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The Geometry of the Projective Action of $\mathbf{SL}(3,\mathbb{R})$ from the Erlangen Perspective

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Abstract. In this paper, we have investigated the projective action of the Lie group $\mathrm{SL}(3,\mathbb{R})$ on the homogeneous space $\mathbb{RP}^2$. In particular, we have studied the action of the subgroups of $\mathrm{SL}(3,\mathbb{R})$ on the non-degenerate conics in the space $\mathbb{RP}^2$. Using the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{R})$, we demonstrate that the isotropy subgroup of the projective unit circle is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$ under certain conditions.

Key Words: Lie Group $\mathrm{SL}(3,\mathbb{R})$, Homogeneous Space, Conics, Exponential Map, Iwasawa Decomposition
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Introduction

Geometry as a group action has dynamic aspects in several branches of mathematics. Starting with Russell, following Klein, this area has been explored by many researchers over time (see, for example, [8, 10, 19, 25]). In this trend, the action of $G = \mathrm{SL}(2,\mathbb{R})$ on the one-dimensional homogeneous space G/H represented by Möbius transformation was extensively studied and investigated by several authors, for example, in [1, 4, 14, 15]. In the series of works [2, 5, 16], the authors proposed advancing a theory of analytic functions inspired by Klein's Erlangen program. In [18], Kisil considered the Möbius action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{RP}^1$ and described different possible realisation of Poincaré extension of Möbius transformation, emphasizing the properties of Möbius-invariant cycles. In the same line, $\mathrm{SL}(2,\mathbb{R})$ invariant geodesic curves and metrics were studied in [3, 6, 9]. These series of works lead to a number of natural and effective generalisations, and hence, it becomes imperative to investigate how things work in the higher dimensional cases. In the parallel line, there is a wide range of literature devoted to the representation theory of Lie groups and symmetric spaces, where the classical cases such as the Lie groups $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})$, $\mathrm{SU}(2)$, etc., have provided powerful tools for studying these groups and other related results, see [11, 22–24]. Moreover, projective geometry is a rich and fascinating field that provides a geometric foundation for a variety of disciplines, including projective algebraic geometry, differential geometry, and the theory of algebraic curves. An integral aspect of this field involves the investigation of the projective action of $\mathrm{SL}(3,\mathbb{R})$ on projective space, providing a potent tool for exploring geometric relationships and structures.

In this manner, following the line of Kisil, we have taken the transformation group as $\mathrm{SL}(3,\mathbb{R})$ and focused on the invariant objects, with the aim of incorporating more invariants in the existing $\mathrm{SL}(2,\mathbb{R})$ as well as in $\mathrm{SL}(3,\mathbb{R})$ geometry. Thus, by studying invariants, we propose to construct geometry of the homogeneous space $\mathbb{RP}^2$ in a systematic way. In this work, as a continuation of [7], the action of the transformation group $\mathrm{SL}(3,\mathbb{R})$ on the two dimensional homogeneous space $\mathbb{RP}^2$ is studied. For $\mathrm{SL}(3,\mathbb{R})$ instance, we examine an extension of the map i associated with $\mathrm{SL}(2,\mathbb{R})$ action. We also investigate the projective action of the subgroups of $\mathrm{SL}(3,\mathbb{R})$ on the non-degenerate conics in the space $\mathbb{RP}^2$. Iwasawa decomposition plays an essential part in the current discussion.

1 Preliminaries

This section reviews standard definitions and theorems related to our work.

A transformation group G can be defined as a non-empty collection of mappings from a set X to itself, adhering to the following conditions: (i) the identity map is an element of G; (ii) if $g_1 \in G$ and $g_2 \in G$, then composition $g_1 g_2 \in G$; (iii) if $g \in G$, then the inverse g^{-1} exists and is a member of G.

Additionally, a group action $\varphi : G \times X \to X$ is termed transitive if for every $x,\ y \in X$, there exists $g \in G$ satisfying $g \cdot x = y$. Furthermore, a homogeneous space is defined as a pair (G, X), where the action of the group G on X is transitive and X is a topological space.

Following notions were introduced in [12]. A matrix Lie group is a subgroup G of $\mathrm{GL}(n,\mathbb{R})$ that exhibits the subsequent characteristic: if A_m denotes a sequence of matrices within G that converges to a matrix A, then either A belongs to G or A is not invertible. For a matrix Lie group G, the associated Lie algebra, represented as $\mathfrak{g}$, is characterized as the collection of matrices X for which $\exp(tX)$ is a member of G for all real numbers t.

If G is a matrix Lie group with Lie algebra $\mathfrak{g}$, then the exponential mapping of G is defined as the map $\exp : \mathfrak{g} \to G$. Hence, the exponential mapping of G is the matrix exponential restricted to the Lie algebra $\mathfrak{g}$ of G.

1.1 Group action on coset spaces

Let G be a Lie group and H be a closed subgroup of G. Then it follows by Cartan's theorem that H is a Lie group (cf. [21]).

Let $G/H = \{gH : g \in G\}$ denotes the space of left cosets of H. In this context, the projection map $p : G \to G/H$ is defined by mapping $g \in G$ to its equivalence class $[g]$, expressed as $p(g) = gH = [g]$. Also, a section s of a projection map p is a right inverse of p, denoted by $s : G/H \to G$, satisfying $p(s(x)) = x$ for all $x \in G/H$.

Theorem 1 *[13] Consider a Lie group G with a closed subgroup H. Let G/H has the quotient topology. Then G/H possesses a unique smooth manifold structure such that the projection map $p : G \to G/H$ is a smooth submersion and G acts smoothly on G/H.*

Remark 1 The action $G \times G/H \to G/H$ defined as $(a, gH) \mapsto agH$ can be viewed as a composition of smooth maps as follows:

$$\phi : G \times G/H \to G/H$$
$$\phi(g, x) = g \cdot x = p(g * s(x)),$$

where $*$ denotes the group operation on G.

1.2 Real projective space $\mathbb{RP}^n$

Let $\mathbb{R}^{n+1} = \{(x_1, x_2, \cdots, x_{n+1}) : x_i \in \mathbb{R}\}$. The real projective space $\mathbb{RP}^n$ consists of points which are equivalence classes of the set $\mathbb{R}^{n+1}\backslash\{0\}$ modulo the equivalence relation $x \sim \lambda x$ for all λ in $\mathbb{R}\backslash\{0\}$.

In particular, the space $\mathbb{RP}^1$ is called the real projective line, while $\mathbb{RP}^2$ is called the real projective plane.

In the real projective plane, a point is represented by a triple (X, Y, Z), referred to as homogeneous coordinates or projective coordinates of the point, where X, Y and Z are not all zero. Since points in $\mathbb{RP}^2$ are equivalence classes, in the homogeneous coordinated setup, the coordinates (X, Y, Z) and $(\lambda X, \lambda Y, \lambda Z)$ are considered to represent the same point for all $\lambda \neq 0$ in $\mathbb{R}$, see [10].

A line L in the projective plane $\mathbb{RP}^2$ can be represented by the homogeneous coordinates $L = (a, b, c)^t$ and is defined by the equation $ax+by+cz = 0$, or in matrix notation $X^tL = 0$, where $X = (x, y, z)^t$ is any point in L.

Similarly, a conic in $\mathbb{RP}^2$ is characterized as the set of points for which a quadratic form on $\mathbb{R}^3$ vanishes. The conic associated with a quadratic form A is given by $\mathbf{C}_A = \{[p] \in \mathbb{RP}^2 : p^tAp = 0\}$, as described in [10].

1.3 Complexification of real Lie algebras and real Lie groups

Consider a finite dimensional real Lie algebra $\mathfrak{g}$, which is essentially a real vector space. The complexification of $\mathfrak{g}$, denoted as $\mathfrak{g}_{\mathbb{C}}$, is defined as the real vector space consisting of linear combinations X_1+iX_2, where X_1 and X_2 are elements of $\mathfrak{g}$. The bracket operation on $\mathfrak{g}$ naturally extends to $\mathfrak{g}_{\mathbb{C}}$, turning it into a complex Lie algebra. $\mathfrak{g}_{\mathbb{C}}$ is referred to as the complexification of the real Lie algebra $\mathfrak{g}$, as introduced in [12].

The complexification of a Lie group G defined over $\mathbb{R}$ is a complex Lie group denoted as $G_{\mathbb{C}}$, which includes G as a real Lie subgroup. This inclusion ensures that the Lie algebra $\mathfrak{g}$ of G is a real form of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$. The Lie group G is termed a real form of the Lie group $G_{\mathbb{C}}$.

1.4 Correspondence between Lie group and Lie algebra homomorphisms

Theorem 2 *[12] Consider matrix Lie groups G and H with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. If $\Phi : G \to H$ is a Lie group homomorphism, there exists a unique linear map $\phi : \mathfrak{g} \to \mathfrak{h}$, satisfying the condition $\Phi(e^X) = e^{\phi(X)}$ for all $X \in \mathfrak{g}$.*

The converse of Theorem 2 holds true under certain condition.

Theorem 3 *[12] Consider matrix Lie groups G and H with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Suppose $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, and G is simply connected. In that case, there exists a unique Lie group homomorphism $\Phi : G \to H$, ensuring that $\Phi(e^X) = e^{\phi(X)}$ for all $X \in \mathfrak{g}$.*

We will utilize the notion of complexification to extend the results of Theorem 3 to the case involving $\mathrm{SL}(2,\mathbb{R})$, even though it does not possess the property of simply connectedness.

Remark 2 Here we discuss some properties of the matrix Lie group $\mathrm{SL}(2,\mathbb{R})$ and the corresponding Lie algebra $\mathfrak{sl}(2)$.
1. The exponential mapping for the matrix Lie group $\mathrm{SL}(2,\mathbb{R})$ is not onto. As an illustration, consider the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

However, there is no $X \in \mathfrak{sl}(2)$ such that $\exp(X) = e^X = A$.
2. $\mathrm{SL}(2,\mathbb{R})$ is not simply connected.
3. If X and Y belong to $\mathfrak{sl}(2)$, then X and Y do not necessarily commute with their commutator $[X,Y]$. Hence, we have $e^Xe^Y \neq e^{X+Y}$ and $e^Xe^Y \neq e^{X+Y+\frac{1}{2}[X,Y]}$.

1.5 Iwasawa decomposition

The Iwasawa decomposition of $\mathrm{SL}(n,\mathbb{R})$ expresses an element g in the group as a product of three matrices, each belonging to a specific subgroup.

Specifically, for $n = 2$, i.e., $\mathrm{SL}(2,\mathbb{R})$, this decomposition takes the form $\mathrm{SL}(2,\mathbb{R}) = KAN$, where

$$A = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : \ t \in \mathbb{R}, \ t > 0 \right\}, \ N = \left\{ \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} : \ \eta \in \mathbb{R} \right\},$$

$$K = \mathrm{SO}(2,\mathbb{R}) = \left\{ M \in \mathrm{SL}(2,\mathbb{R}) : MM^T = M^T M = I_2 \right\}.$$

This unique decomposition, detailed in [20], is known as the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{R})$. It plays a crucial role in connecting the fix subgroup of the projective unit circle to $\mathrm{PSL}(2,\mathbb{R})$ in our exploration.

1.6 Two dimensional homogeneous space $\mathbb{RP}^2$

As in [7], we consider the action of $\mathrm{SL}(3,\mathbb{R})$ on the space of left cosets $X = G/H$, where $G = \mathrm{SL}(3,\mathbb{R})$ and

$$H = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \middle| a_{22}a_{33} - a_{23}a_{32} = \frac{1}{a_{11}}, a_{11} \neq 0 \right\}.$$

Expressed in terms of the parametrization $z = (x, y) \in X$, the set-theoretic action of $\mathrm{SL}(3,\mathbb{R})$ on $\mathrm{SL}(3,\mathbb{R})/H$ can be formulated as a composition of smooth maps as follows:

$$g : z \mapsto g \cdot z = p(g * s(z)).$$

We can define another map $r : G \to H$ such that $r(g) = h$, where $h = s(p(g))^{-1}g$. Hence, g can be uniquely written as $g = s(p(g))r(g)$ (see [7] for the details).

In this set up, the $\mathrm{SL}(3,\mathbb{R})$ action takes the form

$$(x, y) \mapsto \left(\frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}} \right),$$

provided $a_{31}x + a_{32}y + a_{33} \neq 0$.

Now, if we allow $a_{31}x + a_{32}y + a_{33} = 0$, this action indeed gives us a projective transformation of the space $\mathbb{RP}^2$. This $\mathrm{SL}(3,\mathbb{R})$ action on $\mathbb{RP}^2$ is denoted as $g : [p] \mapsto [g \cdot p]$. Let ϕ be the action defined by

$$\phi : \mathrm{SL}(3,\mathbb{R}) \times \mathbb{RP}^2 \to \mathbb{RP}^2 \phi(g, [p]) = [g \cdot p].$$

We consider the projective transformation $\phi_g : \mathbb{RP}^2 \to \mathbb{RP}^2$ such that $\phi_g([p]) = [g \cdot p]$ for all $g \in \mathrm{SL}(3,\mathbb{R})$.

2 Extension of the map i

In the consideration of the Möbius action of $\mathrm{SL}(2,\mathbb{R})$, for the study of the invariant properties of cycles (cf. [17]), the Fillmore Springer Cnops construction (FSCc) is a useful construction that allows the association of a cycle with a 2×2 cycle matrix via the coefficients. The most important component of this construction is the map i, which corresponds to a column vector a row vector by the rule (see [17, 18])

$$i : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto (y, -x).$$

Remark 3 Let us now consider some properties of the map i. We consider the left multiplication of $\mathrm{SL}(2,\mathbb{R})$ on column vector $(x,y)^t \in \mathbb{R}^2$

$$\mathcal{L}_g : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}, \text{ where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}). \tag{1}$$

1. The map i is a linear map. Indeed, for $X = (x_1, y_1)^t$ and $Y = (x_2, y_2)^t$, we have $i(X+Y) = i(X) + i(Y)$ since

$$\begin{aligned} i\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) &= i \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = (y_1 + y_2, -x_1 - x_2) \\ &= (y_1, -x_1) + (y_2, -x_2) = i \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + i \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \end{aligned}$$

and $i(cX) = ci(X)$, which follows from

$$i\left(c \begin{pmatrix} x \\ y \end{pmatrix} \right) = i \begin{pmatrix} cx \\ cy \end{pmatrix} = (cy, -cx) = ci \begin{pmatrix} x \\ y \end{pmatrix} = ci(X).$$

2. The map i intertwines the left multiplication $\mathcal{L}_g$ (cf. equation (1)) and the right multiplication $\mathcal{R}_{g^{-1}}$, that is, $i(\mathcal{L}_g x) = i(x)\mathcal{R}_{g^{-1}}$. Indeed,

$$\begin{aligned} \mathcal{R}_{g^{-1}} : (y, -x) \mapsto (cx+dy, -by-ax) &= (y, -x) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= i\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right). \end{aligned}$$

In order to extend the map i in three-dimensions, we consider the left multiplication of $\mathrm{SL}(3,\mathbb{R})$ on the three-dimensional column vector $(x,y,z)^t \in \mathbb{R}^3$:

$$\mathcal{L}_g : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $g = (a_{ij}) \in \mathrm{SL}(3,\mathbb{R})$.

Proposition 1 *There is no non-zero map i that corresponds a three dimensional column vector to a three-dimensional row vector that is both linear and intertwines the left multiplication $\mathcal{L}_g$ and the right multiplication $\mathcal{R}_{g^{-1}}$, where $g \in SL(3, \mathbb{R})$.*

Proof. Suppose there exists linear map $i : \mathbb{R}^3 \to \mathbb{R}^3$ from three-dimensional real column vector space to three-dimensional real row vector space such that $i(\mathcal{L}_g X) = i(X)\mathcal{R}_{g^{-1}}$, where $g \in \mathrm{SL}(3, \mathbb{R})$, $X \in \mathbb{R}^3$.

Since it is sufficient to know the linear map on basis vectors, let us choose a basis of $\mathbb{R}^3$ as $e_1 = (1, 0, 0)^t$, $e_2 = (0, 1, 0)^t$, $e_3 = (0, 0, 1)^t$. Now, we consider the linear map applied to these basis vectors as follows

$$i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (x_1, x_2, x_3), \ i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (y_1, y_2, y_3) \text{ and } i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (z_1, z_2, z_3).$$

Here, we have

$$i(\mathcal{L}_g e_i) = i(e_i)\mathcal{R}_{g^{-1}} \text{ for all } g \in \mathrm{SL}(3, \mathbb{R}), \ i = 1, 2, 3. \tag{2}$$

We proceed to establish that for certain values of g in $\mathrm{SL}(3, \mathbb{R})$, relation (2) leads to a zero-map.

In particular, consider

$$g = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}(3, \mathbb{R})$$

such that

$$i(\mathcal{L}_g e_1) = i\left(\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (x_1, x_2, x_3)$$

and

$$i(e_1)\mathcal{R}_{g^{-1}} = (x_1, x_2, x_3) \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = (x_1, -x_1 + x_2, -x_2 + x_3).$$

Then from $i(\mathcal{L}_g e_1) = i(e_1)\mathcal{R}_{g^{-1}}$, it follows that

$$(x_1, x_2, x_3) = (x_1, -x_1 + x_2, -x_2 + x_3),$$

and hence, $x_2 = -x_1 + x_2$ and $x_3 = -x_2 + x_3$, that is, $x_1 = 0$ and $x_2 = 0$.

Now, take

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in \mathrm{SL}(3, \mathbb{R}),$$

so that

$$i(\mathcal{L}_g e_1) = i\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (x_1, x_2, x_3)$$

and

$$i(e_1)\mathcal{R}_{g^{-1}} = (x_1, x_2, x_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = (x_1, x_2 - x_3, x_3).$$

Thus, the relation defined in equation (2) yields

$$(x_1, x_2, x_3) = (x_1, x_2 - x_3, x_3),$$

which implies $x_2 = x_2 - x_3$, and hence, $x_3 = 0$.

Therefore, to achieve $i(\mathcal{L}_g e_1) = i(e_1)\mathcal{R}_{g^{-1}}$ for all $g \in \mathrm{SL}(3, \mathbb{R})$, we must possess

$$i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (x_1, x_2, x_3) = (0, 0, 0). \tag{3}$$

In a similar way, if we consider

$$g = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}(3, \mathbb{R}),$$

then $i(\mathcal{L}_g e_2) = i(e_2)\mathcal{R}_{g^{-1}}$ leads to

$$i\left(\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) = (y_1, y_2, y_3) \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which simplifies to

$$i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (y_1 - y_2, y_2, -y_1 + y_3).$$

Hence, $(y_1, y_2, y_3) = (y_1 - y_2, y_2, -y_1 + y_3)$, which implies $y_1 = y_1 - y_2$ and $y_3 = -y_1 + y_3$, and therefore, $y_2 = 0$ and $y_1 = 0$.

When

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad i(\mathcal{L}_g e_2) = i(e_2)\mathcal{R}_{g^{-1}},$$

we find that $y_3 = 0$. Thus,

$$i\begin{pmatrix}0\\1\\0\end{pmatrix} = (y_1, y_2, y_3) = (0, 0, 0). \qquad (4)$$

Similarly, considering

$$i : \begin{pmatrix}0\\0\\1\end{pmatrix} \to (z_1, z_2, z_3),$$

we observe that for

$$g = \begin{pmatrix}1 & 0 & 0\\1 & 1 & 0\\0 & 1 & 1\end{pmatrix},$$

the condition $i(\mathcal{L}_g e_3) = i(e_3)\mathcal{R}_{g^{-1}}$ implies $z_2 = z_3 = 0$. Additionally, if

$$g = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix},$$

then the condition $i(\mathcal{L}_g e_3) = i(e_3)\mathcal{R}_{g^{-1}}$ results in $z_1 = 0$. Hence,

$$i\begin{pmatrix}0\\0\\1\end{pmatrix} = (z_1, z_2, z_3) = (0, 0, 0). \qquad (5)$$

Therefore, combining equations (3), (4) and (5), we obtain

$$\begin{aligned} i\begin{pmatrix}x\\y\\z\end{pmatrix} &= i\left(x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix} \right) \\ &= x(0, 0, 0) + y(0, 0, 0) + z(0, 0, 0) = (0, 0, 0). \end{aligned}$$

Thus, the only linear map which intertwines the left multiplication $\mathcal{L}_g$ and the right multiplication $\mathcal{R}_{g^{-1}}$ ($g \in$ SL(3, ℝ)) is the zero map. □

3 Fixed subgroup of the unit circle

The preceding section indicates that conics in $\mathbb{RP}^2$ cannot be addressed analogously to cycles of SL(2, ℝ) in terms of FSCc. Therefore, we now examine conics through their fixed subgroups.

Let us consider the equation of the conic C (see subsection 1.2) as

$$ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2 = 0$$

or

$$(x\ y\ z)\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,$$

that is, $p^tAp = 0$.

Let A be the matrix associated with the conic C. If we transform the conic projectively, then under the transformation g, the point p on the conic maps to the point $p' = gp$. Therefore, from $(g^{-1}p')^tA(g^{-1}p') = 0$, it follows $p'^t(g^{-1})^tA(g^{-1})p' = 0$. Thus, if a conic is represented by the matrix A, then under the action of g, the transformed conic is represented by the matrix $(g^{-1})^t Ag^{-1}$. Furthermore, it is well-known that all non-degenerate conics are the projective images of a unit circle (see, for example, [10]). Let us now consider the equation of the unit circle C in homogeneous coordinates:

$$x^2 + y^2 = z^2. \tag{6}$$

The matrix associated to the circle C defined by (6) is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where $X^tAX = 0$, $X = (x, y, z)^t$.

Theorem 4 *Under the projective action, the fixed subgroup of the unit circle $M := M(x, y, \theta)$ is a three-dimensional subgroup of $SL(3, \mathbb{R})$ given by*

$$M = \begin{cases} \begin{pmatrix} \frac{xy}{\beta}\cos\theta + \frac{\alpha}{\beta}\sin\theta & \pm\frac{x\alpha}{\beta}\cos\theta \mp \frac{y}{\beta}\sin\theta & \beta\cos\theta \\ \frac{xy}{\beta}\sin\theta - \frac{\alpha}{\beta}\cos\theta & \pm\frac{x\alpha}{\beta}\sin\theta \pm \frac{y}{\beta}\cos\theta & \beta\sin\theta \\ y & \pm\alpha & x \end{pmatrix}, & x^2 \neq 1, \\ \\ \begin{pmatrix} \pm\cos\theta & \mp\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, & x^2 = 1, \end{cases}$$

where $\alpha = \sqrt{x^2 - y^2 - 1}$ and $\beta = \sqrt{x^2 - 1}$.

Proof. To determine the fixed subgroup of the projective unit circle, let us note that under the transformation of $g \in \mathrm{SL}(3, \mathbb{R})$, the unit circle C remains invariant. Therefore, the transformed unit circle must be represented by the matrix

$$A' = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix}$$

for some $a \neq 0$, since it represents the same circle (6). Hence, the fixed subgroup of the circle C is determined by

$$K_{C} = \left\{ g \in \mathrm{SL}(3, \mathbb{R}) \middle| (g^{-1})^t A g^{-1} = A' \right\}.$$

Let

$$h = g^{-1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

then $h^t A h = A'$ gives

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix},$$

and hence,

$$\begin{pmatrix} \begin{matrix} a_{11}^2 + a_{21}^2 \\ -a_{31}^2 \end{matrix} & \begin{matrix} a_{11}a_{12} + a_{21}a_{22} \\ -a_{31}a_{32} \end{matrix} & \begin{matrix} a_{11}a_{13} + a_{21}a_{23} \\ -a_{31}a_{33} \end{matrix} \\ \begin{matrix} a_{11}a_{12} + a_{21}a_{22} \\ -a_{31}a_{32} \end{matrix} & \begin{matrix} a_{12}^2 + a_{22}^2 \\ -a_{32}^2 \end{matrix} & \begin{matrix} a_{12}a_{13} + a_{22}a_{23} \\ -a_{32}a_{33} \end{matrix} \\ \begin{matrix} a_{11}a_{13} + a_{21}a_{23} \\ -a_{31}a_{33} \end{matrix} & \begin{matrix} a_{12}a_{13} + a_{22}a_{23} \\ -a_{32}a_{33} \end{matrix} & \begin{matrix} a_{13}^2 + a_{23}^2 \\ -a_{33}^2 \end{matrix} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix}. \quad (7)$$

Equating a_{ij}-th entry of both sides of equation (7), we get

$$a_{11}^2 + a_{21}^2 - a_{31}^2 = a, \quad (8)$$
$$a_{11}a_{12} + a_{21}a_{22} - a_{31}a_{32} = 0, \quad (9)$$
$$a_{11}a_{13} + a_{21}a_{23} - a_{31}a_{33} = 0, \quad (10)$$
$$a_{12}^2 + a_{22}^2 - a_{32}^2 = a, \quad (11)$$
$$a_{12}a_{13} + a_{22}a_{23} - a_{32}a_{33} = 0, \quad (12)$$
$$a_{13}^2 + a_{23}^2 - a_{33}^2 = -a, \quad (13)$$

such that

$$\det((a_{ij})) = 1. \quad (14)$$

Depending on the value of a, there are two cases to follow.
Case 1: a is positive. Let $a = m^2 > 0$. Referring to equation (13), it is evident that $a_{33} \neq 0$, or else $a_{13}^2 + a_{23}^2 = -a = -m^2$, which leads to a contradiction. Let $a_{33} = x \neq 0$. From equation (13), we derive $a_{13}^2 + a_{23}^2 = x^2 - a$, which further yields

$$a_{13} = \sqrt{x^2 - a}\ \cos\theta, \quad a_{23} = \sqrt{x^2 - a}\ \sin\theta.$$

Additionally, considering equation (8) and assigning $a_{31} = y$, we obtain $a_{11}^2 + a_{21}^2 = a + y^2$, resulting in the expressions

$$a_{11} = \sqrt{y^2 + a}\ \cos\phi, \tag{15}$$

$$a_{21} = \sqrt{y^2 + a}\ \sin\phi. \tag{16}$$

Subsequently, from equation (10), we get $a_{11}a_{13} + a_{21}a_{23} - a_{31}a_{33} = 0$, which implies

$$\sqrt{y^2 + a}\cos\phi\sqrt{x^2 - a}\cos\theta + \sqrt{y^2 + a}\sin\phi\sqrt{x^2 - a}\sin\theta - xy = 0,$$

or

$$\cos(\theta - \phi) = \frac{xy}{\sqrt{(y^2 + a)(x^2 - a)}}$$

provided $x^2 \neq a$, and hence,

$$\phi = \theta - \cos^{-1}\left(\frac{xy}{\sqrt{(y^2 + a)(x^2 - a)}}\right).$$

Therefore, substituting the value of ϕ into equations (15) and (16), and omitting cumbersome but trivial calculations, we obtain

$$\begin{aligned} a_{11} &= \sqrt{y^2 + a}\cos\left(\theta - \cos^{-1}\left(\frac{xy}{\sqrt{(y^2 + a)(x^2 - a)}}\right)\right) \\ &= \frac{xy}{\sqrt{(x^2 - a)}}\cos\theta + \frac{\sqrt{ax^2 - ay^2 - a^2}}{\sqrt{(x^2 - a)}}\sin\theta \end{aligned} \tag{17}$$

and

$$\begin{aligned} a_{21} &= \sqrt{y^2 + a}\sin\left(\theta - \cos^{-1}\left(\frac{xy}{\sqrt{(y^2 + a)(x^2 - a)}}\right)\right) \\ &= \frac{xy}{\sqrt{(x^2 - a)}}\sin\theta - \frac{\sqrt{ax^2 - ay^2 - a^2}}{\sqrt{(x^2 - a)}}\cos\theta. \end{aligned} \tag{18}$$

Again, from equation (11), we have $a_{12}^2 + a_{22}^2 - a_{32}^2 = a$. Let $a_{32} = z$. Given that $a_{12}a_{13} + a_{22}a_{23} - a_{32}a_{33} = 0$, we can replace y by z in equations

(17) and (18) to derive the values of a_{12} and a_{22}. Thus, we get

$$a_{12} = \frac{xz}{\sqrt{(x^2-a)}}\cos\theta + \frac{\sqrt{ax^2-az^2-a^2}}{\sqrt{(x^2-a)}}\sin\theta, \tag{19}$$

$$a_{22} = \frac{xz}{\sqrt{(x^2-a)}}\sin\theta - \frac{\sqrt{ax^2-az^2-a^2}}{\sqrt{(x^2-a)}}\cos\theta. \tag{20}$$

Using equations (9), (17)–(20), we find $a_{11}a_{12} + a_{22}a_{21} - a_{31}a_{32} = 0$, which implies

$$\left(\frac{\sqrt{(ax^2-az^2-a^2)(ax^2-ay^2-a^2)}}{x^2-a}\right)^2 = \left(yz - \frac{x^2yz}{x^2-a}\right)^2,$$

and, consequently, we have

$$a^2z^2(x^2-a) = a^2(x^2-a)(x^2-y^2-a),$$

that is, $z^2 = x^2 - y^2 - a$. Hence, we obtain

$$a_{12} = \pm\frac{x\sqrt{x^2-y^2-a}}{\sqrt{x^2-a}}\cos\theta \mp \frac{\sqrt{a}y}{\sqrt{x^2-a}}\sin\theta,$$

$$a_{22} = \pm\frac{x\sqrt{x^2-y^2-a}}{\sqrt{x^2-a}}\sin\theta \pm \frac{\sqrt{a}y}{\sqrt{x^2-a}}\cos\theta.$$

In particular, if $a_{33}^2 = x^2 = a$, from equation (13), we derive $a_{13}^2 + a_{23}^2 - a = -a$, which implies $a_{13}^2 + a_{23}^2 = 0$, and thus, $a_{13} = a_{23} = 0$.

Considering equation (10), we find $a_{11}a_{13} + a_{21}a_{23} - a_{31}a_{33} = 0$ implies $a_{31} = 0$. Similarly, from equation (12), $a_{12}a_{13} + a_{22}a_{23} - a_{32}a_{33} = 0$ refers $a_{32} = 0$. Therefore, both $a_{11}^2 + a_{21}^2 = a = a_{12}^2 + a_{22}^2$ and $a_{11}a_{12} + a_{21}a_{22} = 0$ lead to (cf. [17])

$$h = \begin{pmatrix} \pm\sqrt{a}\cos\eta & \mp\sqrt{a}\sin\eta & 0 \\ \sqrt{a}\sin\eta & \sqrt{a}\cos\eta & 0 \\ 0 & 0 & \pm\sqrt{a} \end{pmatrix}.$$

Thus, either

$$h = \begin{pmatrix} \frac{xy}{\beta_1}\cos\theta + \frac{\sqrt{a}\alpha_1}{\beta_1}\sin\theta & \pm\frac{x\alpha_1}{\beta_1}\cos\theta \mp \frac{\sqrt{a}y}{\beta_1}\sin\theta & \beta_1\cos\theta \\ \frac{xy}{\beta_1}\sin\theta - \frac{\sqrt{a}\alpha_1}{\beta_1}\cos\theta & \pm\frac{x\alpha_1}{\beta_1}\sin\theta \pm \frac{\sqrt{a}y}{\beta_1}\cos\theta & \beta_1\sin\theta \\ y & \pm\alpha_1 & x \end{pmatrix}, \tag{21}$$

where $\alpha_1 = \sqrt{x^2-y^2-a}$ and $\beta_1 = \sqrt{x^2-a}$, or

$$h = \begin{pmatrix} \pm\sqrt{a}\cos\eta & \mp\sqrt{a}\sin\eta & 0 \\ \sqrt{a}\sin\eta & \sqrt{a}\cos\eta & 0 \\ 0 & 0 & \pm\sqrt{a} \end{pmatrix}. \tag{22}$$

Let us now consider equation (14), where we have $\det((a_{ij})) = \det(h) = 1$. When $x^2 \neq a$ (a is real), and conducting some straightforward calculations of $\det(h)$ from equation (21), we obtain

$$\det(h) = \frac{\sqrt{a}x^2y^2}{x^2-a} + \frac{\sqrt{a}x^2(x^2-y^2-a)}{x^2-a} - \sqrt{a}y^2 - \sqrt{a}(x^2-y^2-a),$$

which simplifies to $\det(h) = a\sqrt{a} = 1$, implying $a = 1$.

When $x^2 = a$, computing $\det(h)$ from equation (22), we get

$$\det(h) = a\sqrt{a}(\cos^2\eta + \sin^2\eta),$$

which implies $\det(h) = a\sqrt{a}$ and thus, $a = 1$.

Therefore, either

$$h = \begin{pmatrix} \frac{xy}{\beta}\cos\theta + \frac{\alpha}{\beta}\sin\theta & \pm\frac{x\alpha}{\beta}\cos\theta \mp \frac{y}{\beta}\sin\theta & \beta\cos\theta \\ \frac{xy}{\beta}\sin\theta - \frac{\alpha}{\beta}\cos\theta & \pm\frac{x\alpha}{\beta}\sin\theta \pm \frac{y}{\beta}\cos\theta & \beta\sin\theta \\ y & \pm\alpha & x \end{pmatrix},$$

where $\alpha = \sqrt{x^2-y^2-1}$ and $\beta = \sqrt{x^2-1}$, or

$$h = \begin{pmatrix} \pm\cos\eta & \mp\sin\eta & 0 \\ \sin\eta & \cos\eta & 0 \\ 0 & 0 & \pm\sqrt{a} \end{pmatrix}.$$

Case 2: a is negative. Let $a = -n^2 < 0$. Referring to equation (8), we obtain $a_{11}^2 + a_{21}^2 - a_{31}^2 = a = -n^2$, implying $a_{31} \neq 0$. Let $a_{31} = u \neq 0$, leading to

$$a_{11} = \sqrt{u^2-n^2}\cos\eta, \quad a_{21} = \sqrt{u^2-n^2}\sin\eta.$$

Similarly, from equation (11), where $a_{12}^2 + a_{22}^2 - a_{32}^2 = a = -n^2$, we deduce $a_{32} \neq 0$. Setting $a_{32} = v \neq 0$, we find

$$a_{12} = \sqrt{v^2-n^2}\cos\zeta, \quad a_{22} = \sqrt{v^2-n^2}\sin\zeta.$$

Now, equation (9) yields $a_{11}a_{12} + a_{21}a_{22} - a_{31}a_{32} = 0$, implying

$$\sqrt{u^2-n^2}\cos\eta\sqrt{v^2-n^2}\cos\zeta + \sqrt{u^2-n^2}\sin\eta\sqrt{v^2-n^2}\sin\zeta - uv = 0,$$

which simplifies to

$$\cos(\eta-\zeta) = \frac{uv}{\sqrt{(u^2-n^2)(v^2-n^2)}},$$

provided $u^2,\ v^2 \neq n^2$. Consequently, we deduce

$$\zeta = \eta - \cos^{-1}\left(\frac{uv}{\sqrt{(u^2-n^2)(v^2-n^2)}}\right).$$

Thus,

$$a_{12} = \sqrt{v^2 - n^2}\cos\zeta = \frac{uv}{\sqrt{(u^2 - n^2)}}\cos\eta + \frac{\sqrt{n^4 - n^2u^2 - n^2v^2}}{\sqrt{(u^2 - n^2)}}\sin\eta, \quad (23)$$

$$a_{22} = \sqrt{v^2 - n^2}\sin\zeta = \frac{uv}{\sqrt{(u^2 - n^2)}}\sin\eta - \frac{\sqrt{n^4 - n^2u^2 - n^2v^2}}{\sqrt{(u^2 - n^2)}}\cos\eta. \quad (24)$$

Again, from equation (13), we have $a_{13}^2 + a_{23}^2 - a_{33}^2 = -a$. Let $a_{33} = w$. Since $a_{11}a_{13} + a_{21}a_{23} - a_{31}a_{33} = 0$, we can similarly obtain the values of a_{13} and a_{23}. Consequently, we have

$$a_{13} = \frac{uw}{\sqrt{(u^2 - n^2)}}\cos\eta + \frac{\sqrt{n^2u^2 - n^2w^2 - n^4}}{\sqrt{(u^2 + a)}}\sin\eta, \quad (25)$$

$$a_{23} = \frac{uw}{\sqrt{(u^2 - n^2)}}\sin\eta - \frac{\sqrt{n^2u^2 - n^2w^2 - n^4}}{\sqrt{(u^2 - n^2)}}\cos\eta. \quad (26)$$

Using equations (12), (23) and (24), and simplifying, we get that

$$a_{12}a_{13} + a_{22}a_{23} - a_{32}a_{33} = 0$$

implies

$$n^4w^2(u^2 - n^2) = n^4(u^2 - n^2)(u^2 + v^2 - n^2),$$

and therefore, $w^2 = u^2 + v^2 - n^2$. Hence, from equations (25) and (26), we find

$$a_{13} = \frac{u\sqrt{u^2 + v^2 - n^2}}{\sqrt{u^2 - n^2}}\cos\eta - \frac{\sqrt{-n^2v^2}}{\sqrt{u^2 - n^2}}\sin\eta,$$

$$a_{23} = \frac{u\sqrt{u^2 + v^2 - n^2}}{\sqrt{u^2 - n^2}}\sin\eta + \frac{\sqrt{-n^2v^2}}{\sqrt{u^2 - n^2}}\cos\eta,$$

which are absurd since $\sqrt{-n^2} \notin \mathbb{R}$.

Next, we assume that $a_{31}^2 = u^2 = n^2$, then $a_{11}^2 + a_{21}^2 - n^2 = -n^2$ implies $a_{11} = a_{21} = 0$. Moreover, from equation (9),we obtain $a_{32} = 0$. Additionally, we have $a_{12}^2 + a_{22}^2 - a_{32}^2 = -n^2$, and thus, $a_{12}^2 + a_{22}^2 = -n^2$, which is a contradiction.

We can obtain similar contradiction for the case $a_{32}^2 = v^2 = n^2$, since this case results in $a_{11}^2 + a_{21}^2 = -n^2$. Thus, Case 2, i.e., $a < 0$, is not possible.

Therefore, the fixed subgroup of the unit circle is given by the matrix M. □

Remark 4 We can also obtain the possible values of a by evaluating the determinant of both the circles. Specifically, under the action of $g \in \mathrm{SL}(3, \mathbb{R})$, the unit circle A transforms to $A' = (g^{-1})^t A g^{-1}$. As $\det(g^t) = \det(g)$ and $\det(g^{-1}) = 1$, we have

$$\begin{aligned}\det((g^{-1})^t A g^{-1}) &= \det((g^{-1})^t)\det(A)\det(g^{-1})\\ &= \det(g^{-1})\det(A)\det(g^{-1}) = \det(A).\end{aligned} \tag{27}$$

Therefore, if

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with $\det(A) = -1$, then A' should be represented by the same matrix: $A' = A$. Otherwise, if

$$A' = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix}$$

for some $a < 0$, then $\det(A') = -a^3 > 0$, contradicting equation (27).

4 Relation between fixed subgroup of the unit circle and SL$(2, \mathbb{R})$

In this section, we develop the interrelationship between the fixed subgroup of the unit circle and SL$(2, \mathbb{R})$, which gives further insights into this subject. Let M be the fixed subgroup of the projective unit circle as mentioned in Theorem 4.

Lemma 1 *The Lie subalgebra* $\mathfrak{m}$ *of the fixed Lie subgroup* M *is isomorphic to* $\mathfrak{sl}(2)$*, the Lie algebra of the special linear group* $SL(2, \mathbb{R})$*.*

Proof. Let us consider the Lie subalgebra $\mathfrak{m}$ of the fixed Lie subgroup M of the projective unit circle. Then, $\mathfrak{m}$ is given by

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}.$$

To show that, for each $X \in \mathfrak{m}$, we will verify that $\exp(tX) \in M$ for all $t \in \mathbb{R}$.

Let

$$X = \begin{pmatrix} 0 & -a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix},$$

then the eigenvalues of tX are 0, $t\sqrt{b^2 + c^2 - a^2}$, and $-t\sqrt{b^2 + c^2 - a^2}$. There are two main cases.

Case 1: $b^2+c^2-a^2=0$. In this case, the only eigenvalue is 0, and it has an algebraic multiplicity of 3. Additionally, matrix X is nilpotent, specifically, $X^3=0$. Therefore,

$$\exp(tX)=\sum_{n=0}^{\infty}\frac{t^nX^n}{n!}=I+tX+\frac{t^2X^2}{2!}$$

$$=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&-at&bt\\at&0&ct\\bt&ct&0\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}(b^2-a^2)t^2&bct^2&-act^2\\bct^2&(c^2-a^2)t^2&abt^2\\act^2&-abt^2&(b^2+c^2)t^2\end{pmatrix}$$

$$=\begin{pmatrix}1+((b^2-a^2)t^2)/2&-at+(bct^2)/2&bt-(act^2)/2\\at+(bct^2)/2&1+((c^2-a^2)t^2)/2&ct+(abt^2)/2\\bt+(act^2)/2&ct-(abt^2)/2&1+((b^2+c^2)t^2)/2\end{pmatrix}.$$

We verify that $\exp(tX)$ satisfies all equations (8)−(14), establishing that $\exp(tX)$ fixes the unit circle. Consequently, $\exp(tX)$ belongs to the Lie subgroup M for all $t\in\mathbb{R}$.
Case 2: $b^2+c^2-a^2\neq 0$.
i) Let $b^2+c^2-a^2=k^2>0$. In this case, tX possesses three distinct eigenvalues: 0, tk and $-tk$, implying that tX is diagonalizable. Consequently, we can express $tX=P^{-1}DP$, where columns of P are formed by the eigenvectors of tX, and D is the diagonal matrix formed by the eigenvalues. Explicitly, when $a\neq 0$,

$$P=\begin{pmatrix}\dfrac{b(b^2+c^2)-a(ab+ck)}{k(b^2+c^2)}&\dfrac{a(ab-ck)-b(b^2+c^2)}{k(b^2+c^2)}&-\dfrac{c}{a}\\\dfrac{ab+ck}{b^2+c^2}&\dfrac{ab-ck}{b^2+c^2}&\dfrac{b}{a}\\1&1&1\end{pmatrix}$$

In the case $a=0$, we observe that $b^2+c^2>0$. Therefore, either

$$P=\begin{pmatrix}b/k&-b/k&-c/b\\c/k&-c/k&1\\1&1&0\end{pmatrix},$$

provided $b\neq 0$, or

$$P=\begin{pmatrix}c/k&-c/k&-b/c\\b/k&-b/k&1\\1&1&0\end{pmatrix},$$

provided $c \neq 0$. Also,

$$D = \begin{pmatrix} tk & 0 & 0 \\ 0 & -tk & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\exp(tX) = P^{-1} \exp(D) P$$

$$= \frac{1}{2k^2} \begin{pmatrix} (b^2-a^2)(e^{tk}+e^{-tk}) & bc(e^{tk}+e^{-tk})-2bc & bk(e^{tk}-e^{-tk})+2ac \\ +2c^2 & -ak(e^{tk}-e^{-tk}) & -ac(e^{tk}+e^{-tk}) \\ & & \\ bc(e^{tk}+e^{-tk})-2bc & (c^2-a^2)(e^{tk}+e^{-tk}) & ck(e^{tk}-e^{-tk})-2ab \\ +ak(e^{tk}-e^{-tk}) & +2b^2 & +ab(e^{tk}+e^{-tk}) \\ & & \\ ac(e^{tk}+e^{-tk})-2ac & 2ab+ck(e^{tk}-e^{-tk}) & (b^2+c^2)(e^{tk}+e^{-tk}) \\ +bk(e^{tk}-e^{-tk}) & -ab(e^{tk}+e^{-tk}) & -2a^2 \end{pmatrix}.$$

In this case, as well, we confirm that $\exp(tX)$ satisfies all equations (8)−(14). Thus, $\exp(tX)$ belongs to the Lie subgroup M for all $t \in \mathbb{R}$.
ii) Let $b^2 + c^2 - a^2 = -k^2 < 0$. The eigenvalues are 0, tki, $-tki$. Thus, $tX = P^{-1}DP$, where

$$P = \begin{pmatrix} \dfrac{b(b^2+c^2)-a(ab+ick)}{ik(b^2+c^2)} & \dfrac{a(ab-ick)-b(b^2+c^2)}{ik(b^2+c^2)} & -\dfrac{c}{a} \\ \dfrac{ab+ick}{b^2+c^2} & \dfrac{ab-ick}{b^2+c^2} & \dfrac{b}{a} \\ 1 & 1 & 1 \end{pmatrix}, \ a \neq 0,$$

and

$$D = \begin{pmatrix} tki & 0 & 0 \\ 0 & -tki & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\exp(tX) = P^{-1}\exp(D)P$$

$$= \frac{1}{k^2}\begin{pmatrix} \begin{matrix}(a^2-b^2)\cos(tk)\\ -c^2\end{matrix} & \begin{matrix}bc-bc\cos(kt)\\ -ak\sin(kt)\end{matrix} & \begin{matrix}ac\cos(kt)-ac\\ +bk\sin(kt)\end{matrix} \\ & & \\ \begin{matrix}bc-bc\cos(kt)\\ +ak\sin(kt)\end{matrix} & \begin{matrix}(a^2-c^2)\cos(kt)\\ -b^2\end{matrix} & \begin{matrix}ck\sin(kt)+ab\\ -ab\cos(kt)\end{matrix} \\ & & \\ \begin{matrix}ac-ac\cos(kt)\\ +bk\sin(kt)\end{matrix} & \begin{matrix}-ab+ab\cos(kt)\\ +ck\sin(kt)\end{matrix} & \begin{matrix}-(b^2+c^2)\cos(kt)\\ +a^2\end{matrix} \end{pmatrix}.$$

In this case, too, we verify that $\exp(tX)$ satisfies all equations $(8)-(14)$, and thus, $\exp(tX) \in M$ for any $t \in \mathbb{R}$.

Hence, we validate that the Lie subalgebra of the fixed subgroup M of the unit circle is given by $\mathfrak{m}$.

Let us now consider a basis $\mathfrak{B}$ for the Lie subalgebra $\mathfrak{m}$. We choose

$$X = \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix},\ Z = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

in $\mathfrak{m}$. Then $\mathfrak{B} = \{X,\ Y,\ Z\}$ forms a basis with the commutator relations $[X,Y] = Z$, $[Y,Z] = -X$, $[Z,X] = Y$, where the Lie bracket is given by $[X,Y] = XY - YX$.

Furthermore, let us consider the Lie algebra of the special linear group, $\mathfrak{sl}(2)$, the Lie algebra of 2×2 trace less real matrices. In $\mathfrak{sl}(2)$, we select the basis vectors as

$$e_1 = \frac{1}{2}\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix},\ e_2 = \frac{1}{2}\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix},\ e_3 = \frac{1}{2}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

Here, $[e_1, e_2] = e_3,\ [e_2, e_3] = -e_1,\ [e_3, e_1] = e_2$.

Therefore, we establish an isomorphism of Lie algebras defined on the basis elements from $\mathfrak{sl}(2)$ to $\mathfrak{m}$ that can be uniquely extended to any element:

$$\phi : \mathfrak{sl}(2) \to \mathfrak{m}$$

such that

$$\phi(e_1) = X,\quad \phi(e_2) = Y,\quad \phi(e_3) = Z \tag{28}$$

with

$$\phi([x,y]) = [\phi(x), \phi(y)] \text{ for all } x,\ y \in \mathfrak{sl}(2).$$

Hence, the result follows. □

Despite $\mathrm{SL}(2,\mathbb{R})$ not being simply connected, we obtain the following result.

Proposition 2 *Consider a matrix Lie group G having the Lie algebra $\mathfrak{g}$. If $\phi : \mathfrak{sl}(2) \to \mathfrak{g}$ stands as a Lie algebra homomorphism, then a corresponding Lie group homomorphism $\Phi : SL(2,\mathbb{R}) \to G$ exists, satisfying $\Phi(\exp X) = \exp\phi(X)$ for all $X \in \mathfrak{sl}(2)$.*

Proof. Here, the central idea is to use the complexification of lie algebras and groups to obtain the notion of simply connectedness. Hence, we consider the complexification of the Lie algebra $\mathfrak{sl}(2)$, denoted as $\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}}$, which is isomorphic to the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ (see [12] for details).

Let $G_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$ denote the complexifications of matrix Lie group G (hence, a closed subgroup of $\mathrm{GL}(n,\mathbb{C})$) and Lie algebra $\mathfrak{g}$, respectively. Since $\phi : \mathfrak{sl}(2) \to \mathfrak{g}$ is a Lie algebra homomorphism, we can extend this homomorphism to the complexification of the Lie algebras (see the subsection 1.3)

$$\tilde{\phi} : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}$$

defined by

$$\tilde{\phi}(X_1 + iX_2) = \phi(X_1) + i\phi(X_2).$$

Consequently, since $\mathrm{SL}(2,\mathbb{C})$ is simply connected, we get a unique Lie group homomorphism (cf. Theorem 3)

$$\tilde{\Phi} : \mathrm{SL}(2,\mathbb{C}) \to G_{\mathbb{C}}$$

satisfying

$$\tilde{\Phi}(\exp(\tilde{X})) = \exp(\phi(\tilde{X})) \text{ for all } \tilde{X} \in \mathfrak{sl}(2,\mathbb{C}).$$

Now, in order to obtain the required map on $\mathrm{SL}(2,\mathbb{R})$ (non-simply connected group), we examine the restriction of $\tilde{\Phi}$ to the real form of $\mathrm{SL}(2,\mathbb{C})$, i.e., to $\mathrm{SL}(2,\mathbb{R})$ (cf. [12]). This restriction is defined as

$$\Phi : \mathrm{SL}(2,\mathbb{R}) \to G$$
$$\Phi(\exp(X)) = \tilde{\Phi}(\exp(X)) \text{ for all } X \in \mathfrak{sl}(2).$$

Therefore, $\Phi(\exp(X)) = \tilde{\Phi}(\exp(X)) = \exp(\phi(X))$ for all $X \in \mathfrak{sl}(2)$. Thus, the result holds. □

Theorem 5 *The fixed subgroup of the projective unit circle $M := M(x,y,\theta)$ such that $x > 1$ is isomorphic to the Lie group $PSL(2,\mathbb{R})$.*

Proof. Using Lemma 1, we consider the isomorphism of the Lie algebras denoted as $\phi : \mathfrak{sl}(2) \to \mathfrak{m}$. Further, Proposition 2 guarantees the existence of a Lie group homomorphism $\Phi : \mathrm{SL}(2,\mathbb{R}) \to M$ such that $\Phi(\exp X) = \exp\phi(X)$ holds for all $X \in \mathfrak{sl}(2)$.

Next, we take into account the Iwasawa decomposition of $g \in \mathrm{SL}(2,\mathbb{R})$ (cf. subsection 1.5), which gives a unique decomposition of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as

$$\begin{aligned} g = KAN &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{a}{\sqrt{a^2+c^2}} & -\frac{c}{\sqrt{a^2+c^2}} \\ \frac{c}{\sqrt{a^2+c^2}} & \frac{a}{\sqrt{a^2+c^2}} \end{pmatrix} \begin{pmatrix} \sqrt{a^2+c^2} & 0 \\ 0 & \frac{1}{\sqrt{a^2+c^2}} \end{pmatrix} \begin{pmatrix} 1 & \frac{ab+cd}{a^2+c^2} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Thus, taking into account that Φ is a group homomorphism and using the definition of exponential map, we can write

$$\begin{aligned} \Phi(g) = \Phi(KAN) = \Phi(K)\Phi(A)\Phi(N) &= \Phi(\exp(k))\Phi(\exp(a))\Phi(\exp(n)) \\ &= \exp(\phi(k))\exp(\phi(a))\exp(\phi(n)), \end{aligned}$$

where

$$k = \begin{pmatrix} 0 & -\cos^{-1}\left(\frac{a}{\sqrt{a^2+c^2}}\right) \\ \cos^{-1}\left(\frac{a}{\sqrt{a^2+c^2}}\right) & 0 \end{pmatrix}, \ n = \begin{pmatrix} 0 & \frac{ad+bc}{a^2+c^2} \\ 0 & 0 \end{pmatrix},$$

and

$$a = \begin{pmatrix} \log\left(\sqrt{a^2+c^2}\right) & 0 \\ 0 & -\log\left(\sqrt{a^2+c^2}\right) \end{pmatrix}.$$

We now explicitly compute

$$\Phi(g) = \exp(\phi(k))\exp(\phi(a))\exp(\phi(n)) \tag{29}$$

to express the map Φ in terms of $a,\ b,\ c,\ d$. Here,

$$k = \cos^{-1}\left(\frac{a}{\sqrt{a^2+c^2}}\right)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus, using equation (28), we obtain

$$\phi(k) = \cos^{-1}\left(\frac{a}{\sqrt{a^2+c^2}}\right)\phi\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = \cos^{-1}\left(\frac{a}{\sqrt{a^2+c^2}}\right)\begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and hence,

$$\exp(\phi(k)) = e^{\phi(k)} = \begin{pmatrix} \frac{a^2-c^2}{a^2+c^2} & -\frac{2ac}{a^2+c^2} & 0 \\ \frac{2ac}{a^2+c^2} & \frac{a^2-c^2}{a^2+c^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{30}$$

Again,

$$a = \begin{pmatrix} \log\left(\sqrt{a^2+c^2}\right) & 0 \\ 0 & -\log\left(\sqrt{a^2+c^2}\right) \end{pmatrix} = \log\left(\sqrt{a^2+c^2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, using equation (28), we get

$$\phi(a) = \log\left(\sqrt{a^2+c^2}\right)\phi\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \log\left(\sqrt{a^2+c^2}\right)\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\exp(\phi(a)) = \begin{pmatrix} \frac{1}{2}\left(a^2+c^2+\frac{1}{a^2+c^2}\right) & 0 & \frac{1}{2}\left(a^2+c^2-\frac{1}{a^2+c^2}\right) \\ 0 & 1 & 0 \\ \frac{1}{2}\left(a^2+c^2-\frac{1}{a^2+c^2}\right) & 0 & \frac{1}{2}\left(a^2+c^2+\frac{1}{a^2+c^2}\right) \end{pmatrix}. \quad (31)$$

Also,

$$n = \begin{pmatrix} 0 & \frac{ab+cd}{a^2+c^2} \\ 0 & 0 \end{pmatrix} = \frac{ab+cd}{a^2+c^2}\left[-\frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right].$$

Thus, taking into account equation (28), we obtain

$$\begin{aligned} \phi(n) &= \frac{ab+cd}{a^2+c^2}\left[-\phi\left(\frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) + \phi\left(\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\right] \\ &= \frac{ab+cd}{a^2+c^2}\left[-\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right] \\ &= \frac{ab+cd}{a^2+c^2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \end{aligned}$$

and therefore,

$$\exp(\phi(n)) = \begin{pmatrix} 1-\frac{(ab+cd)^2}{2(a^2+c^2)^2} & \frac{ab+cd}{(a^2+c^2)} & \frac{(ab+cd)^2}{2(a^2+c^2)^2} \\ -\frac{(ab+cd)}{(a^2+c^2)} & 1 & \frac{(ab+cd)}{(a^2+c^2)} \\ -\frac{(ab+cd)^2}{2(a^2+c^2)^2} & \frac{(ab+cd)}{(a^2+c^2)} & 1+\frac{(ab+cd)^2}{2(a^2+c^2)^2} \end{pmatrix}. \quad (32)$$

Combining equations (29), (30), (31) and (32) and performing the explicit computation of $\Phi(g)$, results in obtaining the Lie group homomorphism $\Phi : \mathrm{SL}(2,\mathbb{R}) \to M$ given by

$$\Phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{2(a^2+c^2)} \times$$

$$\begin{pmatrix} \dfrac{(a^2-c^2)(a^2+c^2)^2 - (a^2-c^2)(ab+cd)^2 + (a^2-c^2) + 4ac(ab+cd)}{(a^2+c^2)} & 2(a^2-c^2)ab + 2(a^2-c^2)cd - 4ac & \dfrac{(a^2-c^2)(a^2+c^2)^2 + (a^2-c^2)(ab+cd)^2 - (a^2-c^2) - 4ac(ab+cd)}{(a^2+c^2)} \\ \dfrac{2ac(a^2+c^2)^2 - 2ac(ab+cd)^2 + 2ac - 2(ab+cd)(a^2-c^2)}{(a^2+c^2)} & 4ac(ab+cd) + 2(a^2-c^2) & \dfrac{2ac(a^2+c^2)^2 + 2ac(ab+cd)^2 - 2ac + 2(ab+cd)(a^2-c^2)}{(a^2+c^2)} \\ (a^2+c^2)^2 - 1 - (ab+cd)^2 & 2ab(a^2+c^2) + 2cd(a^2+c^2) & (a^2+c^2)^2 + 1 + (ab+cd)^2 \end{pmatrix}.$$

Here, we see that the a_{33}-th entry

$$\frac{(a^2+c^2)^2+1+(ab+cd)^2}{2(a^2+c^2)} > 0. \tag{33}$$

We also have $\Phi^{-1} : M \to \mathrm{SL}(2,\mathbb{R})$ such that

$$\Phi^{-1}\left(\begin{bmatrix} \dfrac{xy}{\sqrt{x^2-1}}\cos\theta + \dfrac{\sqrt{x^2-y^2-1}}{\sqrt{x^2-1}}\sin\theta & \pm\dfrac{x\sqrt{x^2-y^2-1}}{\sqrt{x^2-1}}\cos\theta \mp \dfrac{y}{\sqrt{x^2-1}}\sin\theta & \sqrt{x^2-1}\cos\theta \\ \dfrac{xy}{\sqrt{x^2-1}}\sin\theta - \dfrac{\sqrt{x^2-y^2-1}}{\sqrt{x^2-1}}\cos\theta & \pm\dfrac{x\sqrt{x^2-y^2-1}}{\sqrt{x^2-1}}\sin\theta \pm \dfrac{y}{\sqrt{x^2-1}}\cos\theta & \sqrt{x^2-1}\sin\theta \\ y & \pm\sqrt{x^2-y^2-1} & x \end{bmatrix}\right)$$

$$= \begin{pmatrix} \sqrt{\frac{(\sqrt{x^2-1}+x)(\sqrt{x^2-1}+y)}{2\sqrt{x^2-1}}}\cos\frac{\theta}{2} & \pm\sqrt{\frac{(\sqrt{x^2-1}+x)(\sqrt{x^2-1}-y)}{2\sqrt{x^2-1}}}\cos\frac{\theta}{2} \\ \pm\sqrt{\frac{(\sqrt{x^2-1}-y)}{2(\sqrt{x^2-1}+x)\sqrt{x^2-1}}}\sin\frac{\theta}{2} & -\sqrt{\frac{(\sqrt{x^2-1}+y)}{2(\sqrt{x^2-1}+x)\sqrt{x^2-1}}}\sin\frac{\theta}{2} \\ & \\ \sqrt{\frac{(\sqrt{x^2-1}+x)(\sqrt{x^2-1}+y)}{2\sqrt{x^2-1}}}\sin\frac{\theta}{2} & \pm\sqrt{\frac{(\sqrt{x^2-1}+x)(\sqrt{x^2-1}-y)}{2\sqrt{x^2-1}}}\sin\frac{\theta}{2} \\ \mp\sqrt{\frac{(\sqrt{x^2-1}-y)}{2(\sqrt{x^2-1}+x)\sqrt{x^2-1}}}\cos\frac{\theta}{2} & +\sqrt{\frac{(\sqrt{x^2-1}+y)}{2(\sqrt{x^2-1}+x)\sqrt{x^2-1}}}\cos\frac{\theta}{2} \end{pmatrix},$$

and

$$\Phi^{-1}\left(\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\right) = \begin{pmatrix}\cos(\theta/2) & -\sin(\theta/2)\\ \sin(\theta/2) & \cos(\theta/2)\end{pmatrix}.$$

Hence, for real values, it is necessary that $x^2-1>0$. Additionally, equation (33) implies $x>0$. Consequently, satisfying both conditions, namely $x^2-1>0$ and $x>0$, leads to the conclusion that $x>1$.

Again, the kernel of Φ is defined as

$$\mathrm{Ker}(\Phi) = \{g \in \mathrm{SL}(2,\mathbb{R}) \mid \Phi(g) = I_{3\times 3} \in M\}.$$

If $\Phi(g) = I_{3\times3}$, then we must have

$$ab+cd=0,\ \frac{(a^2-c^2)(ab+cd)-2ac}{a^2+c^2}=0,\ \frac{(a^2+c^2)^2-1-(ab+cd)^2}{2(a^2+c^2)}=0,$$

and

$$\frac{2ac(ab+cd)+(a^2-c^2)}{a^2+c^2}=1.$$

Consequently, the derived conditions lead to the conclusion $a^2=1$, implying $a=\pm1$, $b=0$, $c=0$. Thus,

$$\begin{pmatrix}\pm1 & 0\\ 0 & \pm1\end{pmatrix} \in Ker(\Phi).$$

Hence, we obtain the homomorphism $\Phi : \mathrm{SL}(2,\mathbb{R}) \to M(x,y,\theta)$ with kernel $\{\pm I\}$, and the inverse map Φ^{-1} exists for every $m(x,y,\theta) \in M$, where $x>1$. In particular,

$$M := M(x,y,\theta) \cong \mathrm{SL}(2,\mathbb{R})/\{\pm I\} = \mathrm{PSL}(2,\mathbb{R})$$

where $x>1$. Thus, the theorem follows. $\square$

Remark 5 To generate the inverse map, we have decomposed any element $m \in M$ in the following manner, which significantly simplifies the computations of the inverse map. For any

$$m = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{xy}{\sqrt{x^2-1}} & \pm\frac{x\sqrt{x^2-y^2-1}}{\sqrt{x^2-1}} & \sqrt{x^2-1} \\ -\frac{\sqrt{x^2-y^2-1}}{\sqrt{x^2-1}} & \pm\frac{y}{\sqrt{x^2-1}} & 0 \\ y & \pm\sqrt{x^2-y^2-1} & x \end{pmatrix}$$

in M, we have $\Phi^{-1} : M \to \mathrm{SL}(2,\mathbb{R})$ such that

$$\Phi^{-1}\left(\begin{bmatrix} \frac{xy}{\sqrt{x^2-1}} & \frac{x\sqrt{x^2-y^2-1}}{\sqrt{x^2-1}} & \sqrt{x^2-1} \\ -\frac{\sqrt{x^2-y^2-1}}{\sqrt{x^2-1}} & \frac{y}{\sqrt{x^2-1}} & 0 \\ y & \sqrt{x^2-y^2-1} & x \end{bmatrix} \right)$$

$$= \begin{pmatrix} \sqrt{\frac{(\sqrt{x^2-1}+x)(\sqrt{x^2-1}+y)}{2\sqrt{x^2-1}}} & \pm\sqrt{\frac{(\sqrt{x^2-1}+x)(\sqrt{x^2-1}-y)}{2\sqrt{x^2-1}}} \\ \mp\sqrt{\frac{(\sqrt{x^2-1}-y)}{2(\sqrt{x^2-1}+x)\sqrt{x^2-1}}} & \sqrt{\frac{(\sqrt{x^2-1}+y)}{2(\sqrt{x^2-1}+x)\sqrt{x^2-1}}} \end{pmatrix},$$

and

$$\Phi^{-1}\left(\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

This gives the description of $\Phi^{-1}(m)$ since Φ is a homomorphism.

5 Conclusion

We have explored the action of the transformation group $\mathrm{SL}(3,\mathbb{R})$ on the two dimensional homogeneous space $\mathbb{RP}^2$. Our inquiry has involved an extension of the map i associated with $\mathrm{SL}(2,\mathbb{R})$ action, and we have established the non-existence of such a map for $\mathrm{SL}(3,\mathbb{R})$ action. Following this, we have investigated the isotropy subgroup of the projective unit circle, providing a detailed and explicit expression for it. Furthermore, we use a factorization of $\mathrm{SL}(2,\mathbb{R})$, specifically the Iwasawa decomposition, to show that, under certain conditions, the isotropy subgroup of unit circle under $\mathrm{SL}(3,\mathbb{R})$ action is isomorphic to the projective special linear group $\mathrm{PSL}(2,\mathbb{R})$.

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