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Minimal prime ideals of $\sigma(*)$-rings and their extensions

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Abstract

Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$ ($\mathbb{Q}$ the field of rational numbers). Let σ be an automorphism of R and δ a σ-derivation of R. Let further σ be such that $a\sigma(a) \in P(R)$ implies that $a \in P(R)$ for $a \in R$, where $P(R)$ is the prime radical of R. In this paper we study minimal prime ideals of Ore extension $R[x; \sigma, \delta]$ and we prove the following in this direction:

Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ and δ be as above. Then P is a minimal prime ideal of $R[x; \sigma, \delta]$ if and only if there exists a minimal prime ideal U of R with $P = U[x; \sigma, \delta]$.

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Introduction and preliminaries

Notation: All rings are associative with identity. Throughout this paper R denotes a ring with identity $1 \neq 0$. The prime radical of R is denoted by $P(R)$. The field of rational numbers is denoted by $\mathbb{Q}$. The set of prime ideals of R is denoted by $Spec(R)$, the set of minimal prime ideals of R is denoted by $Min.Spec(R)$.

Let R be a right Noetherian ring. Let K be an ideal of a ring R such that $\sigma^m(K) = K$ for some integer $m \geq 1$, we denote $\cap_{i=1}^{m} \sigma^i(K)$ by K^0.

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Ore extensions: Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R ($\delta : R \to R$ is an additive map with $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$).

For example let σ be an endomorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be defined by
$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}$, for all $r \in R$. Then ϕ is a ring homomorphism if and only if δ is a σ-derivation of R.

In case σ is the identity map, δ is called just a derivation of R. For example let $R = F[x]$, where F is a field. The $\delta : R \to R$ defined by $\delta(f(x)) = \frac{d}{dx}(f(x))$ for all $f(x) \in F[X]$ is a derivation of R.

We denote the Ore extension $R[x; \sigma, \delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable; i.e. $\sigma(I) = I$ and I is δ-invariant; i.e. $\delta(I) \subseteq I$, then we denote $I[x; \sigma, \delta]$ by $O(I)$. We would like to mention that $R[x; \sigma, \delta]$ is the usual set of polynomials with coefficients in R, i.e. $\{\sum_{i=0}^{n} x^i a_i,\ a_i \in R\}$ in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take coefficients of the polynomials on the right as followed in McConnell and Robson [8].

In case δ is the zero map, we denote the skew polynomial ring $R[x; \sigma]$ by $S(R)$ and for any ideal I of R with $\sigma(I) = I$, we denote $I[x; \sigma]$ by $S(I)$.
In case σ is the identity map, we denote the differential operator ring $R[x; \delta]$ by $D(R)$ and for any ideal J of R with $\delta(J) \subseteq J$, we denote $J[x; \delta]$ by $D(J)$.

Ore-extensions (skew-polynomial rings and differential operator rings) have been of interest to many authors. For example see [1, 2, 3, 4, 6, 7, 8].

Minimal Prime ideals: This article concerns the study of minimal prime ideals of Ore extensions (skew polynomial rings). Recall that a minimal prime ideal in a ring R is any prime ideal of R that does not properly contain any other prime ideal. Regarding minimal prime ideals, we have the following:

Proposition (3.3) of [6]: Any prime ideal U in a ring R contains a minimal prime ideal.

Theorem (3.4) of [6]: In a right Noetherian ring R, there exist only finitely many minimal prime ideals, and there is a finite product of minimal prime ideals (repetition allowed) that equals zero.

Lemma (3.20) of [6]: Let R be a ring, δ a derivation of R. Let U be a minimal prime ideal of R such that R/U has characteristic zero. Then $\delta(U) \subseteq U$.

Lemma (3.4) of [5]: Let R be a right Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let δ be a derivation of R and U a minimal prime ideal of R. Then $\delta(U) \subseteq U$.

The following result regarding contraction of a minimal prime ideal of differential operator ring $R[x;\delta]$ is due to Gabriel [5].

Proposition (3.3)(b) of [5]: Let R be a right Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let δ be a derivation of R and P a minimal prime ideal of $D(R) = R[x;\delta]$. Then $P \cap R$ is a minimal prime ideal of R.

Much is not known about the minimal prime ideals of $S(R) = R[x;\sigma]$ or the full Ore extension $O(R) = R[x;\sigma,\delta]$. But we state some facts as follows:

We recall from 10.5.4 of McConnell and Robson [8] that an ideal U of a ring R is called σ-prime (σ is an automorphism of R) if U is σ-stable (i.e. $\sigma(U) = U$) and for σ-stable ideals I, J of R; $IJ \subseteq U$ implies that $I \subseteq U$ or $I \subseteq U$. The set of σ-prime ideals of R is denoted by $\sigma - Spec(R)$.

Lemma (10.6.4)(ii, iii, iv) of [8]: Let R be a ring and σ an automorphism of R. Then

1. $P \in Spec(S(R))$ and $x \notin P$ implies that $P \in \sigma - Spec(S(R))$
2. $0 \neq P \in Spec(S(R))$ and $x \notin P$ implies that $P \cap R \in \sigma - Spec(R)$
3. $U \in \sigma - Spec(R)$ implies that $U(S(R)) \in \sigma - Spec(S(R))$.

Let R be a right Noetherian ring. We know that $Min.Spec(R)$ is finite (Theorem (3.4) of [6]) and $\sigma^j(U) \in Min.Spec(R)$ for any $U \in Min.Spec(R)$, and for all integers $j \geq 1$, therefore, there exists an integer $m \geq 1$ such that $\sigma^m(U) = U$ for all $U \in Min.Spec(R)$. We denote $\cap_{i=1}^{m}\sigma^i(U)$ by U^0 and note that U^0 is σ-stable and is called σ-cyclic.

Proposition (10.6.12) of [8]: Let R be a right Noetherian ring, σ an automorphism of R and $U \in \sigma - Spec(R)$. Then U is σ-cyclic and $U(S(R)) \in \sigma - Spec(S(R))$.

In Theorems 2.4 and 3.7 of [1] the following has been proved regarding minimal prime ideals of $S(R)$ and $D(R)$ respectively:

1. Let R be a right Noetherian ring and σ an automorphism of R. Then $P \in Min.Spec(S(R))$ if and only if there exists $U \in Min.Spec(R)$ Such that $S(P \cap R) = P$ and $P \cap R = U^0$.
2. Let R be a right Noetherian $\mathbb{Q}$-algebra and δ a derivation of R. Then $P \in Min.Spec(D(R))$ if and only if $P = D(P \cap R)$ and $P \cap R \in Min.Spec(R)$.

Before we state the main result, we require the following:

$\sigma(*)$-rings: Recall that in [7], Kwak defines a $\sigma(*)$-ring R to be a ring in which $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$, where σ is an endomorphism of R.

Example 1 *Let* $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$*, where F is a field. Then* $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ *Let $\sigma : R \to R$ be defined by* $\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. *Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring.*

Example 2 *Let $R = \mathbb{C}$, the field of complex numbers. Then $\sigma : R \to R$ defined by $\sigma(a+ib) = a - ib$ is an automorphism of R and R is a $\sigma(*)$-ring.*

We note that if R is a ring and σ is an endomorphism of R such that R is a $\sigma(*)$-ring, then R is 2-primal (i.e. the set of nilpotent elements of R and $P(R)$ coincide).

1 Main Results

We now state the main result in the form of the following Theorem:

Theorem A: Let R be a right Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ be a σ-derivation of R. Then $P \in Min.Spec(O(R))$ if and only if there exists $U \in Min.Spec(R)$ such that $O(P \cap R) = P$ and $(P \cap R) = U$.

Towards the proof of the above Theorem, we require the following:

Recall that an ideal I of a ring R is said to be completely semiprime if $a^2 \in I$ implies that $a \in I$.

Proposition 1 *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R. Then $\sigma(U) = U$ and $\delta(U) \subseteq U$ for all $U \in Min.Spec(R)$.*

Proof. See Proposition (2.1) of Bhat [3]. To make the paper self contained, we include the proof of this Proposition.

We will first show that $P(R)$ is completely semiprime. Let $a \in R$ be such that $a^2 \in P(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R).$$

Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$.

We now show that $\sigma(U) = U$ for all $U \in Min.Spec(R)$. Let $U = U_1$ be a minimal prime ideal of R. Let $U_2, U_3, ..., U_n$ be the other minimal primes of R. Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Now P(R) is completely semiprime implies that $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$.

Let now $T = \{a \in U \mid \text{such that } \delta^k(a) \in U \text{ for all integers } k \geq 1\}$. First of all, we will show that T is an ideal of R. Let a, $b \in T$. Then $\delta^k(a) \in U$ and $\delta^k(b) \in U$ for all integers $k \geq 1\}$. Now $\delta^k(a - b) = \delta^k(a) - \delta^k(b) \in U$ for all $k \geq 1\}$. Therefore $a - b \in T$. Therefore T is a δ-invariant ideal of R.

We will now show that $T \in Spec(R)$. Suppose $T \notin Spec(R)$. Let $a \notin T$, $b \notin T$ be such that $aRb \subseteq T$. Let t, s be least such that $\delta^t(a) \notin U$ and $\delta^s(b) \notin U$. Now there exists $c \in R$ such that $\delta^t(a)c\sigma^t(\delta^s(b)) \notin U$. Let $d = \sigma^{-t}(c)$. Now $\delta^{t+s}(adb) \in U$ as $aRb \subseteq T$. This implies on simplification that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) + u \in U$, where u is sum of terms involving $\delta^l(a)$ or $\delta^m(b)$, where $l < t$ and $m < s$. Therefore by assumption $u \in U$ which implies that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) \in U$. This is a contradiction. Therefore, our supposition must be wrong. Hence $T \in Spec(R)$. Now $T \subseteq U$, so $T = U$ as $U \in Min.Spec(R)$. Hence $\delta(U) \subseteq U$.□

Recall that an ideal P of a ring R is completely prime if R/P is a domain, i.e. $ab \in P$ implies $a \in P$ or $b \in P$ for a, $b \in R$ (McCoy [9]). In commutative sense completely prime and prime have the same meaning. We also note that every completely prime ideal of a ring R is a prime ideal, but the converse need not be true.

The following example shows that a prime ideal need not be a completely prime ideal.

Example 3 *(Example 1.1 of Bhat [4]): Let* $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbb{Z})$. *If p is a prime number, then the ideal $P = M_2(p\mathbb{Z})$ is a prime ideal of R, but is not completely prime, since for* $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, *we have $ab \in P$, even though $a \notin P$ and $b \notin P$.*

Theorem 1 *Let R be a Noetherian ring, and σ an automorphism of R. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R.*

Proof. See Theorem (2.4) of [2]. □

Let σ be an endomorphism of a ring R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then σ can be extended to an endomorphism (say $\overline{\sigma}$) of $R[x; \sigma, \delta]$ by $\overline{\sigma}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i)$. Also δ can be extended to a $\overline{\sigma}$-derivation (say $\overline{\delta}$) of $R[x; \sigma, \delta]$

by $\overline{\delta}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \delta(a_i)$.
We note that if $\sigma(\delta(a)) \neq \delta(\sigma(a))$ for all $a \in R$, then the above does not hold. For example let $f(x) = xa$ and $g(x) = xb$, $a, b \in R$. Then

$$\overline{\delta}(f(x)g(x)) = x^2\{\delta(\sigma(a))\sigma(b) + \sigma(a)\delta(b)\} + x\{\delta^2(a)\sigma(b) + \delta(a)\sigma(b)\},$$

but

$$\overline{\delta}(f(x))\overline{\sigma}(g(x)) + f(x)\overline{\delta}(g(x)) = x^2\{\sigma(\delta(a))\sigma(b) + \sigma(a)\delta(b)\} + x\{\delta^2(a)\sigma(b) + \delta(a)\sigma(b)\}.$$

Theorem 2 *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then R is a $\sigma(*)$-ring implies that $O(R) = R[x; \sigma, \delta]$ is a Noetherian $\overline{\sigma}(*)$-ring.*

Proof. Let R be a Noetherian ring and σ an automorphism of R such that R is a $\sigma(*)$-ring. We shall prove that $O(R) = R[x; \sigma, \delta]$ is a Noetherian $\overline{\sigma}(*)$-ring. For this we will show that any minimal $P \in Min.Spec(O(R))$ is completely prime and $\overline{\sigma}(P) = P$.

Let $P \in Min.Spec(O(R))$. Then by Lemma (2.2) of Bhat [3] $P \cap R \in Min.Spec(R)$. Now R is a $\sigma(*)$-ring implies that $\sigma(P \cap R) = P \cap R$ and $P \cap R$ is a completely prime ideal of R by Theorem (1). Now Proposition (1) implies that $\delta(P \cap R) \subseteq P \cap R$. Now Theorem (2.4) of Bhat [4] implies that $O(P \cap R)$ is a completely prime ideal of $O(R)$. Now $O(P \cap R) \subseteq P$ implies that $O(P \cap R) = P$ as P is minimal. Now $\sigma(P \cap R) = P \cap R$ implies that $\overline{\sigma}(P) = P$.

Thus $\overline{\sigma}(P) = P$ and P is completely prime for all $P \in Min.Spec(O(R))$. Moreover $O(R) = R[x; \sigma, \delta]$ is Noetherian by Theorem (2.6) of Goodearl and Warfield [6]. Hence by Theorem (1) $R[x; \sigma, \delta]$ is a $\overline{\sigma}(*)$-ring. $\square$

We are now in a position to prove Theorem A in the form of Theorem (3) below:

Theorem 3 *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R. Then*

1. *If U is a minimal prime ideal of R, then $O(U)$ is a minimal prime ideal of of $O(R)$ and $O(U) \cap R = U$.*
2. *If P is a minimal prime ideal of $O(R)$, then $P \cap R$ is a minimal prime ideal of R.*

Proof. **(1)** $\sigma(U) = U$ and $\delta(U) \subseteq U$ by Proposition (1). Now it can be easily seen that $O(U) \in Spec(O(R))$.

(2) We note that σ can be extended to an endomorphism (say $\overline{\sigma}$) of $R[x; \sigma, \delta]$ by $\overline{\sigma}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i)$. Also δ can be extended to a $\overline{\sigma}$-derivation (say $\overline{\delta}$) of $R[x; \sigma, \delta]$ by $\overline{\delta}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \delta(a_i)$.

Now Theorem (2) implies that $O(R) = R[x; \sigma, \delta]$ is a Noetherian $\overline{\sigma}(*)$-ring. Therefore, Proposition (1) implies that $\overline{\sigma}(P) = P$ and $\overline{\delta}(P) \subseteq P$. So $\sigma(P \cap R) = P \cap R$ and $\delta(P \cap R) \subseteq P \cap R$. Now it can be seen that $P \cap R \in Spec(R)$ and, therefore, $O(P \cap R) \in Spec(O(R))$. Now $O(P \cap R) \subseteq P$ implies that $O(P \cap R) = P$. $\square$

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