The Geometry of the Projective Action of $\text{SL}(3, \mathbb{R})$ from the Erlangen Perspective

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Abstract. In this paper, we have investigated the projective action of the Lie group $\text{SL}(3, \mathbb{R})$ on the homogeneous space $\mathbb{RP}^2$. In particular, we have studied the action of the subgroups of $\text{SL}(3, \mathbb{R})$ on the non-degenerate conics in the space $\mathbb{RP}^2$. Using the Iwasawa decomposition of $\text{SL}(2, \mathbb{R})$, we demonstrate that the isotropy subgroup of the projective unit circle is isomorphic to $\text{PSL}(2, \mathbb{R})$ under certain conditions.

Key Words: Lie Group $\text{SL}(3, \mathbb{R})$, Homogeneous Space, Conics, Exponential Map, Iwasawa Decomposition

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Introduction

Geometry as a group action has dynamic aspects in several branches of mathematics. Starting with Russell, following Klein, this area has been explored by many researchers over time (see, for example, [8, 10, 19, 25]). In this trend, the action of $G = \text{SL}(2, \mathbb{R})$ on the one-dimensional homogeneous space $G/H$ represented by Möbius transformation was extensively studied and investigated by several authors, for example, in [1, 4, 14, 15]. In the series of works [2, 5, 16], the authors proposed advancing a theory of analytic functions inspired by Klein’s Erlangen program. In [18], Kisil considered the Möbius action of $\text{SL}(2, \mathbb{R})$ on $\mathbb{RP}^1$ and described different possible realisation of Poincaré extension of Möbius transformation, emphasizing the properties of Möbius-invariant cycles. In the same line, $\text{SL}(2, \mathbb{R})$ invariant geodesic curves and metrics were studied in [3, 6, 9]. These series of works lead to a number of natural and effective generalisations, and hence, it becomes imperative to investigate how things work in the higher dimensional cases. In the parallel line, there is a wide range of literature devoted to the representation theory of Lie groups and symmetric spaces, where the classical
cases such as the Lie groups SL(2, R), SL(3, R), SL(2, C), SU(2), etc., have provided powerful tools for studying these groups and other related results, see [11, 22–24]. Moreover, projective geometry is a rich and fascinating field that provides a geometric foundation for a variety of disciplines, including projective algebraic geometry, differential geometry, and the theory of algebraic curves. An integral aspect of this field involves the investigation of the projective action of SL(3, R) on projective space, providing a potent tool for exploring geometric relationships and structures.

In this manner, following the line of Kisil, we have taken the transformation group as SL(3, R) and focused on the invariant objects, with the aim of incorporating more invariants in the existing SL(2, R) as well as in SL(3, R) geometry. Thus, by studying invariants, we propose to construct geometry of the homogeneous space \( \mathbb{RP}^2 \) in a systematic way. In this work, as a continuation of [7], the action of the transformation group SL(3, R) on the two dimensional homogeneous space \( \mathbb{RP}^2 \) is studied. For SL(3, R) instance, we examine an extension of the map \( i \) associated with SL(2, R) action. We also investigate the projective action of the subgroups of SL(3, R) on the non-degenerate conics in the space \( \mathbb{RP}^2 \). Iwasawa decomposition plays an essential part in the current discussion.

1 Preliminaries

This section reviews standard definitions and theorems related to our work.

A transformation group \( G \) can be defined as a non-empty collection of mappings from a set \( X \) to itself, adhering to the following conditions: (i) the identity map is an element of \( G \); (ii) if \( g_1 \in G \) and \( g_2 \in G \), then composition \( g_1g_2 \in G \); (iii) if \( g \in G \), then the inverse \( g^{-1} \) exists and is a member of \( G \).

Additionally, a group action \( \varphi : G \times X \to X \) is termed transitive if for every \( x, y \in X \), there exists \( g \in G \) satisfying \( g \cdot x = y \). Furthermore, a homogeneous space is defined as a pair \( (G, X) \), where the action of the group \( G \) on \( X \) is transitive and \( X \) is a topological space.

Following notions were introduced in [12]. A matrix Lie group is a subgroup \( G \) of GL(n, R) that exhibits the subsequent characteristic: if \( A_m \) denotes a sequence of matrices within \( G \) that converges to a matrix \( A \), then either \( A \) belongs to \( G \) or \( A \) is not invertible. For a matrix Lie group \( G \), the associated Lie algebra, represented as \( \mathfrak{g} \), is characterized as the collection of matrices \( X \) for which \( \exp(tX) \) is a member of \( G \) for all real numbers \( t \).

If \( G \) is a matrix Lie group with Lie algebra \( \mathfrak{g} \), then the exponential mapping of \( G \) is defined as the map \( \exp : \mathfrak{g} \to G \). Hence, the exponential mapping of \( G \) is the matrix exponential restricted to the Lie algebra \( \mathfrak{g} \) of \( G \).
1.1 Group action on coset spaces

Let \( G \) be a Lie group and \( H \) be a closed subgroup of \( G \). Then it follows by Cartan’s theorem that \( H \) is a Lie group (cf. [21]).

Let \( G/H = \{ gH : g \in G \} \) denotes the space of left cosets of \( H \). In this context, the projection map \( p : G \to G/H \) is defined by mapping \( g \in G \) to its equivalence class \([g]\), expressed as \( p(g) = gH = [g] \). Also, a section \( s \) of a projection map \( p \) is a right inverse of \( p \), denoted by \( s : G/H \to G \), satisfying \( p(s(x)) = x \) for all \( x \in G/H \).

**Theorem 1** [13] Consider a Lie group \( G \) with a closed subgroup \( H \). Let \( G/H \) has the quotient topology. Then \( G/H \) possesses a unique smooth manifold structure such that the projection map \( p : G \to G/H \) is a smooth submersion and \( G \) acts smoothly on \( G/H \).

**Remark 1** The action \( G \times G/H \to G/H \) defined as \((a, gH) \mapsto agH\) can be viewed as a composition of smooth maps as follows:

\[
\phi : G \times G/H \to G/H \\
\phi(g, x) = g \cdot x = p(g \ast s(x)),
\]

where \( \ast \) denotes the group operation on \( G \).

1.2 Real projective space \( \mathbb{R}P^n \)

Let \( \mathbb{R}^{n+1} = \{(x_1, x_2, \ldots, x_{n+1}) : x_i \in \mathbb{R}\} \). The real projective space \( \mathbb{R}P^n \) consists of points which are equivalence classes of the set \( \mathbb{R}^{n+1} \setminus \{0\} \) modulo the equivalence relation \( x \sim \lambda x \) for all \( \lambda \) in \( \mathbb{R} \setminus \{0\} \).

In particular, the space \( \mathbb{R}P^1 \) is called the real projective line, while \( \mathbb{R}P^2 \) is called the real projective plane.

In the real projective plane, a point is represented by a triple \((X, Y, Z)\), referred to as homogeneous coordinates or projective coordinates of the point, where \( X, Y \) and \( Z \) are not all zero. Since points in \( \mathbb{R}P^2 \) are equivalence classes, in the homogeneous coordinated setup, the coordinates \((X, Y, Z)\) and \((\lambda X, \lambda Y, \lambda Z)\) are considered to represent the same point for all \( \lambda \neq 0 \) in \( \mathbb{R} \), see [10].

A line \( L \) in the projective plane \( \mathbb{R}P^2 \) can be represented by the homogeneous coordinates \( L = (a, b, c)^t \) and is defined by the equation \( ax + by + cz = 0 \), or in matrix notation \( X^tL = 0 \), where \( X = (x, y, z)^t \) is any point in \( L \).

Similarly, a conic in \( \mathbb{R}P^2 \) is characterized as the set of points for which a quadratic form on \( \mathbb{R}^3 \) vanishes. The conic associated with a quadratic form \( A \) is given by \( C_A = \{ [p] \in \mathbb{R}P^2 : p^tAp = 0 \} \), as described in [10].
1.3 Complexification of real Lie algebras and real Lie groups

Consider a finite dimensional real Lie algebra \( g \), which is essentially a real vector space. The complexification of \( g \), denoted as \( g_C \), is defined as the real vector space consisting of linear combinations \( X_1 + iX_2 \), where \( X_1 \) and \( X_2 \) are elements of \( g \). The bracket operation on \( g \) naturally extends to \( g_C \), turning it into a complex Lie algebra. \( g_C \) is referred to as the complexification of the real Lie algebra \( g \), as introduced in [12].

The complexification of a Lie group \( G \) defined over \( \mathbb{R} \) is a complex Lie group denoted as \( G_C \), which includes \( G \) as a real Lie subgroup. This inclusion ensures that the Lie algebra \( g \) of \( G \) is a real form of the Lie algebra \( g_C \) of \( G_C \). The Lie group \( G \) is termed a real form of the Lie group \( G_C \).

1.4 Correspondence between Lie group and Lie algebra homomorphisms

\textbf{Theorem 2} [12] Consider matrix Lie groups \( G \) and \( H \) with corresponding Lie algebras \( g \) and \( h \). If \( \Phi : G \rightarrow H \) is a Lie group homomorphism, there exists a unique linear map \( \phi : g \rightarrow h \), satisfying the condition \( \Phi(e^X) = e^{\phi(X)} \) for all \( X \in g \).

The converse of Theorem 2 holds true under certain condition.

\textbf{Theorem 3} [12] Consider matrix Lie groups \( G \) and \( H \) with corresponding Lie algebras \( g \) and \( h \). Suppose \( \phi : g \rightarrow h \) is a Lie algebra homomorphism, and \( G \) is simply connected. In that case, there exists a unique Lie group homomorphism \( \Phi : G \rightarrow H \), ensuring that \( \Phi(e^X) = e^{\phi(X)} \) for all \( X \in g \).

We will utilize the notion of complexification to extend the results of Theorem 3 to the case involving \( \text{SL}(2,\mathbb{R}) \), even though it does not possess the property of simply connectedness.

\textbf{Remark 2} Here we discuss some properties of the matrix Lie group \( \text{SL}(2,\mathbb{R}) \) and the corresponding Lie algebra \( \mathfrak{sl}(2) \).

1. The exponential mapping for the matrix Lie group \( \text{SL}(2,\mathbb{R}) \) is not onto. As an illustration, consider the matrix

\[
A = \begin{pmatrix}
-1 & 1 \\
0 & -1
\end{pmatrix} \in \text{SL}(2,\mathbb{R}).
\]

However, there is no \( X \in \mathfrak{sl}(2) \) such that \( \exp(X) = e^X = A \).

2. \( \text{SL}(2,\mathbb{R}) \) is not simply connected.

3. If \( X \) and \( Y \) belong to \( \mathfrak{sl}(2) \), then \( X \) and \( Y \) do not necessarily commute with their commutator \( [X,Y] \). Hence, we have \( e^X e^Y \neq e^{X+Y} \) and \( e^X e^Y \neq e^{X+Y+\frac{1}{2}[X,Y]} \).
1.5 Iwasawa decomposition

The Iwasawa decomposition of SL(n, R) expresses an element g in the group as a product of three matrices, each belonging to a specific subgroup.

Specifically, for n = 2, i.e., SL(2, R), this decomposition takes the form

\[ SL(2, \mathbb{R}) = KAN \]

where

\[ A = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{R}, \ t > 0 \right\}, \]

\[ N = \left\{ \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} : \ \eta \in \mathbb{R} \right\}, \]

\[ K = SO(2, \mathbb{R}) = \left\{ M \in SL(2, \mathbb{R}) : MM^T = M^T M = I_2 \right\}. \]

This unique decomposition, detailed in [20], is known as the Iwasawa decomposition of SL(2, R). It plays a crucial role in connecting the fix subgroup of the projective unit circle to PSL(2, R) in our exploration.

1.6 Two dimensional homogeneous space \( \mathbb{RP}^2 \)

As in [7], we consider the action of SL(3, R) on the space of left cosets \( X = G/H \), where \( G = SL(3, \mathbb{R}) \) and

\[ H = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \middle| a_{22}a_{33} - a_{23}a_{32} = \frac{1}{a_{11}}, a_{11} \neq 0 \right\}. \]

Expressed in terms of the parametrization \( z = (x, y) \in X \), the set-theoretic action of SL(3, R) on SL(3, R)/H can be formulated as a composition of smooth maps as follows:

\[ g : z \mapsto g \cdot z = p(g * s(z)). \]

We can define another map \( r : G \to H \) such that \( r(g) = h \), where \( h = s(p(g))^{-1}g \). Hence, \( g \) can be uniquely written as \( g = s(p(g))r(g) \) (see [7] for the details).

In this set up, the SL(3, R) action takes the form

\[ (x, y) \mapsto \left( \begin{array}{c} a_{11}x + a_{12}y + a_{13} \\ a_{21}x + a_{22}y + a_{23} \\ a_{31}x + a_{32}y + a_{33} \end{array} \right), \]

provided \( a_{31}x + a_{32}y + a_{33} \neq 0 \).

Now, if we allow \( a_{31}x + a_{32}y + a_{33} = 0 \), this action indeed gives us a projective transformation of the space \( \mathbb{RP}^2 \). This SL(3, R) action on \( \mathbb{RP}^2 \) is denoted as \( g : [p] \mapsto [g \cdot p] \). Let \( \phi \) be the action defined by

\[ \phi : SL(3, \mathbb{R}) \times \mathbb{RP}^2 \to \mathbb{RP}^2 \phi(g, [p]) = [g \cdot p]. \]

We consider the projective transformation \( \phi_g : \mathbb{RP}^2 \to \mathbb{RP}^2 \) such that \( \phi_g([p]) = [g \cdot p] \) for all \( g \in SL(3, \mathbb{R}) \).
2 Extension of the map $i$

In the consideration of the M"obius action of $\text{SL}(2, \mathbb{R})$, for the study of the invariant properties of cycles (cf. [17]), the Fillmore Springer Cnops construction (FSCc) is a useful construction that allows the association of a cycle with a $2 \times 2$ cycle matrix via the coefficients. The most important component of this construction is the map $i$, which corresponds to a column vector a row vector by the rule (see [17,18])

$$i: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x \end{pmatrix}.$$

**Remark 3** Let us now consider some properties of the map $i$. We consider the left multiplication of $\text{SL}(2, \mathbb{R})$ on column vector $(x, y)^t \in \mathbb{R}^2$

$$\mathcal{L}_g: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}, \text{ where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}).$$

1. The map $i$ is a linear map. Indeed, for $X = (x_1, y_1)^t$ and $Y = (x_2, y_2)^t$, we have $i(X + Y) = i(X) + i(Y)$ since

$$i \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} y_1 + y_2, -x_1 - x_2 \end{pmatrix} = (y_1, -x_1) + (y_2, -x_2) = i \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right) + i \left( \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right),$$

and $i(cX) = ci(X)$, which follows from

$$i \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} cy \\ -cx \end{pmatrix} = ci \left( \begin{pmatrix} x \\ y \end{pmatrix} \right).$$

2. The map $i$ intertwines the left multiplication $\mathcal{L}_g$ (cf. equation [1]) and the right multiplication $\mathcal{R}_g^{-1}$, that is, $i(\mathcal{L}_g x) = i(x)\mathcal{R}_g^{-1}$. Indeed,

$$\mathcal{R}_g^{-1}: (y, -x) \mapsto (cx + dy, -by - ax) = (y, -x) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = i \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right).$$

In order to extend the map $i$ in three-dimensions, we consider the left multiplication of $\text{SL}(3, \mathbb{R})$ on the three-dimensional column vector $(x, y, z)^t \in \mathbb{R}^3$:

$$\mathcal{L}_g: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $g = (a_{ij}) \in \text{SL}(3, \mathbb{R})$. 
**Proposition 1** There is no non-zero map $i$ that corresponds a three-dimensional column vector to a three-dimensional row vector that is both linear and intertwines the left multiplication $L_g$ and the right multiplication $R_{g^{-1}}$, where $g \in \text{SL}(3, \mathbb{R})$.

**Proof.** Suppose there exists linear map $i : \mathbb{R}^3 \to \mathbb{R}^3$ from three-dimensional real column vector space to three-dimensional real row vector space such that $i(L_g X) = i(X) R_{g^{-1}}$, where $g \in \text{SL}(3, \mathbb{R})$, $X \in \mathbb{R}^3$.

Since it is sufficient to know the linear map on basis vectors, let us choose a basis of $\mathbb{R}^3$ as $e_1 = (1, 0, 0)^t$, $e_2 = (0, 1, 0)^t$, $e_3 = (0, 0, 1)^t$. Now, we consider the linear map applied to these basis vectors as follows

$$i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (x_1, x_2, x_3), \quad i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (y_1, y_2, y_3) \text{ and } i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (z_1, z_2, z_3).$$

Here, we have

$$i(L_g e_i) = i(e_i) R_{g^{-1}} \text{ for all } g \in \text{SL}(3, \mathbb{R}), \; i = 1, 2, 3. \quad (2)$$

We proceed to establish that for certain values of $g$ in $\text{SL}(3, \mathbb{R})$, relation (2) leads to a zero-map.

In particular, consider

$$g = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}(3, \mathbb{R})$$

such that

$$i(L_g e_1) = i \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (x_1, x_2, x_3)$$

and

$$i(e_1) R_{g^{-1}} = (x_1, x_2, x_3) \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = (x_1, -x_1 + x_2, -x_2 + x_3).$$

Then from $i(L_g e_1) = i(e_1) R_{g^{-1}}$, it follows that

$$(x_1, x_2, x_3) = (x_1, -x_1 + x_2, -x_2 + x_3),$$

and hence, $x_2 = -x_1 + x_2$ and $x_3 = -x_2 + x_3$, that is, $x_1 = 0$ and $x_2 = 0$.

Now, take

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in \text{SL}(3, \mathbb{R}),$$
so that

\[ i(\mathcal{L}_{e_1}) = i \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (x_1, x_2, x_3) \]

and

\[ i(e_1)R_{g^{-1}} = (x_1, x_2, x_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = (x_1, x_2 - x_3, x_3). \]

Thus, the relation defined in equation (2) yields

\[ (x_1, x_2, x_3) = (x_1, x_2 - x_3, x_3), \]

which implies \( x_2 = x_2 - x_3 \), and hence, \( x_3 = 0 \).

Therefore, to achieve \( i(\mathcal{L}_{e_1}) = i(e_1)R_{g^{-1}} \) for all \( g \in \text{SL}(3, \mathbb{R}) \), we must possess

\[ i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (x_1, x_2, x_3) = (0, 0, 0). \] (3)

In a similar way, if we consider

\[ g = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}(3, \mathbb{R}), \]

then \( i(\mathcal{L}_{e_2}) = i(e_2)R_{g^{-1}} \) leads to

\[ i \left( \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (y_1, y_2, y_3) \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

which simplifies to

\[ i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (y_1 - y_2, y_2, -y_1 + y_3). \]

Hence, \((y_1, y_2, y_3) = (y_1 - y_2, y_2, -y_1 + y_3)\), which implies \( y_1 = y_1 - y_2 \) and \( y_3 = -y_1 + y_3 \), and therefore, \( y_2 = 0 \) and \( y_1 = 0 \).

When

\[ g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad i(\mathcal{L}_{e_2}) = i(e_2)R_{g^{-1}}, \]
we find that \( y_3 = 0 \). Thus,

\[
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} = (y_1, y_2, y_3) = (0, 0, 0).
\]

(4)

Similarly, considering

\[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} \rightarrow (z_1, z_2, z_3),
\]

we observe that for

\[
g = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix},
\]

the condition \( i(L_g e_3) = i(e_3)R_{g^{-1}} \) implies \( z_2 = z_3 = 0 \). Additionally, if

\[
g = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

then the condition \( i(L_g e_3) = i(e_3)R_{g^{-1}} \) results in \( z_1 = 0 \). Hence,

\[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} = (z_1, z_2, z_3) = (0, 0, 0).
\]

(5)

Therefore, combining equations (3), (4) and (5), we obtain

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
x \\
1 \\
0
\end{pmatrix} + \begin{pmatrix}
y \\
0 \\
1
\end{pmatrix} + \begin{pmatrix}
z \\
0 \\
1
\end{pmatrix} = x(0, 0, 0) + y(0, 0, 0) + z(0, 0, 0) = (0, 0, 0).
\]

Thus, the only linear map which intertwines the left multiplication \( L_g \) and the right multiplication \( R_{g^{-1}} \) \((g \in SL(3, \mathbb{R}))\) is the zero map. □

3 Fixed subgroup of the unit circle

The preceding section indicates that conics in \( \mathbb{RP}^2 \) cannot be addressed analogously to cycles of \( SL(2, \mathbb{R}) \) in terms of FSCc. Therefore, we now examine conics through their fixed subgroups.

Let us consider the equation of the conic \( C \) (see subsection 1.2) as

\[
ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + f z^2 = 0
\]
or
\[
(x \ y \ z) \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,
\]
that is, \(p'Ap = 0\).

Let \(A\) be the matrix associated with the conic \(C\). If we transform the conic projectively, then under the transformation \(g\), the point \(p\) on the conic maps to the point \(p' = gp\). Therefore, from \((g^{-1}p')^tA(g^{-1}p') = 0\), it follows \(p''(g^{-1})^tA(g^{-1})p' = 0\). Thus, if a conic is represented by the matrix \(A\), then under the action of \(g\), the transformed conic is represented by the matrix \((g^{-1})^tAg^{-1}\). Furthermore, it is well-known that all non-degenerate conics are the projective images of a unit circle (see, for example, [10]). Let us now consider the equation of the unit circle \(C\) in homogeneous coordinates:
\[
x^2 + y^2 = z^2.
\]
(6)
The matrix associated to the circle \(C\) defined by (6) is
\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]
where \(X^tAX = 0, \ X = (x, y, z)^t\).

**Theorem 4** Under the projective action, the fixed subgroup of the unit circle \(M := M(x, y, \theta)\) is a three-dimensional subgroup of \(SL(3, \mathbb{R})\) given by
\[
M = \begin{cases} 
\left( \begin{array}{ccc} \frac{xy}{\beta} \cos \theta + \frac{\alpha}{\beta} \sin \theta & \pm \frac{x\alpha}{\beta} \cos \theta \mp \frac{y}{\beta} \sin \theta & \beta \cos \theta \\
\frac{xy}{\beta} \sin \theta - \frac{\alpha}{\beta} \cos \theta & \pm \frac{x\alpha}{\beta} \sin \theta \pm \frac{y}{\beta} \cos \theta & \beta \sin \theta \\
y & \pm \alpha & x \end{array} \right), & x^2 \neq 1, \\
\left( \begin{array}{ccc} \pm \cos \theta & \mp \sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & \pm 1 \end{array} \right), & x^2 = 1,
\end{cases}
\]
where \(\alpha = \sqrt{x^2 - y^2 - 1}\) and \(\beta = \sqrt{x^2 - 1}\).

**Proof.** To determine the fixed subgroup of the projective unit circle, let us note that under the transformation of \(g \in SL(3, \mathbb{R})\), the unit circle \(C\) remains
invariant. Therefore, the transformed unit circle must be represented by the matrix

\[ A' = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix} \]

for some \( a \neq 0 \), since it represents the same circle \( \text{(6)} \). Hence, the fixed subgroup of the circle \( C \) is determined by

\[ K_C = \left\{ g \in \text{SL}(3, \mathbb{R}) \mid (g^{-1})' A g^{-1} = A' \right\}. \]

Let

\[ h = g^{-1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \]

then \( h^t A h = A' \) gives

\[ \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix}, \]

and hence,

\[ \begin{pmatrix} a_{11}^2 + a_{21}^2 & a_{11} a_{12} + a_{21} a_{22} & a_{11} a_{13} + a_{21} a_{23} \\ -a_{31}^2 & -a_{31} a_{32} & -a_{31} a_{33} \\ -a_{31} a_{32} & a_{12}^2 + a_{22}^2 & a_{12} a_{13} + a_{22} a_{23} \\ -a_{32}^2 & -a_{32} a_{33} & -a_{32} a_{33} \\ -a_{31} a_{33} & a_{13}^2 + a_{23}^2 & a_{13} a_{13} + a_{23} a_{23} \\ -a_{33}^2 & -a_{33} a_{33} & -a_{33} a_{33} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix}. \]  (7)

Equating \( a_{ij} \)-th entry of both sides of equation \( \text{(7)} \), we get

\[ a_{11}^2 + a_{21}^2 - a_{31}^2 = a, \]  (8)
\[ a_{11} a_{12} + a_{21} a_{22} - a_{31} a_{32} = 0, \]  (9)
\[ a_{11} a_{13} + a_{21} a_{23} - a_{31} a_{33} = 0, \]  (10)
\[ a_{12}^2 + a_{22}^2 - a_{32}^2 = a, \]  (11)
\[ a_{12} a_{13} + a_{22} a_{23} - a_{32} a_{33} = 0, \]  (12)
\[ a_{13}^2 + a_{23}^2 - a_{33}^2 = -a, \]  (13)

such that

\[ \det((a_{ij})) = 1. \]  (14)
Depending on the value of $a$, there are two cases to follow.

Case 1: $a$ is positive. Let $a = m^2 > 0$. Referring to equation (13), it is evident that $a_{33} \neq 0$, or else $a_{13}^2 + a_{23}^2 = -a = -m^2$, which leads to a contradiction. Let $a_{33} = x \neq 0$. From equation (13), we derive $a_{13}^2 + a_{23}^2 = x^2 - a$, which further yields

$$a_{13} = \sqrt{x^2 - a} \ \cos \theta, \quad a_{23} = \sqrt{x^2 - a} \ \sin \theta.$$  

Additionally, considering equation (8) and assigning $a_{31} = y$, we obtain

$$a_{11}^2 + a_{21}^2 = a + y^2,$$  

resulting in the expressions

$$a_{11} = \sqrt{y^2 + a} \ \cos \phi,$$
$$a_{21} = \sqrt{y^2 + a} \ \sin \phi.$$  

(15)  

(16)

Subsequently, from equation (10), we get $a_{11} a_{13} + a_{21} a_{23} - a_{31} a_{33} = 0$, which implies

$$\sqrt{y^2 + a} \cos \phi \sqrt{x^2 - a} \ \cos \theta + \sqrt{y^2 + a} \sin \phi \sqrt{x^2 - a} \ \sin \theta - xy = 0,$$

or

$$\cos(\theta - \phi) = \frac{xy}{\sqrt{(y^2 + a)(x^2 - a)}}$$

provided $x^2 \neq a$, and hence,

$$\phi = \theta - \cos^{-1} \left( \frac{xy}{\sqrt{(y^2 + a)(x^2 - a)}} \right).$$

Therefore, substituting the value of $\phi$ into equations (15) and (16), and omitting cumbersome but trivial calculations, we obtain

$$a_{11} = \sqrt{y^2 + a} \cos \left( \theta - \cos^{-1} \left( \frac{xy}{\sqrt{(y^2 + a)(x^2 - a)}} \right) \right)$$

$$= \frac{xy}{\sqrt{(x^2 - a)}} \ \cos \theta + \sqrt{ax^2 - ay^2 - a^2} \ \sin \theta$$  

(17)

and

$$a_{21} = \sqrt{y^2 + a} \sin \left( \theta - \cos^{-1} \left( \frac{xy}{\sqrt{(y^2 + a)(x^2 - a)}} \right) \right)$$

$$= \frac{xy}{\sqrt{(x^2 - a)}} \ \sin \theta - \sqrt{ax^2 - ay^2 - a^2} \ \cos \theta.$$  

(18)

Again, from equation (11), we have $a_{12}^2 + a_{22}^2 - a_{32}^2 = a$. Let $a_{32} = z$. Given that $a_{12} a_{13} + a_{22} a_{23} - a_{32} a_{33} = 0$, we can replace $y$ by $z$ in equations
(17) and (18) to derive the values of $a_{12}$ and $a_{22}$. Thus, we get
\begin{align*}
a_{12} &= \frac{xz}{\sqrt{(x^2 - a)}} \cos \theta + \frac{\sqrt{a x^2 - a z^2 - a^2}}{\sqrt{(x^2 - a)}} \sin \theta, \\
a_{22} &= \frac{xz}{\sqrt{(x^2 - a)}} \sin \theta - \frac{\sqrt{a x^2 - a z^2 - a^2}}{\sqrt{(x^2 - a)}} \cos \theta. \tag{20}
\end{align*}
Using equations (9), (17)–(20), we find $a_{11}a_{12} + a_{22}a_{21} - a_{31}a_{32} = 0$, which implies
\[(\sqrt{(a x^2 - a z^2 - a^2)(a x^2 - a y^2 - a^2)} \bigg/ x^2 - a\)^2 = \left( yz - \frac{x^2 yz}{x^2 - a} \right)^2,
\]
and, consequently, we have
\[a^2 z^2(x^2 - a) = a^2(x^2 - a)(x^2 - y^2 - a),\]
that is, $z^2 = x^2 - y^2 - a$. Hence, we obtain
\begin{align*}
a_{12} &= \pm x \frac{\sqrt{x^2 - y^2 - a}}{\sqrt{x^2 - a}} \cos \theta \mp \frac{\sqrt{a y}}{\sqrt{x^2 - a}} \sin \theta, \\
a_{22} &= \pm x \frac{\sqrt{x^2 - y^2 - a}}{\sqrt{x^2 - a}} \sin \theta \mp \frac{\sqrt{a y}}{\sqrt{x^2 - a}} \cos \theta.
\end{align*}
In particular, if $a_{33}^2 = x^2 = a$, from equation (13), we derive $a_{13}^2 + a_{23}^2 - a = -a$, which implies $a_{13}^2 + a_{23}^2 = 0$, and thus, $a_{13} = a_{23} = 0$.

Considering equation (10), we find $a_{11}a_{13} + a_{21}a_{23} - a_{31}a_{33} = 0$ implies $a_{31} = 0$. Similarly, from equation (12), $a_{12}a_{13} + a_{22}a_{23} - a_{32}a_{33} = 0$ refers $a_{32} = 0$. Therefore, both $a_{11}^2 + a_{21}^2 = a = a_{12}^2 + a_{22}^2$ and $a_{11}a_{12} + a_{21}a_{22} = 0$ lead to (cf. (17))
\[h = \begin{pmatrix} \pm \sqrt{a} \cos \eta & \mp \sqrt{a} \sin \eta & 0 \\ \sqrt{a} \sin \eta & \sqrt{a} \cos \eta & 0 \\ 0 & 0 & \pm \sqrt{a} \end{pmatrix}.
\]
Thus, either
\[h = \begin{pmatrix} x y & \frac{\sqrt{a} \alpha_1}{\beta_1} \cos \theta + \sqrt{a} \alpha_1 \frac{\beta_1}{\beta_1} \sin \theta & \pm \frac{x \alpha_1}{\beta_1} \cos \theta \mp \sqrt{a} \alpha_1 \frac{\beta_1}{\beta_1} \sin \theta & \pm \frac{a y}{\beta_1} \cos \theta \mp \sqrt{a} \alpha_1 \frac{\beta_1}{\beta_1} \sin \theta & \beta_1 \cos \theta \\ x y & \frac{\sqrt{a} \alpha_1}{\beta_1} \sin \theta \frac{\beta_1}{\beta_1} \cos \theta & \pm \frac{x \alpha_1}{\beta_1} \cos \theta \pm \sqrt{a} \alpha_1 \frac{\beta_1}{\beta_1} \sin \theta & \pm \frac{a y}{\beta_1} \cos \theta \pm \sqrt{a} \alpha_1 \frac{\beta_1}{\beta_1} \sin \theta & \beta_1 \sin \theta \\ \frac{\sqrt{a} \alpha_1}{\beta_1} \sin \theta & \beta_1 \cos \theta & \pm \frac{x \alpha_1}{\beta_1} \sin \theta & \pm \frac{a y}{\beta_1} \cos \theta & \beta_1 \sin \theta \end{pmatrix}, \tag{21}
\]
where $\alpha_1 = \sqrt{x^2 - y^2 - a}$ and $\beta_1 = \sqrt{x^2 - a}$, or
\[h = \begin{pmatrix} \pm \sqrt{a} \cos \eta & \mp \sqrt{a} \sin \eta & 0 \\ \sqrt{a} \sin \eta & \sqrt{a} \cos \eta & 0 \\ 0 & 0 & \pm \sqrt{a} \end{pmatrix}. \tag{22}
\]
Let us now consider equation (14), where we have \( \det((a_{ij})) = \det(h) = 1 \). When \( x^2 \neq a \) (\( a \) is real), and conducting some straightforward calculations of \( \det(h) \) from equation (21), we obtain

\[
\det(h) = \frac{\sqrt{ax^2y^2}}{x^2-a} + \frac{\sqrt{ax^2(y^2-a)}}{x^2-a} - \sqrt{ay^2} - \sqrt{a(x^2-y^2-a)},
\]

which simplifies to \( \det(h) = a\sqrt{a} = 1 \), implying \( a = 1 \).

When \( x^2 = a \), computing \( \det(h) \) from equation (22), we get

\[
\det(h) = a\sqrt{a}(\cos^2 \eta + \sin^2 \eta),
\]

which implies \( \det(h) = a\sqrt{a} \) and thus, \( a = 1 \).

Therefore, either

\[
h = \begin{pmatrix}
\frac{xy}{\beta} & \frac{x}{\beta} \cos \theta + \frac{y}{\beta} \sin \theta & \pm \frac{x}{\beta} \cos \theta & \frac{y}{\beta} \sin \theta & \beta \cos \theta \\
\frac{xy}{\beta} & \frac{x}{\beta} \sin \theta - \frac{y}{\beta} \cos \theta & \pm \frac{x}{\beta} \sin \theta & \frac{y}{\beta} \cos \theta & \beta \sin \theta \\
y & \pm \alpha & 0 & 0 & x
\end{pmatrix},
\]

where \( \alpha = \sqrt{x^2 - y^2 - 1} \) and \( \beta = \sqrt{x^2 - 1} \), or

\[
h = \begin{pmatrix}
\pm \cos \eta & \mp \sin \eta & 0 \\
\sin \eta & \cos \eta & 0 \\
0 & 0 & \pm \sqrt{a}
\end{pmatrix},
\]

Case 2: \( a \) is negative. Let \( a = -n^2 < 0 \). Referring to equation (8), we obtain

\[
a_{11}^2 + a_{21}^2 - a_{31}^2 = a = -n^2, \quad \text{implying} \quad a_{31} \neq 0.
\]

Let \( a_{31} = u \neq 0 \), leading to

\[
a_{11} = \sqrt{u^2 - n^2} \cos \eta, \quad a_{21} = \sqrt{u^2 - n^2} \sin \eta.
\]

Similarly, from equation (11), where \( a_{12}^2 + a_{22}^2 - a_{32}^2 = a = -n^2 \), we deduce \( a_{32} \neq 0 \). Setting \( a_{32} = v \neq 0 \), we find

\[
a_{12} = \sqrt{v^2 - n^2} \cos \zeta, \quad a_{22} = \sqrt{v^2 - n^2} \sin \zeta.
\]

Now, equation (9) yields

\[
a_{11}a_{12} + a_{21}a_{22} - a_{31}a_{32} = 0,
\]

implying

\[
\sqrt{u^2 - n^2} \cos \eta \sqrt{v^2 - n^2} \cos \zeta + \sqrt{u^2 - n^2} \sin \eta \sqrt{v^2 - n^2} \sin \zeta - uv = 0,
\]

which simplifies to

\[
\cos(\eta - \zeta) = \frac{uv}{\sqrt{(u^2 - n^2)(v^2 - n^2)}},
\]

provided \( u^2, v^2 \neq n^2 \). Consequently, we deduce

\[
\zeta = \eta - \cos^{-1} \left( \frac{uv}{\sqrt{(u^2 - n^2)(v^2 - n^2)}} \right).
\]
Thus,

\[ a_{12} = \sqrt{v^2 - n^2} \cos \zeta = \frac{uw}{\sqrt{(u^2 - n^2)}} \cos \eta + \frac{\sqrt{n^4 - n^2u^2 - n^2v^2}}{\sqrt{(u^2 - n^2)}} \sin \eta, \quad (23) \]

\[ a_{22} = \sqrt{v^2 - n^2} \sin \zeta = \frac{uw}{\sqrt{(u^2 - n^2)}} \sin \eta - \frac{\sqrt{n^4 - n^2u^2 - n^2v^2}}{\sqrt{(u^2 - n^2)}} \cos \eta. \quad (24) \]

Again, from equation (13), we have \( a_{13}^2 + a_{23}^2 - a_{33}^2 = -a \). Let \( a_{33} = w \). Since \( a_{11}a_{13} + a_{21}a_{23} - a_{31}a_{33} = 0 \), we can similarly obtain the values of \( a_{13} \) and \( a_{23} \). Consequently, we have

\[ a_{13} = \frac{uw}{\sqrt{(u^2 - n^2)}} \cos \eta + \frac{\sqrt{n^4u^2 - n^2w^2 - n^4}}{\sqrt{(u^2 + a)}} \sin \eta, \quad (25) \]

\[ a_{23} = \frac{uw}{\sqrt{(u^2 - n^2)}} \sin \eta - \frac{\sqrt{n^4u^2 - n^2w^2 - n^4}}{\sqrt{(u^2 - n^2)}} \cos \eta. \quad (26) \]

Using equations (12), (23) and (24), and simplifying, we get that

\[ a_{12}a_{13} + a_{22}a_{23} - a_{32}a_{33} = 0 \]

implies

\[ n^4w^2(u^2 - n^2) = n^4(u^2 - n^2)(u^2 + v^2 - n^2), \]

and therefore, \( w^2 = u^2 + v^2 - n^2 \). Hence, from equations (25) and (26), we find

\[ a_{13} = \frac{u\sqrt{u^2 + v^2 - n^2}}{\sqrt{u^2 - n^2}} \cos \eta - \frac{\sqrt{-n^2v^2}}{\sqrt{u^2 - n^2}} \sin \eta, \]

\[ a_{23} = \frac{u\sqrt{u^2 + v^2 - n^2}}{\sqrt{u^2 - n^2}} \sin \eta + \frac{\sqrt{-n^2v^2}}{\sqrt{u^2 - n^2}} \cos \eta, \]

which are absurd since \( \sqrt{-n^2} \notin \mathbb{R} \).

Next, we assume that \( a_{31}^2 = u^2 = n^2 \), then \( a_{11}^2 + a_{21}^2 - n^2 = -n^2 \) implies \( a_{11} = a_{21} = 0 \). Moreover, from equation (9), we obtain \( a_{32} = 0 \). Additionally, we have \( a_{12}^2 + a_{22}^2 - a_{32}^2 = -n^2 \), and thus, \( a_{12}^2 + a_{22}^2 = -n^2 \), which is a contradiction.

We can obtain similar contradiction for the case \( a_{32}^2 = v^2 = n^2 \), since this case results in \( a_{12}^2 + a_{21}^2 = -n^2 \). Thus, Case 2, i.e., \( a < 0 \), is not possible.

Therefore, the fixed subgroup of the unit circle is given by the matrix \( M \).

\[ \square \]

**Remark 4** We can also obtain the possible values of \( a \) by evaluating the determinant of both the circles. Specifically, under the action of \( g \in \text{SL}(3, \mathbb{R}) \),
the unit circle $A$ transforms to $A' = (g^{-1})^tAg^{-1}$. As $\det(g^t) = \det(g)$ and $\det(g^{-1}) = 1$, we have

$$\det((g^{-1})^tAg^{-1}) = \det((g^{-1})^t) \det(A) \det(g^{-1})$$

$$= \det(g^{-1}) \det(A) \det(g^{-1}) = \det(A).$$

(27)

Therefore, if

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with $\det(A) = -1$, then $A'$ should be represented by the same matrix: $A' = A$. Otherwise, if

$$A' = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix}$$

for some $a < 0$, then $\det(A') = -a^3 > 0$, contradicting equation (27).

4 Relation between fixed subgroup of the unit circle and $\text{SL}(2, \mathbb{R})$

In this section, we develop the interrelationship between the fixed subgroup of the unit circle and $\text{SL}(2, \mathbb{R})$, which gives further insights into this subject. Let $M$ be the fixed subgroup of the projective unit circle as mentioned in Theorem 4.

**Lemma 1** The Lie subalgebra $\mathfrak{m}$ of the fixed Lie subgroup $M$ is isomorphic to $\mathfrak{sl}(2)$, the Lie algebra of the special linear group $\text{SL}(2, \mathbb{R})$.

**Proof.** Let us consider the Lie subalgebra $\mathfrak{m}$ of the fixed Lie subgroup $M$ of the projective unit circle. Then, $\mathfrak{m}$ is given by

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}.$$

To show that, for each $X \in \mathfrak{m}$, we will verify that $\exp(tX) \in M$ for all $t \in \mathbb{R}$. Let

$$X = \begin{pmatrix} 0 & -a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix},$$

then the eigenvalues of $tX$ are $0$, $t\sqrt{b^2 + c^2 - a^2}$, and $-t\sqrt{b^2 + c^2 - a^2}$. There are two main cases.
Case 1: $b^2 + c^2 - a^2 = 0$. In this case, the only eigenvalue is 0, and it has an algebraic multiplicity of 3. Additionally, matrix $X$ is nilpotent, specifically, $X^3 = 0$. Therefore,

$$\exp(tX) = \sum_{n=0}^{\infty} \frac{t^nX^n}{n!} = I + tX + \frac{t^2X^2}{2!}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -at & bt \\ at & 0 & ct \\ bt & ct & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} (b^2 - a^2)t^2 & bct^2 & -act^2 \\ bct^2 & (c^2 - a^2)t^2 & abt^2 \\ act^2 & -abt^2 & (b^2 + c^2)t^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + ((b^2 - a^2)t^2)/2 & -at + (bct^2)/2 & bt - (act^2)/2 \\ at + (bct^2)/2 & 1 + ((c^2 - a^2)t^2)/2 & ct + (abt^2)/2 \\ bt + (act^2)/2 & ct - (abt^2)/2 & 1 + ((b^2 + c^2)t^2)/2 \end{pmatrix}.$$
provided $c \neq 0$. Also,

$$D = \begin{pmatrix} tk & 0 & 0 \\ 0 & -tk & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$

Thus,

$$\exp(tX) = P^{-1} \exp(D) P $$

$$= \frac{1}{2k^2} \begin{pmatrix} (b^2 - a^2) (e^{tk} + e^{-tk}) + 2c^2 & bc(e^{tk} + e^{-tk}) - 2bc & bk(e^{tk} - e^{-tk}) + 2ac \\ -ak(e^{tk} - e^{-tk}) & c^2 - a^2 (e^{tk} + e^{-tk}) + 2b^2 & ck(e^{tk} - e^{-tk}) + 2ab \\ -bk(e^{tk} - e^{-tk}) & -ab(e^{tk} + e^{-tk}) & (b^2 + c^2)(e^{tk} + e^{-tk}) - 2a^2 \end{pmatrix}. $$

In this case, as well, we confirm that $\exp(tX)$ satisfies all equations (8)–(14).

Thus, $\exp(tX)$ belongs to the Lie subgroup $M$ for all $t \in \mathbb{R}$.

ii) Let $b^2 + c^2 - a^2 = -k^2 < 0$. The eigenvalues are 0, $tki$, $-tki$. Thus, $tX = P^{-1}DP$, where

$$P = \begin{pmatrix} \frac{b(b^2 + c^2) - a(ab + ick)}{ik(b^2 + c^2)} & \frac{a(ab - ick) - b(b^2 + c^2)}{ik(b^2 + c^2)} & -c \\ ab + ick & ab - ick & b \\ \frac{b}{b^2 + c^2} & \frac{b}{b^2 + c^2} & \frac{a}{a} \end{pmatrix}, a \neq 0,$$

and

$$D = \begin{pmatrix} tki & 0 & 0 \\ 0 & -tki & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$
Hence,
\[
\exp(tX) = P^{-1} \exp(D) P
\]

\[
= \frac{1}{k^2} \begin{pmatrix}
(a^2 - b^2) \cos(tk) & bc - bc \cos(kt) & ac \cos(kt) - ac \\
-c^2 & -ak \sin(kt) & +bk \sin(kt) \\
bc - bc \cos(kt) & (a^2 - c^2) \cos(kt) & ck \sin(kt) + ab \\
+ak \sin(kt) & -b^2 & -ab \cos(kt) \\
ac - ac \cos(kt) & -ab + ab \cos(kt) & -(b^2 + c^2) \cos(kt) \\
+bk \sin(kt) & +ck \sin(kt) & +a^2
\end{pmatrix}.
\]

In this case, too, we verify that \( \exp(tX) \) satisfies all equations (8)–(14), and thus, \( \exp(tX) \in M \) for any \( t \in \mathbb{R} \).

Hence, we validate that the Lie subalgebra of the fixed subgroup \( M \) of the unit circle is given by \( m \).

Let us now consider a basis \( \mathfrak{B} \) for the Lie subalgebra \( m \). We choose

\[
X = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad Y = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad Z = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

in \( m \). Then \( \mathfrak{B} = \{X, Y, Z\} \) forms a basis with the commutator relations

\([X,Y] = Z, [Y,Z] = -X, [Z,X] = Y\), where the Lie bracket is given by

\([X,Y] = XY - YX\).

Furthermore, let us consider the Lie algebra of the special linear group, \( \mathfrak{sl}(2) \), the Lie algebra of \( 2 \times 2 \) trace less real matrices. In \( \mathfrak{sl}(2) \), we select the basis vectors as

\[
e_1 = \frac{1}{2} \begin{pmatrix}
0 & -1 \\
1 & 0 \\
0 & 0
\end{pmatrix}, \quad e_2 = \frac{1}{2} \begin{pmatrix}
1 & 0 \\
0 & -1 \\
0 & 0
\end{pmatrix}, \quad e_3 = \frac{1}{2} \begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & 0
\end{pmatrix}.
\]

Here, \([e_1, e_2] = e_3, [e_2, e_3] = -e_1, [e_3, e_1] = e_2\).

Therefore, we establish an isomorphism of Lie algebras defined on the basis elements from \( \mathfrak{sl}(2) \) to \( m \) that can be uniquely extended to any element:

\[
\phi: \mathfrak{sl}(2) \to m
\]

such that

\[
\phi(e_1) = X, \quad \phi(e_2) = Y, \quad \phi(e_3) = Z
\]

(28)

with

\[
\phi([x,y]) = [\phi(x), \phi(y)] \text{ for all } x, y \in \mathfrak{sl}(2).
\]

Hence, the result follows. \( \square \)
Despite $\text{SL}(2, \mathbb{R})$ not being simply connected, we obtain the following result.

**Proposition 2** Consider a matrix Lie group $G$ having the Lie algebra $\mathfrak{g}$. If $\phi : \mathfrak{sl}(2) \to \mathfrak{g}$ stands as a Lie algebra homomorphism, then a corresponding Lie group homomorphism $\Phi : \text{SL}(2, \mathbb{R}) \to G$ exists, satisfying $\Phi(\exp X) = \exp \phi(X)$ for all $X \in \mathfrak{sl}(2)$.

**Proof.** Here, the central idea is to use the complexification of Lie algebras and groups to obtain the notion of simply connectedness. Hence, we consider the complexification of the Lie algebra $\mathfrak{sl}(2)$, denoted as $\mathfrak{sl}(2, \mathbb{C})$, which is isomorphic to the complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ (see [12] for details).

Let $G_\mathbb{C}$ and $\mathfrak{g}_\mathbb{C}$ denote the complexifications of matrix Lie group $G$ (hence, a closed subgroup of $\text{GL}(n, \mathbb{C})$) and Lie algebra $\mathfrak{g}$, respectively. Since $\phi : \mathfrak{sl}(2) \to \mathfrak{g}$ is a Lie algebra homomorphism, we can extend this homomorphism to the complexification of the Lie algebras (see the subsection 1.3)

$$\tilde{\phi} : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}_\mathbb{C}$$

defined by

$$\tilde{\phi}(X_1 + iX_2) = \phi(X_1) + i\phi(X_2).$$

Consequently, since $\text{SL}(2, \mathbb{C})$ is simply connected, we get a unique Lie group homomorphism (cf. Theorem 3)

$$\tilde{\Phi} : \text{SL}(2, \mathbb{C}) \to G_\mathbb{C}$$

satisfying

$$\tilde{\Phi}(\exp(\tilde{X})) = \exp(\phi(\tilde{X}))$$

for all $\tilde{X} \in \mathfrak{sl}(2, \mathbb{C})$.

Now, in order to obtain the required map on $\text{SL}(2, \mathbb{R})$ (non-simply connected group), we examine the restriction of $\tilde{\Phi}$ to the real form of $\text{SL}(2, \mathbb{C})$, i.e., to $\text{SL}(2, \mathbb{R})$ (cf. [12]). This restriction is defined as

$$\Phi : \text{SL}(2, \mathbb{R}) \to G$$

$$\Phi(\exp(X)) = \tilde{\Phi}(\exp(X))$$

for all $X \in \mathfrak{sl}(2)$. Thus, the result holds. $\square$

**Theorem 5** The fixed subgroup of the projective unit circle $M := M(x, y, \theta)$ such that $x > 1$ is isomorphic to the Lie group $\text{PSL}(2, \mathbb{R})$.

**Proof.** Using Lemma 1, we consider the isomorphism of the Lie algebras denoted as $\phi : \mathfrak{sl}(2) \to \mathfrak{m}$. Further, Proposition 2 guarantees the existence of a Lie group homomorphism $\Phi : \text{SL}(2, \mathbb{R}) \to M$ such that $\Phi(\exp X) = \exp \phi(X)$ holds for all $X \in \mathfrak{sl}(2)$.
Next, we take into account the Iwasawa decomposition of \( g \in \text{SL}(2, \mathbb{R}) \) (cf. subsection 1.5), which gives a unique decomposition of \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) as

\[
g = KAN = \begin{pmatrix} \cos \theta & - \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} \frac{a}{\sqrt{a^2 + c^2}} & \frac{c}{\sqrt{a^2 + c^2}} \\ -\frac{c}{\sqrt{a^2 + c^2}} & \frac{a}{\sqrt{a^2 + c^2}} \end{pmatrix} \begin{pmatrix} \sqrt{a^2 + c^2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & ab + cd \\ 0 & a^2 + c^2 \end{pmatrix}.
\]

Thus, taking into account that \( \Phi \) is a group homomorphism and using the definition of exponential map, we can write

\[
\Phi(g) = \Phi(KAN) = \Phi(K)\Phi(A)\Phi(N) = \Phi(\exp(k))\Phi(\exp(a))\Phi(\exp(n)) = \exp(\phi(k))\exp(\phi(a))\exp(\phi(n)),
\]

where

\[
k = \begin{pmatrix} 0 & -\cos^{-1} \left( \frac{a}{\sqrt{a^2 + c^2}} \right) \\ \cos^{-1} \left( \frac{a}{\sqrt{a^2 + c^2}} \right) & 0 \end{pmatrix},
\]

\[
n = \begin{pmatrix} 0 & ad + bc \\ 0 & a^2 + c^2 \end{pmatrix},
\]

and

\[
a = \begin{pmatrix} \log \left( \sqrt{a^2 + c^2} \right) \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -\log \left( \sqrt{a^2 + c^2} \right) \end{pmatrix}.
\]

We now explicitly compute

\[
\Phi(g) = \exp(\phi(k))\exp(\phi(a))\exp(\phi(n))
\]

to express the map \( \Phi \) in terms of \( a, b, c, d \). Here,

\[
k = \cos^{-1} \left( \frac{a}{\sqrt{a^2 + c^2}} \right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Thus, using equation (28), we obtain

\[
\phi(k) = \cos^{-1} \left( \frac{a}{\sqrt{a^2 + c^2}} \right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \cos^{-1} \left( \frac{a}{\sqrt{a^2 + c^2}} \right) \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

and hence,

\[
\exp(\phi(k)) = e^{\phi(k)} = \frac{a^2 - c^2}{a^2 + c^2} - \frac{2ac}{a^2 + c^2} \begin{pmatrix} 0 \\ a^2 + c^2 \\ a^2 + c^2 \end{pmatrix}.
\]
Again,
\[
a = \begin{pmatrix} \log \left( \sqrt{a^2 + c^2} \right) & 0 \\ 0 & -\log \left( \sqrt{a^2 + c^2} \right) \end{pmatrix} = \log \left( \sqrt{a^2 + c^2} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Therefore, using equation (28), we get
\[
\phi(a) = \log \left( \sqrt{a^2 + c^2} \right) \phi \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \log \left( \sqrt{a^2 + c^2} \right) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Hence,
\[
\exp(\phi(a)) = \begin{pmatrix} \frac{1}{2} \left( a^2 + c^2 + \frac{1}{a^2 + c^2} \right) & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \left( a^2 + c^2 - \frac{1}{a^2 + c^2} \right).
\]

Also,
\[
n = \begin{pmatrix} 0 & \frac{ab + cd}{a^2 + c^2} \\ 0 & 0 \end{pmatrix} = \frac{ab + cd}{a^2 + c^2} \left[ -\frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right].
\]

Thus, taking into account equation (28), we obtain
\[
\phi(n) = \frac{ab + cd}{a^2 + c^2} \left[ -\phi \left( \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) + \phi \left( \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \right]
\]
\[
= \frac{ab + cd}{a^2 + c^2} \left[ -\left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) + \left( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \right]
\]
\[
= \frac{ab + cd}{a^2 + c^2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

and therefore,
\[
\exp(\phi(n)) = \begin{pmatrix} 1 - \frac{(ab + cd)^2}{2(a^2 + c^2)^2} & \frac{ab + cd}{a^2 + c^2} & \frac{(ab + cd)^2}{2(a^2 + c^2)^2} \\ -\frac{ab + cd}{a^2 + c^2} & 1 & \frac{ab + cd}{a^2 + c^2} \\ -\frac{(ab + cd)^2}{2(a^2 + c^2)^2} & \frac{ab + cd}{a^2 + c^2} & 1 + \frac{(ab + cd)^2}{2(a^2 + c^2)^2} \end{pmatrix}.
\]
Combining equations (29), (30), (31) and (32) and performing the explicit computation of \( \Phi(g) \), results in obtaining the Lie group homomorphism \( \Phi : SL(2, \mathbb{R}) \rightarrow M \) given by:

\[
\Phi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \frac{1}{2(a^2 + c^2)^2} \times
\]

\[
\begin{pmatrix}
(a^2 - c^2)(a^2 + c^2)^2 - \\
(a^2 - c^2)(ab + cd)^2 + \\
(a^2 - c^2) + 4ac(ab + cd) \\
(a^2 + c^2)
\end{pmatrix}
\begin{pmatrix}
2(a^2 - c^2)ab + \\
2(a^2 - c^2)cd - \\
-4ac
\end{pmatrix}
\begin{pmatrix}
(a^2 - c^2)(a^2 + c^2)^2 + \\
(a^2 - c^2)(ab + cd)^2 - \\
(a^2 - c^2) - 4ac(ab + cd) \\
(a^2 + c^2)
\end{pmatrix}

2ac(a^2 + c^2)^2 - \\
2ac(ab + cd)^2 + 2ac \\
-2(ab + cd)(a^2 - c^2) \\
(a^2 + c^2)

4ac(ab + cd) + 2(a^2 - c^2) \\
2ac(a^2 + c^2)^2 + \\
2ac(ab + cd)^2 - 2ac \\
(a^2 + c^2)

4ac(ab + cd) \\
2cd(a^2 + c^2) \\
(a^2 + c^2)

(a^2 + c^2)^2 - 1 \\
- (ab + cd)^2 \\
2ab(a^2 + c^2) + \\
2cd(a^2 + c^2) \\
(a^2 + c^2)^2 + 1 \\
+ (ab + cd)^2
\]

Here, we see that the \( a_{33} \)-th entry

\[
\frac{(a^2 + c^2)^2 + 1 + (ab + cd)^2}{2(a^2 + c^2)} > 0.
\]

(33)

We also have \( \Phi^{-1} : M \rightarrow SL(2, \mathbb{R}) \) such that

\[
\Phi^{-1} \left( \begin{bmatrix}
\frac{xy}{\sqrt{x^2 - 1}} \cos \theta + \frac{\sqrt{x^2 - y^2 - 1}}{\sqrt{x^2 - 1}} \sin \theta \\
\frac{x \sqrt{x^2 - y^2 - 1}}{\sqrt{x^2 - 1}} \cos \theta + \frac{y}{\sqrt{x^2 - 1}} \sin \theta \\
\frac{xy}{\sqrt{x^2 + 1}} \sin \theta - \frac{\sqrt{x^2 - y^2 - 1}}{\sqrt{x^2 - 1}} \cos \theta \\
y \pm \frac{x \sqrt{x^2 - y^2 - 1}}{\sqrt{x^2 - 1}} \sin \theta + \frac{y}{\sqrt{x^2 - 1}} \cos \theta
\end{bmatrix} \right)
\]
\[
\begin{align*}
\sqrt{\left(\sqrt{x^2 - 1} + \sqrt{x^2 - 1 + y}\right)} \cos \frac{\theta}{2} & \pm \sqrt{\left(\sqrt{x^2 - 1 - y}\right) \cos \frac{\theta}{2}} \sin \frac{\theta}{2} \\
\sqrt{\left(\sqrt{x^2 - 1} + \sqrt{x^2 - 1 + y}\right)} \sin \frac{\theta}{2} & \pm \sqrt{\left(\sqrt{x^2 - 1 - y}\right) \cos \frac{\theta}{2}} \sin \frac{\theta}{2}
\end{align*}
\]

and

\[
\Phi^{-1}\left(\begin{array}{ccc}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}\right) = \left(\begin{array}{ccc}
\cos(\theta/2) & -\sin(\theta/2) & 0 \\
\sin(\theta/2) & \cos(\theta/2) & 0
\end{array}\right).
\]

Hence, for real values, it is necessary that \(x^2 - 1 > 0\). Additionally, equation (33) implies \(x > 0\). Consequently, satisfying both conditions, namely \(x^2 - 1 > 0\) and \(x > 0\), leads to the conclusion that \(x > 1\).

Again, the kernel of \(\Phi\) is defined as

\[
\text{Ker}(\Phi) = \{g \in \text{SL}(2, \mathbb{R}) \mid \Phi(g) = I_{3 \times 3} \in M\}.
\]

If \(\Phi(g) = I_{3 \times 3}\), then we must have

\[
ab + cd = 0, \quad \frac{(a^2 - c^2)(ab + cd) - 2ac}{a^2 + c^2} = 0, \quad \frac{(a^2 + c^2)^2 - 1 - (ab + cd)^2}{2(a^2 + c^2)} = 0,
\]

and

\[
\frac{2ac(ab + cd) + (a^2 - c^2)}{a^2 + c^2} = 1.
\]

Consequently, the derived conditions lead to the conclusion \(a^2 = 1\), implying \(a = \pm 1, b = 0, c = 0\). Thus,

\[
\left(\begin{array}{cc}
\pm 1 & 0 \\
0 & \pm 1
\end{array}\right) \in \text{Ker}(\Phi).
\]

Hence, we obtain the homomorphism \(\Phi : \text{SL}(2, \mathbb{R}) \to M(x, y, \theta)\) with kernel \(\{\pm I\}\), and the inverse map \(\Phi^{-1}\) exists for every \(m(x, y, \theta) \in M\), where \(x > 1\). In particular,

\[
M := M(x, y, \theta) \cong \text{SL}(2, \mathbb{R})/\{\pm I\} = \text{PSL}(2, \mathbb{R})
\]

where \(x > 1\). Thus, the theorem follows. □
Remark 5 To generate the inverse map, we have decomposed any element $m \in M$ in the following manner, which significantly simplifies the computations of the inverse map. For any

$$m = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{xy}{\sqrt{x^2 - 1}} & \pm \frac{x\sqrt{x^2 - y^2 - 1}}{\sqrt{x^2 - 1}} & \sqrt{x^2 - 1} \\ \mp \frac{x\sqrt{x^2 - y^2 - 1}}{\sqrt{x^2 - 1}} & \mp \frac{y}{\sqrt{x^2 - 1}} & 0 \\ y & \pm \sqrt{x^2 - y^2 - 1} & x \end{pmatrix}$$

in $M$, we have $\Phi^{-1} : M \to \text{SL}(2, \mathbb{R})$ such that

$$\Phi^{-1} \left( \begin{array}{c} \frac{xy}{\sqrt{x^2 - 1}} \\ \mp \frac{x\sqrt{x^2 - y^2 - 1}}{\sqrt{x^2 - 1}} \\ y \\ \pm \sqrt{x^2 - y^2 - 1} \end{array} \right) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. $$

This gives the description of $\Phi^{-1}(m)$ since $\Phi$ is a homomorphism.

5 Conclusion

We have explored the action of the transformation group $\text{SL}(3, \mathbb{R})$ on the two dimensional homogeneous space $\mathbb{R}P^2$. Our inquiry has involved an extension of the map $i$ associated with $\text{SL}(2, \mathbb{R})$ action, and we have established the non-existence of such a map for $\text{SL}(3, \mathbb{R})$ action. Following this, we have investigated the isotropy subgroup of the projective unit circle, providing a
detailed and explicit expression for it. Furthermore, we use a factorization of SL(2, $\mathbb{R}$), specifically the Iwasawa decomposition, to show that, under certain conditions, the isotropy subgroup of unit circle under SL(3, $\mathbb{R}$) action is isomorphic to the projective special linear group PSL(2, $\mathbb{R}$).

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