^a_2 = \pi_{ij}a_ia_j, \quad i \geq j,$$

where

$$\pi_{ij} = \begin{cases} 0,\ i-j=0,1; \\ 1,\ i-j=2,4,\dots,2\left\lfloor \frac{N}{2} \right\rfloor; \\ -1,\ i-j=3,5,\dots,2\left\lfloor \frac{N-1}{2} \right\rfloor+1; \end{cases}$$

and the system (53) admits the following Hamiltonian representation:

$$\dot{b}_1 = \{H_1^b, b_i\}_2^b,$$

whith $H_1^b := \sum_i b_i$ and Poisson bracket $\{\cdot,\cdot\}_2^b$ with nonvanishing elements:

$$\{b_{i+1}, b_i\}_2^b = b_{i+1}b_i, \quad \{b_{i+2}, b_i\}_2^b = b_{i+2}b_i.$$

It should be noted that $\{\cdot,\cdot\}_2^b$ is a local bracket (i.e., $\{\cdot, b_i\}_2^b$ depends only on the neighboring coordinates $b_{i+2}, b_{i+1}, b_{i-1}, b_{i-2},$) while $\{\cdot,\cdot\}_2^a$ is a nonlocal one. Obviously, the Miura mapping (34) transforms H_2^a to H_1^b and $\{\cdot,\cdot\}_2^a$ to $\{\cdot,\cdot\}_2^b$, so it may be useful to apply the above results to the study of Hamiltonian properties of Bogoyavlensky lattices.

Also, as we have seen, equations (25) and (29) are equivalent, in a certain sense, to the original systems (1)–(2). It may be of interest to consider (25) and (29) from the point of view of the theory of integrable systems.

As known [8, 29, 32], alongside with (1)–(2), the family of Bogoyavlensky lattices contains the system

$$\dot{c}_i = c_i^2 \left(\prod_{j=1}^{r} c_{i+j} - \prod_{j=1}^{r} c_{i-j} \right). \tag{68}$$

The operator L, which appears in the Lax representation for (68), differs sufficiently from the above operators $L1, L2$ and M, and the inverse spectral problems for such operators are less studied. Some recent results in this area are contained in [27]. The Miura transformation between (68) and (2)

$$b_i = c_i c_{i+1} \cdots c_{i+r}$$

was obtained by Bogoyavlensky [8], and its description in terms of the inverse spectral data is another interesting task.

All the above issues can be addressed for future work.

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