The relations between matroids of arbitrary cardinality and independence spaces[1]

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Abstract

This paper deals with the relationships between two classes of infinite matroids–the classes of matroids of arbitrary cardinality and of independence spaces primarily with the help of hyperplane set approach and sometimes of closure operator approach.

Key Words: matroid of arbitrary cardinality; independence space; hyperplane closure operator; infinite matroids

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1   Introduction and Preliminaries

Oxley said in [3] that there is no single class of structures that one calls infinite matroids. Rather, various authors with differing motivations have studied a variety of class of matroid-like structures on infinite sets. Several of these classes differ quite markedly in the properties possessed by their members and, in some cases, the precise relationship between particular classes is still not known. The purpose of this paper is to indicate the links between two of the more frequently studied classes of infinite matroids–the classes of matroids of arbitrary cardinality(cf.[1]) and of independence spaces(cf.[2, 3]). It will discuss primarily with the hyperplane set approach. Some details of the closure operator approach will also be needed.

We firstly provides the hyperplane axioms for a matroid of arbitrary cardinality in this paper. Afterwards, it will discuss the relationships between matroids of arbitrary cardinality and independence spaces.

[9] presents independent axioms of matroids of arbitrary cardinality; [2, 3] tells us the definition of independence spaces by the way of independent sets. Comparing the two results above in [9] and [2, 3], we get that a matroid of arbitrary cardinality is an independence space, but not pledge to the converse. It seems that this paper is not valuable to be done. Actually, we know that finite matroid theory is essential to the study on combinatorics and discrete mathematics, one of the important reasons is that there are many equivalent axioms for finite matroids. As a main part of matroid theory, infinite matroids are also hoped to obtain many equivalent axioms as finite matroids have. In this way, it could generalize the applicable ranges of infinite matroid theory. We find out that [2] gives the closure operator axioms for independence spaces. In my knowledge fields, no man had found out the hyperplane axioms for matroids of arbitrary cardinality, though [1] presents the definition of matroids of arbitrary cardinality by the way of closure sets. On the other hand, the importance of hyperplanes of matroids of arbitrary cardinality has been exposed in [1]. We believe that the hyperplane axioms will be contributory to the future work on infinite matroids especially matroids of arbitrary cardinality. Based on these motivations, we do this paper.

In what follows, we assume that E is some arbitrary–possibly infinite–set.

Definition 1 [1] Assume $m \in \mathbb{N}_0$ and $\mathcal{F} \subseteq \mathcal{P}(E)$. Then the pair $M := (E, \mathcal{F})$ is called a *matroid of rank m with $\mathcal{F}$ as its closed sets*, if the following axioms hold:
(F1) $E \in \mathcal{F}$;
(F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$;
(F3) Assume $F_0 \in \mathcal{F}$ and $x_1, x_2 \in E \setminus F_0$. Then one has either

$\{F \in \mathcal{F} | F_0 \cup \{x_1\} \subseteq F\} = \{F \in \mathcal{F} | F_0 \cup \{x_2\} \subseteq F\}$ or

$F_1 \cap F_2 = F_0$ for certain $F_1, F_2 \in \mathcal{F}$ containing $F_0 \cup \{x_1\}$ or $F_0 \cup \{x_2\}$, respectively;
(F4) $m = max\{n \in \mathbb{N}_0 |$ there exist $F_0, F_1, ..., F_n \in \mathcal{F}$ with $F_0 \subset F_1 \subset ... \subset F_n = E\}$.

The set of *hyperplanes* of M consists exactly of the maximal closed sets in M (with respect to inclusion) different from E. The *closure operator* $\sigma = \sigma_M : \mathcal{P}(E) \to \mathcal{F}$ of M is defined by $\sigma(A) = \sigma_M(A) := \bigcap_{\substack{F \in \mathcal{F} \\ A \subseteq F}} F$. The *rank function* $\rho = \rho_M : \mathcal{P}(E) \to \{0, 1, \ldots, m\}$ of M is defined by $\rho(A) := max\{k \in \mathbb{N}_0 |$ there exist $F_0, F_1, \ldots, F_k \in \mathcal{F}$ with $F_0 \subset F_1 \subset \ldots \subset F_k = \sigma(A)\}$. $\mathcal{F} \ni F_2$ *covers* $F_1 \in \mathcal{F}$ if $F_1 \subset F_2$ and there does not exist some $F \in \mathcal{F}$ with $F_1 \subset F \subset F_2$. Assume $F, F' \in \mathcal{F}$ satisfy $F \subseteq F'$. A chain $F = F_0 \subset F_1 \subset \ldots \subset F_n = F'$ where $F_i \in \mathcal{F}(i = 0, 1, \ldots, n)$ is called a *maximal chain* between F and F', if each $F_j, 1 \leq j \leq n$, covers F_{j-1}, n is called the *length* of the given chain.

Sometimes one calls $M = (E, \mathcal{F})$ given in Definition 1 a matroid of arbitrary cardinality

and simply denotes it by M.

Lemma 1 [1] Assume $M := (E, \mathcal{F})$ is a matroid of arbitrary cardinality with σ_M as its closure operator. Then

(1) For any family $(F_i)_{i \in I}$ of closed sets in M, one has also $F := \bigcap_{i \in I} F_i \in \mathcal{F}$.

(2) For any $A \subseteq E$, the set $\sigma_M(A)$ is the smallest set in $\mathcal{F}$ containing A. In particular, one has $\sigma_M(A) = A$ if and only if $A \in \mathcal{F}$. Moreover, σ_M satisfies the following conditions:
$$A \subseteq \sigma_M(A) = \sigma_M(\sigma_M(A)) \text{ for all } A \subseteq E; \quad \text{for } A \subseteq B \subseteq E, \text{ one has } \sigma_M(A) \subseteq \sigma_M(B).$$
Furthermore, σ_M satisfies the following exchange condition:

For $A \subseteq E$ and $x, y \in E \setminus \sigma_M(A)$, one has $y \in \sigma_M(A \cup \{x\})$ if and only if $x \in \sigma_M(A \cup \{y\})$.

(3) For $F_1, F_2 \in \mathcal{F}$ with $F_1 \subset F_2$, the following statements (i)-(iii) are equivalent:
(i) F_2 covers F_1. (ii) For all $x \in F_2 \setminus F_1$, one has $\sigma_M(F_1 \cup \{x\}) = F_2$.
(iii) There exists some $x \in F_2 \setminus F_1$ with $\sigma_M(F_1 \cup \{x\}) = F_2$.

(4) Suppose $F \subseteq F'$ and $F, F' \in \mathcal{F}$. Then all the maximal chains of closed sets between F and F' have the same length.

(5) If $\mathcal{F} \ni F'$ covers $F \in \mathcal{F}$, then $\rho(F') = \rho(F) + 1$, where ρ is the rank function of M.

Definition 2 ([2, pp.385-387 & 3,p.74]) An *independence space* $M_P(E)$ is a set E together with a collection $\mathcal{I}$ of subsets of E (called *independent sets*) such that
(i1) $\mathcal{I} \neq \emptyset$;
(i2) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$;
(i3) If $A, B \in \mathcal{I}$ and $|A|, |B| < \infty$ with $|A| = |B| + 1$, then $\exists a \in A \setminus B$ fits $B \cup \{a\} \in \mathcal{I}$;
(i4) If $A \subseteq E$ and every finite subset of A is a member of $\mathcal{I}$, then $A \in \mathcal{I}$.

A subset of E is *dependent* if it is not independent. Every maximal independent subset of $M_P(E)$ is a *base*. A *hyperplane* of $M_P(E)$ is a maximal set not containing any base and a *circuit* to be a minimal dependent. The *closure operator* σ of $M_P(E)$ is defined by $x \in \sigma(A)$ if $x \in A$ or there exists a circuit C with $x \in C \subseteq A \cup \{x\}$. A set X is *closed* if $\sigma(X) = X$.

Lemma 2([2, pp.387-389&3,p.75]) A function $\sigma : \mathcal{P}(E) \to \mathcal{P}(E)$ is the closure operator of an independence space on E if and only if for X, Y subsets of E, and $x, y \in E$:
(s1) $X \subseteq \sigma(X)$;
(s2) $Y \subseteq X \Rightarrow \sigma(Y) \subseteq \sigma(X)$;
(s3) $\sigma(X) = \sigma(\sigma(X))$;
(s4) $y \in \sigma(X \cup \{x\}) \setminus \sigma(X) \Rightarrow x \in \sigma(X \cup \{y\})$;
(s5) $a \in \sigma(X) \Rightarrow a \in \sigma(X_f)$ for some finite subset X_f of X.

Lemma 3 [6] Let $M = (E, \mathcal{F})$ be a matroid defined as in Definition 1. Then $L(M) = (\mathcal{F}, \subseteq)$ is a geometric lattice with finite length.

In [6], for a matroid of arbitrary cardinality $M = (E, \mathcal{F})$, it discusses the relations between M and $L(M)$ (i.e. $(\mathcal{F}, \subseteq)$). For the detail about the relations between M and $L(M)$, we refer to [6]. [7, 8] tells us that an interval of a geometric lattice is geometric. For more knowledge of lattice theory are cf.[7, 8].

On the behalf of convenience for the sequel, we provide some definitions as follows.

Definition 3 Let $\mathcal{H} = \{H_\tau \subseteq E : \tau \in \mathcal{T}\}$ be a subset of $\mathcal{P}(E)$ satisfying $H_1 \nsubseteq H_2$ for $H_1, H_2 \in \mathcal{H}$ and $E \supseteq X \subseteq H_\alpha$ for $H_\alpha \in \mathcal{H}, (\alpha \in \mathcal{A} \subseteq \mathcal{T})$, where $\mathcal{A} = \{i \in \mathcal{T} : X \subseteq H_i$ and $H_i \in \mathcal{H}\}$. One defines $\sigma_\mathcal{H} : \mathcal{P}(E) \to \mathcal{P}(E)$ as a map satisfies:

(i) If $\mathcal{A} \neq \emptyset$, then one defines $\sigma_\mathcal{H}(X) = \bigcap\limits_{\alpha \in \mathcal{A}} H_\alpha$.

(ii) If $\mathcal{A} = \emptyset$, i.e., $E \supseteq X \nsubseteq H$ for any $H \in \mathcal{H}$, then one defines $\sigma_\mathcal{H}(X) = E$.

Let $X, Y \subseteq E$. "$\sigma_\mathcal{H}(Y)$ *covers* $\sigma_\mathcal{H}(X)$" means that $\sigma_\mathcal{H}(X) \subseteq \sigma_\mathcal{H}(Y)$ and none of $\sigma_\mathcal{H}(Z)$ for $Z \subseteq E$ satisfies $\sigma_\mathcal{H}(X) \subset \sigma_\mathcal{H}(Z) \subset \sigma_\mathcal{H}(Y)$.

Set $X_j \subseteq E$, $j = 0, 1, \ldots, k$. If a chain $\sigma_\mathcal{H}(X_0) \subset \sigma_\mathcal{H}(X_1) \subset \ldots \subset \sigma_\mathcal{H}(X_k)$ satisfies that each $\sigma_\mathcal{H}(X_j)$ covers $\sigma_\mathcal{H}(X_{j-1}), (1 \leq j \leq k)$, then k is called the *length* of the given chain.

It is obvious that in Definition 3, if $k < \infty$, the length is finite, otherwise infinite.

It is not difficult to obtain from the above Definition 3 the following result:

Corollary 1 (1) For any $H \in \mathcal{H}, \sigma_\mathcal{H}(H)$ exists and $\sigma_\mathcal{H}(H) = H$.

(2) Let $X, Y \subseteq E$. Then $X \subseteq \sigma_\mathcal{H}(X)$; $X \subseteq Y \Rightarrow \sigma_\mathcal{H}(X) \subseteq \sigma_\mathcal{H}(Y)$; $\sigma_\mathcal{H}(\sigma_\mathcal{H}(Y)) = \sigma_\mathcal{H}(Y)$.

2   Hyperplane axioms

The many different axiom systems such as hyperplane axioms for finite matroids are given in [2, 4, 5], but according to my knowledge, none of researchers studied on hyperplane axioms for matroids of arbitrary cardinality. This section will set up the hyperplane axioms for matroids of arbitrary cardinality.

Theorem 1 (Hyperplane axioms) A collection $\mathcal{H}$ of subsets of E is the set of hyperplanes of a matroid of arbitrary cardinality on E if and only if the conditions (H1)–(H4) hold.

(H1) If $H_1, H_2 \in \mathcal{H}$ with $H_1 \neq H_2$, then $H_1 \nsubseteq H_2$;

(H2) Let $X \subseteq E$. Then $\sigma_\mathcal{H}(X)$ exists;

(H3) Let $F \subseteq E$ and $x, y \in E$. Then $y \in \sigma_\mathcal{H}(F \cup \{x\}) \Rightarrow x \in \sigma_\mathcal{H}(F \cup \{y\})$;

(H4) Let $X, Y \subseteq E$. Then $max\{t \in \mathbb{N}_0|$ there exist $X_j, j = 1, \ldots, t$ such that $Y = X_t$ and

$\sigma_\mathcal{H}(X) \subset \sigma_\mathcal{H}(X_1) \subset \ldots \subset \sigma_\mathcal{H}(X_t)\} < \infty$.

Proof ($\Longrightarrow$) Let $\mathcal{H} = \{H_t : t \in \mathcal{T}\}$ be the set of hyperplanes of a matroid $M = (E, \mathcal{F})$ of arbitrary cardinality of rank m with σ_M, ρ as its closure operator and rank function respectively. Let $H_1, H_2 \in \mathcal{H}$ and $H_1 \neq H_2$. By the maximality of H_1 and H_2, (H1) holds. Furthermore, (H2) holds.

Before proceeding, we firstly prove that for $X \subseteq E$, $\sigma_{\mathcal{H}}(X) = \sigma_M(X)$ is right. Let $\mathcal{A} = \{t \in \mathcal{T} : X \subseteq H_t\}$.

When $X \nsubseteq H$ for any $H \in \mathcal{H}$. Then $\sigma_M(X) = E$ holds by Definition 1 and at the same time, $\sigma_{\mathcal{H}}(X) = E$ is correct. Thus, $\sigma_{\mathcal{H}}(X) = \sigma_M(X)$ is true.

When $X \subseteq H_\alpha \in \mathcal{H}$ ($\alpha \in \mathcal{A} \subseteq \mathcal{T}$ with $|\mathcal{A}| \neq 0$), where $\mathcal{A} \subseteq \mathcal{T}$ satisfies $X \subseteq H_\alpha$ ($\alpha \in \mathcal{A}$) and $X \nsubseteq H$ if $H \in \mathcal{H} \setminus \{H_\alpha : \alpha \in \mathcal{A}\}$. Next by induction on $m = \rho(E)$ to prove $\sigma_{\mathcal{H}}(X) = \sigma_M(X)$. Obviously, $\rho(\sigma_M(X)) \leq \rho(H_\alpha) = m - 1, (\forall \alpha \in \mathcal{A})$. Distinguishing two cases for discussion.

Case 1: $m = 1, \sigma_{\mathcal{H}}(X) = \sigma_M(X)$ is obvious.

Case 2: $2 \leq m$. Let $F = \sigma_M(X)$.

In the status of $\rho(F) = m - 1$, $F \in \mathcal{H}$ is evident. In addition, under this status, by (H1), Lemma 1 and Corollary 1, $\mathcal{F} \ni \sigma_{\mathcal{H}}(X) \subseteq \sigma_{\mathcal{H}}(\sigma_M(X)) = \sigma_M(X)$ is correct. Because of $X \subseteq \sigma_{\mathcal{H}}(X)$ and Lemma 1, $\sigma_M(X) \subseteq \sigma_M(\sigma_{\mathcal{H}}(X)) = \sigma_{\mathcal{H}}(X)$ holds. Thus $\sigma_M(X) = \sigma_{\mathcal{H}}(X)$.

Assume that for any $Y \subseteq E$, if $k \leq \rho(D) \leq m - 1$, then $\sigma_{\mathcal{H}}(Y) = \sigma_M(Y)$ holds, where $D = \sigma_M(Y)$. Now let $\rho(F) = k - 1 \geq 0$.

Let $\{F_j \in \mathcal{F} : j \in \mathcal{J}\}$ be the set of the members that covers F in M. Then by Lemma 1(5) and Lemma 3, $\rho(F_j) = \rho(F) + 1 = k$ ($j \in \mathcal{J}$) and $F = \bigcap_{j \in \mathcal{J}} F_j$. Besides, by the assumption and Lemma 1, $\sigma_{\mathcal{H}}(F_j) = \sigma_M(F_j) = F_j$ $(j \in \mathcal{J})$.

In addition, for $H \in \mathcal{H}$ and $F_i \in \{F_j \in \mathcal{F} : j \in \mathcal{J}\}$, if $F_i \subset H$, then it is evidently $H \in \{H_\alpha : \alpha \in \mathcal{A}\}$ in light of $F \subset F_i$.

On the other hand, when $k = m - 1$, it is obviously $\rho(F) = m - 2$, besides, for any $H \in \{H_\alpha : \alpha \in \mathcal{A}\}$, H covers F. Hence $\{H_\alpha : \alpha \in \mathcal{A}\} = \{F_j \in \mathcal{F} : j \in \mathcal{J}\}$. When $k < m - 1$. Suppose there exists $H \in \{H_\alpha : \alpha \in \mathcal{A}\}$ such that $F_j \nsubseteq H$ ($\forall j \in \mathcal{J}$). By Lemma 3, one obtains that the interval $[F, H]$ in $L(M)$ is still a geometric lattice with finite length, and hence there is $F_t \in [F, H]$ such that F_t covers F in $L(M)$, and further by the definition of $L(M)$, F_t covers F in M, and hence $F_t \in \{F_j \in \mathcal{F} : j \in \mathcal{J}\}$. However, by the supposition, $F_t \notin \{F_j \in \mathcal{F} : j \in \mathcal{J}\}$, a contradiction. That is to say, for any $H \in \{H_\alpha : \alpha \in \mathcal{A}\}$, there exists $F_i \in \{F_j \in \mathcal{F} : j \in \mathcal{J}\}$ satisfying $F_i \subset H$.

The above two hands shows us $\bigcap_{j \in \mathcal{J}} F_j \subseteq \bigcap_{\alpha \in \mathcal{A}} H_\alpha$.

Furthermore, if $|\mathcal{J}| = 1$, then the interval $[F, E]$ in $L(M)$ is not geometric, a contradiction to Lemma 3 and the properties of geometric lattices. Thus $|\mathcal{J}| > 1$. By the (3) and (4) in Lemma 1, for $F_1, F_2 \in \{F_j : j \in \mathcal{J}\}$, one has $F_1 \nsubseteq F_2$. This implies $\bigcap_{j \in \mathcal{J}} F_j \subset F_j = \bigcap_{\substack{\alpha_j \in \mathcal{A}_j \subseteq \mathcal{A} \\ H_{\alpha_j} \in \mathcal{H}}} H_{\alpha_j}$ $(j \in \mathcal{J})$, where $\mathcal{A}_j = \{i : F_j \subseteq H_i \in \{H_\alpha : \alpha \in \mathcal{A}\}\}$ $(j \in \mathcal{J})$. We

notice that by the above two hands, $\bigcap_{j \in \mathcal{J}} (\bigcap_{\substack{\alpha_j \in \mathcal{A}_j \subseteq \mathcal{A} \\ H_{\alpha_j} \in \mathcal{H}}} H_{\alpha_j}) = \bigcap_{\alpha \in \mathcal{A}} H_\alpha$. Of course, by Lemma 1,

$\mathcal{F} \ni \bigcap_{j \in \mathcal{J}} (\bigcap_{\substack{\alpha_j \in \mathcal{A}_j \subseteq \mathcal{A} \\ H_{\alpha_j} \in \mathcal{H}}} H_{\alpha_j})$. Besides, $F \subseteq \bigcap_{j \in \mathcal{J}} (\bigcap_{\substack{\alpha_j \in \mathcal{A}_j \subseteq \mathcal{A} \\ H_{\alpha_j} \in \mathcal{H}}} H_{\alpha_j}) \subset F_j$ and $\rho(F_j) = \rho(F) + 1$ taken

together follows $\rho(F) = \rho(\bigcap_{j \in \mathcal{J}} (\bigcap_{\substack{\alpha_j \in \mathcal{A}_j \subseteq \mathcal{A} \\ H_{\alpha_j} \in \mathcal{H}}} H_{\alpha_j}) = \bigcap_{\alpha \in \mathcal{A}} H_\alpha)$, and so $\bigcap_{j \in \mathcal{J}} F_j = \bigcap_{\alpha \in \mathcal{A}} H_\alpha$. Hence

$\sigma_\mathcal{H}(X) = \bigcap_{\alpha \in \mathcal{A}} H_\alpha = \bigcap_{j \in \mathcal{J}} F_j = F = \sigma_M(X)$.

Summing up, by Definition 1 and Lemma 1, one gets that $\mathcal{H}$ satisfies (H3)and (H4).

($\Longleftarrow$) Suppose the collection $\mathcal{H}$ satisfies (H1)-(H4). Let $\mathcal{F} = \{X \subseteq E \mid X = \sigma_\mathcal{H}(X)\}$. We prove that $(E, \mathcal{F})$ is a matroid of arbitrary cardinality. $E \in \mathcal{F}$ is evident, i.e. (F1) holds for $(E, \mathcal{F})$.

Let $F_1, F_2 \in \mathcal{F}$. According to (H2) and Corollary 1, one has $F_1 \cap F_2 = \sigma_\mathcal{H}(F_1) \cap \sigma_\mathcal{H}(F_2) \subseteq \sigma_\mathcal{H}(F_1 \cap F_2) \subseteq \sigma_\mathcal{H}(F_j), (j = 1, 2)$, further, $F_1 \cap F_2 \subseteq \sigma_\mathcal{H}(F_1 \cap F_2) \subseteq \sigma_\mathcal{H}(F_1) \cap \sigma_\mathcal{H}(F_2) = F_1 \cap F_2$. Thus $F_1 \cap F_2 = \sigma_\mathcal{H}(F_1 \cap F_2)$, and so, $F_1 \cap F_2 \in \mathcal{F}$. Hence (F2) holds for $(E, \mathcal{F})$.

Let $F_0 \in \mathcal{F}, x_1, x_2 \in E \setminus F_0$. One sees easily that $\{F \in \mathcal{F} | F_0 \cup \{x_1\} \subseteq F\} = \{F \in \mathcal{F} | F_0 \cup \{x_2\} \subseteq F\} \Leftrightarrow \sigma_\mathcal{H}(F_0 \cup \{x_1\}) = \sigma_\mathcal{H}(F_0 \cup \{x_2\})$. Suppose $\{F \in \mathcal{F} | F_0 \cup \{x_1\} \subseteq F\} \neq \{F \in \mathcal{F} | F_0 \cup \{x_2\} \subseteq F\}$ and $F_j = \sigma_\mathcal{H}(F_0 \cup \{x_j\}), (j = 1, 2)$. By (F2), $(F_1 \cap F_2) \in \mathcal{F}$, besides, $F_0 \subseteq (F_1 \cap F_2)$. If $F_0 \subset (F_1 \cap F_2)$. Let $y \in (F_1 \cap F_2) \setminus F_0$. Then $y \in F_1 = \sigma_\mathcal{H}(F_0 \cup \{x_1\})$, and so by (H3), $x_1 \in \sigma_\mathcal{H}(F_0 \cup \{y\})$. Similarly, $y \in F_2 \Rightarrow x_2 \in \sigma_\mathcal{H}(F_0 \cup \{y\})$. In addition, $F_0 \subseteq \sigma_\mathcal{H}(F_0 \cup \{y\})$. Hence $F_0 \cup \{x_j\} \subseteq \sigma_\mathcal{H}(F_0 \cup \{y\}), (j = 1, 2)$, so that $F_j \subseteq \sigma_\mathcal{H}(F_0 \cup \{y\}), (j = 1, 2)$, and furthermore, $F_1 \cap F_2 \subseteq F_j \subseteq \sigma_\mathcal{H}(F_0 \cup \{y\}) \subseteq F_1 \cap F_2, (j = 1, 2)$. Say, $F_1 \cap F_2 = F_1 = F_2 = \sigma_\mathcal{H}(F_0 \cup \{y\})$. This result follows $\{F \in \mathcal{F} | F_0 \cup \{x_1\} \subseteq F\} = \{F \in \mathcal{F} | F_0 \cup \{x_2\} \subseteq F\}$, a contradiction. Equivalently to say, $(F_1 \cap F_2) \setminus F_0 = \emptyset$. So, $F_0 = F_1 \cap F_2$, i.e. (F3) is satisfied by $(E, \mathcal{F})$.

Let $H' \in \mathcal{H}, \emptyset = X_0$, and $\sigma_\mathcal{H}(X_0) \subset \ldots \subset \sigma_\mathcal{H}(X_{j-1}) \subset \sigma_\mathcal{H}(X_j) \subset \ldots \subset H' = \sigma_\mathcal{H}(X_\mathbf{A})$ be a chain from $\sigma_\mathcal{H}(X_0)$ to H' such that $\sigma_\mathcal{H}(X_j)$ covers $\sigma_\mathcal{H}(X_{j-1}), (1 \leq j \leq \mathbf{A})$. Then by (H4), $|\mathbf{A}| \leq max\{t \in \mathbb{N}_0 | \exists Y_j \subseteq E, (j = 1, \ldots, t)$ with $\sigma_\mathcal{H}(X_0) \subset \sigma_\mathcal{H}(Y_1) \subset \sigma_\mathcal{H}(Y_2) \subset \ldots \subset H' = \sigma_\mathcal{H}(Y_t)\} < \infty$. Considering with the definition of $\sigma_\mathcal{H}$ follows that H' is covered by $\sigma_\mathcal{H}(E)$. Hence the length of the chain $\sigma_\mathcal{H}(X_0) \subset \ldots \subset \sigma_\mathcal{H}(X_j) \subset \ldots \subset H' \subset \sigma_\mathcal{H}(E) = E$ is finite. Further, with the arbitrariness of H' and the above discussion, one obtains $max\{k \in \mathbb{N}_0 | \exists F_0, F_1, \ldots, F_k \in \mathcal{F}$ with $F_0 \subset F_1 \subset \ldots \subset F_k = E\} < \infty$, i.e. (F4) holds for $(E, \mathcal{F})$.

Next to prove that $\mathcal{H}$ is the set of hyperplanes of $(E, \mathcal{F})$.

Let $\mathcal{H}_\mathcal{F}$ be the set of hyperplanes of $(E, \mathcal{F})$. For any $H \in \mathcal{H}$, by the definition of $\sigma_\mathcal{H}$, one has $\sigma_\mathcal{H}(H) = H$. Say $H \in \mathcal{F}$, and further $H \subseteq H_F$ for some $H_F \in \mathcal{H}_\mathcal{F}$. If $H \subset H_F$, then by (H1), one gets $\sigma_\mathcal{H}(H_F) = E$, a contradiction with the definition of $\mathcal{F}$ that tells us $\sigma_\mathcal{H}(H_F) = H_F \neq E$. Thus $H = H_F$, and further, $\mathcal{H} \subseteq \mathcal{H}_\mathcal{F}$. Conversely, for $\forall H_F \in \mathcal{H}_\mathcal{F}$, because $\mathcal{H}_\mathcal{F} \subseteq \mathcal{F} \setminus \{E\}, \sigma_\mathcal{H}(H_F) = H_F, \mathcal{H} \subseteq \mathcal{F}$ and the definition of $\sigma_\mathcal{H}$, one obtains that there exists $H \in \mathcal{H}$ satisfying $H_F \subseteq H$, and so $H_F = H$ by the maximality of H_F. Thus $\mathcal{H}_\mathcal{F} \subseteq \mathcal{H}$.

Corollary 2 Let $M_i = (E, \mathcal{F}_i)$ be a matroid of arbitrary cardinality with $\mathcal{H}_i$ as its set of hyperplanes ($i = 1, 2$). Then $\mathcal{H}_1 = \mathcal{H}_2$ if and only if $\sigma_{\mathcal{H}_1}(X) = \sigma_{\mathcal{H}_2}(X)$ for any $X \subseteq E$, in notation $\sigma_{\mathcal{H}_1} = \sigma_{\mathcal{H}_2}$.

Furthermore, a matroid M of arbitrary cardinality is determined by $\sigma_{\mathcal{H}}$ uniquely. In addition, $\sigma_{\mathcal{H}} = \sigma_M$, where $\mathcal{H}$ is the set of hyperplanes of M and σ_M is the closure operator of M.

Proof Routine verification from the proof of Theorem 1.

3   Relations

This section will discuss the relations between a matroid of arbitrary cardinality and an independence space.

Lemma 4 Let $M = (E, \mathcal{F})$ be a matroid of arbitrary cardinality of rank m with $\mathcal{H}$ as its collection of hyperplanes. Then $\sigma_{\mathcal{H}}$ satisfies (s1)-(s5). Furthermore, M is an independence space on E.

Proof By Theorem 1, $\sigma_{\mathcal{H}}$ is the closure operator of M. Therefore $\sigma_{\mathcal{H}}$ satisfies (s1)-(s4) by Lemma 1. One only needs to prove the correctness of (s5) for $\sigma_{\mathcal{H}}$.

Firstly, one proves $\sigma_{\mathcal{H}}(X \cup \{y\}) = \sigma_{\mathcal{H}}(\sigma_{\mathcal{H}}(X) \cup \{y\})$ for $X \subseteq E$ and $y \in E$. By (s1) and (s2), $\sigma_{\mathcal{H}}(X \cup \{y\}) \subseteq \sigma_{\mathcal{H}}(\sigma_{\mathcal{H}}(X) \cup \{y\})$. However, $X \subseteq X \cup \{y\}$ and (s2) together tells us $\sigma_{\mathcal{H}}(X) \subseteq \sigma_{\mathcal{H}}(X \cup \{y\})$, and further, $\sigma_{\mathcal{H}}(X) \cup \{y\} \subseteq \sigma_{\mathcal{H}}(X \cup \{y\})$. Considering with (s2) and (s3), one has $\sigma_{\mathcal{H}}(\sigma_{\mathcal{H}}(X) \cup \{y\}) \subseteq \sigma_{\mathcal{H}}(\sigma_{\mathcal{H}}(X \cup \{y\})) = \sigma_{\mathcal{H}}(X \cup \{y\})$. Therefore $\sigma_{\mathcal{H}}(X \cup \{y\}) = \sigma_{\mathcal{H}}(\sigma_{\mathcal{H}}(X) \cup \{y\})$.

The following is to prove that $\sigma_{\mathcal{H}}$ satisfies (s5). Suppose $X \subseteq E$ and $|X| = \infty, a \in \sigma_{\mathcal{H}}(X)$ satisfy that for all $X_f \subseteq X$ with $|X_f| < \infty, a \notin \sigma_{\mathcal{H}}(X_f)$ holds.

Let $x_1 \in X$ and $X_1 = \{x_1\}$. It is easy to see $a \notin \sigma_{\mathcal{H}}(X_1)$. Since $x_1 \in X$ follows $X \cap \sigma_{\mathcal{H}}(X_1) \neq \emptyset$. If $X \setminus \sigma_{\mathcal{H}}(X_1) = \emptyset$, then $X \subseteq \sigma_{\mathcal{H}}(X_1)$, further $\sigma_{\mathcal{H}}(X) \subseteq \sigma_{\mathcal{H}}(\sigma_{\mathcal{H}}(X_1)) = \sigma_{\mathcal{H}}(X_1)$, a contradiction with $a \in \sigma_{\mathcal{H}}(X) \setminus \sigma_{\mathcal{H}}(X_1)$. Hence $X \setminus \sigma_{\mathcal{H}}(X_1) \neq \emptyset$. Put $x_2 \in X \setminus \sigma_{\mathcal{H}}(X_1)$. Let $X_2 = \{x_1, x_2\}$. Then $\sigma_{\mathcal{H}}(\sigma_{\mathcal{H}}(X_1) \cup \{x_2\}) = \sigma_{\mathcal{H}}(X_1 \cup \{x_2\}) = \sigma_{\mathcal{H}}(\{x_1\} \cup \{x_2\}) = \sigma_{\mathcal{H}}(X_2) \supset \sigma_{\mathcal{H}}(X_1)$ and $a \notin \sigma_{\mathcal{H}}(X_2)$. Besides, $X \setminus \sigma_{\mathcal{H}}(X_2) \neq \emptyset$, otherwise $X \subseteq \sigma_{\mathcal{H}}(X_2)$ follows $a \in \sigma_{\mathcal{H}}(X) \subseteq \sigma_{\mathcal{H}}(X_2)$, a contradiction. Repeated application of this augmentation yields that for all $m \in \mathbb{N}_0$, there exists a set $\{X_j = \{x_1, x_2, \ldots, x_j\} \subseteq X : j = 1, 2, \ldots, m, m+1\}$ (where $x_{j+1} \in X_{j+1} \setminus X_j, (j = 1, \ldots, m, m+1)$) satisfying $X_1 \subset X_2 \subset \ldots \subset X_j \subset X_{j+1} \subset \ldots \subset X_m \subset X_{m+1} \subset X \subseteq E$ and $\sigma_{\mathcal{H}}(X_1) \subset \sigma_{\mathcal{H}}(X_2) \subset \ldots \subset \sigma_{\mathcal{H}}(X_m) \subset \sigma_{\mathcal{H}}(X_{m+1}) \subset \sigma_{\mathcal{H}}(X) \subseteq E$, a contradiction with (F4). So $\sigma_{\mathcal{H}}$ satisfies (s5).

Moreover, by Lemma 2 and Corollary 2, M is an independence space $M_P(E)$ and the closure operator of $M_P(E)$ is just $\sigma_{\mathcal{H}}$.

The following counter example shows that not every independence space is a matroid of arbitrary cardinality.

Example 1 Let $|E = \{x_0, x_1, x_2, x_3, \ldots\}| = \infty$ and $\mathcal{I} = 2^E$. It is easy to check that $\mathcal{I}$ satisfies (i1)-(i4). That is to say, $(E, \mathcal{I}) = M_P(E)$ is an independence space. In addition, evidently, the set of bases of $M_P(E)$ is exactly $\{E\}$. Therefore, the set $\mathcal{H}$ of hyperplanes of $M_P(E)$ is $\{H \subseteq E \mid H = E \setminus \{h\}$, for $h \in E\}$. However $|E| = \infty$ follows $|H| = \infty \ (\forall H \in \mathcal{H})$. Let $H_{2j} = (E = \{x_t : t = 0, 1, 2, \ldots\}) \setminus \{x_{2j}\}, (j = 0, 1, 2, \ldots)$ and $X = \{x_1, x_3, x_5, \ldots, x_{2k+1}, \ldots\}$. Then $X \subset H_{2j} \in \mathcal{H}, (j = 0, 1, 2, \ldots)$, and $\bigcap\limits_{j=0}^{\infty} H_{2j} = X$. Besides,

$$\left(\emptyset = \bigcap_{H \in \mathcal{H}} H\right) \subset \left(\{x_1\} = \bigcap_{x_1 \in H \in \mathcal{H}} H\right) \subset \left(\{x_1, x_3\} = \bigcap_{\{x_1, x_3\} \subseteq H \in \mathcal{H}} H\right) \subset \ldots \subset \left(\bigcap_{j=0}^{\infty} H_{2j} = X = \sigma_{\mathcal{H}}(X)\right)$$

and $\emptyset = \sigma_{\mathcal{H}}(\emptyset), \{x_1\} = \sigma_{\mathcal{H}}(\{x_1\}), \{x_1, x_3\} = \sigma_{\mathcal{H}}(\{x_1, x_3\}), \ldots, X = \sigma_{\mathcal{H}}(X)$. Hence $\sigma_{\mathcal{H}}(\emptyset) \subset \sigma_{\mathcal{H}}(\{x_1\}) \subset \sigma_{\mathcal{H}}(\{x_1, x_3\}) \subset \ldots \subset \sigma_{\mathcal{H}}(X)$ is not a chain with finite length. Consequently, for $\emptyset, X \subseteq E$, (H4) does not hold. Thus by Theorem 1, $(E, \mathcal{I}) = M_p(E)$ is not a matroid of arbitrary cardinality.

Combining with the results in Lemma 4 and Example 1, one gets the following

Theorem 2 Every matroid of arbitrary cardinality on E is an independence space on E, but not vice versa.

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