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COEFFICIENT INEQUALITY FOR CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract

The objective of this paper is to an obtain an upper bound to the second Hankel determinant $|a_2a_4 - a_3^2|$ for the function f, belonging to a certain subclass of analytic functions, using Toeplitz determinants.

Key Words: Analytic function, upper bound, second Hankel determinant, positive real function, Toeplitz determinants.
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Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions.

In 1976, Noonan and Thomas [13] defined the q^{th} Hankel determinant of f for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{2}$$

This determinant has been considered by several authors. For example, Noor [14] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with bounded boundary. Ehrenborg [4] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [9]. One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in$ S. Ali [2] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ to the class of strongly starlike functions of order $\alpha (0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2|. \tag{3}$$

Janteng, Halim and Darus [8] have considered the functional $|a_2a_4 - a_3^2|$ and found a sharp bound for the function f in the subclass RT of S, consisting of functions whose derivative has a positive real part studied by Mac Gregor [10]. In their work, they have shown that if $f \in$ RT then $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. These authors [7] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses of S, namely, starlike and convex functions denoted by ST and CV and shown that $|a_2a_4 - a_3^2| \leq 1$ and $|a_2a_4 - a_3^2| \leq \frac{1}{8}$ respectively. Mishra and Gochhayat [11] have obtained the sharp bound to the non- linear functional $|a_2a_4 - a_3^2|$ for the class of analytic functions denoted by $R_\lambda(\alpha, \rho)(0 \leq \rho \leq 1, 0 \leq \lambda < 1, |\alpha| < \frac{\pi}{2})$, defined as $Re\left\{e^{i\alpha}\frac{\Omega_z^\lambda f(z)}{z}\right\} > \rho\cos\alpha$, using the fractional differential operator denoted by Ω_z^λ, defined by Owa and Srivastava [15]. These authors have shown that, if $f \in R_\lambda(\alpha, \rho)$ then $|a_2a_4 - a_3^2| \leq \left\{\frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2cos^2\alpha}{9}\right\}$. Similarly, the same coefficient inequality was calculated for certain subclasses of analytic functions by many authors ([1], [3], [12]).
Motivated by the above mentioned results obtained by different authors in this direction, in this paper, we consider a certain subclass of analytic functions and obtain an upper bound to the functional $|a_2a_4 - a_3^2|$ for the function f belonging to this class , defined as follows.
Definition1.1.A function $f(z) \in A$ is said to be in the class $Q(\alpha, \beta, \gamma)$ with α , $\beta > 0$ and

$0 \leq \gamma < \alpha + \beta \leq 1$, if it satisfies the condition that

$$Re\left\{\alpha\frac{f(z)}{z} + \beta f'(z)\right\} \geq \gamma, \qquad \forall z \in E \tag{4}$$

This class was considered and studied by Zhi- Gang Wang, Chun-yi gao and Shao-Mou yuan [17].
We first state some preliminary Lemmas required for proving our result.

1 Preliminary Results

Let P denote the class of functions p analytic in E for which Re$\{p(z)\} > 0$,

$$p(z) = (1 + c_1 z + c_2 z^2 + c_3 z^3 + ...) = \left[1 + \sum_{n=1}^{\infty} c_n z^n\right], \forall z \in E. \tag{5}$$

Lemma 1 *([16]) If $p \in \mathrm{P}$, then $|c_k| \leq 2$, for each $k \geq 1$.*

Lemma 2 *([5]) The power series for p given in (5) converges in the unit disc E to a function in* P *if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix} \quad , n = 1, 2, 3....$$

and $c_{-k} = \overline{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(\exp(it_k)z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$. This necessary and sufficient condition is due to Caratheodory and Toeplitz, can be found in [5].
We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$ respectively, we get

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4c_1^2] \geq 0,$$

which is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \qquad \text{for some } x, |x| \leq 1. \tag{6}$$

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2 \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \quad (7)$$

From the relations (6) *and* (7)*, after simplifying, we get*

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\}$$

for some real value of z*, with* $|z| \leq 1$. (8)

2 Main Result

Theorem 1 *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Q(\alpha, \beta, \gamma)$, $(\alpha, \beta > 0 \quad and \quad 0 \leq \gamma < \alpha + \beta \leq 1)$ then*

$$|a_2a_4 - a_3^2| \leq \left[\frac{4(\alpha + \beta - \gamma)^2}{(\alpha + 3\beta)^2}\right].$$

Proof. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Q(\alpha, \beta, \gamma)$, from the definition 1.1, there exists an analytic function $p \in P$ in the unit disc E with $p(0) = 1$ and $\text{Re}\{p(z)\} > 0$ such that

$$\left\{\alpha \frac{f(z)}{z} + \beta f'(z) - \gamma\right\} = p(z) \Rightarrow \left\{\frac{\alpha f(z) + \beta z f'(z) - \gamma z}{(\alpha + \beta - \gamma)z}\right\} = p(z) \quad (9)$$

Replacing $f(z)$, $f'(z)$ and $p(z)$ with their equivalent series expressions in the relation (9), we have

$$\left[\alpha\left\{z + \sum_{n=2}^{\infty} a_n z^n\right\} + \beta z\left\{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right\} - \gamma z\right] = (\alpha + \beta - \gamma)z\left[\left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}\right]$$

Upon simplification, we obtain

$$\left[(\alpha + 2\beta)a_2 + (\alpha + 3\beta)a_3z + (\alpha + 4\beta)a_4z^2 + ...\right]$$
$$= (\alpha + \beta - \gamma)\left[c_1 + c_2z + c_3z^2 + ...\right]. \quad (10)$$

Equating the coefficients of like powers of z^0, z^1 and z^2 respectively on both sides of (10), we get

$$[a_2 = \frac{(\alpha + \beta - \gamma)}{(\alpha + 2\beta)}c_1; a_3 = \frac{(\alpha + \beta - \gamma)}{(\alpha + 3\beta)}c_2; a_4 = \frac{(\alpha + \beta - \gamma)}{(\alpha + 4\beta)}c_3] \quad (11)$$

Considering the second Hankel functional $|a_2a_4 - a_3^2|$ for the function $f \in Q(\alpha, \beta, \gamma)$ and substituting the values of a_2, a_3 and a_4 from the relation (11), we have

$$|a_2a_4 - a_3^2| = \left|\frac{(\alpha + \beta - \gamma)}{(\alpha + 2\beta)}c_1 \times \frac{(\alpha + \beta - \gamma)}{(\alpha + 4\beta)}c_3 - \frac{(\alpha + \beta - \gamma)^2}{(\alpha + 3\beta)^2}c_2^2\right|$$

Upon simplification, we obtain

$$|a_2a_4 - a_3^2| = \frac{(\alpha + \beta - \gamma)^2}{(\alpha + 2\beta)(\alpha + 3\beta)^2(\alpha + 4\beta)} \times |(\alpha + 3\beta)^2c_1c_3 - (\alpha + 2\beta)(\alpha + 4\beta)c_2^2| \quad (12)$$

The expression (12) is equivalent to

$$|a_2a_4 - a_3^2| = \frac{(\alpha+\beta-\gamma)^2}{(\alpha+2\beta)(\alpha+3\beta)^2(\alpha+4\beta)} \times \left|d_1c_1c_3 + d_2c_2^2\right|. \tag{13}$$

Where

$$\{d_1 = (\alpha+3\beta)^2; d_2 = -(\alpha+2\beta)(\alpha+4\beta)\} \tag{14}$$

Substituting the values of c_2 and c_3 from (6) and (8) respectively from Lemma 2 in the right hand side of (13), we have

$$\left|d_1c_1c_3 + d_2c_2^2\right| = |d_1c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} + d_2 \times \frac{1}{4}\{c_1^2 + x(4-c_1^2)\}^2|.$$

Using the facts that $|z| < 1$ and $|xa + yb| \leq |x||a| + |y||b|$, where x, y, a and b are real numbers, after simplifying, we get

$$4\left|d_1c_1c_3 + d_2c_2^2\right| \leq |(d_1+d_2)c_1^4 + 2d_1c_1(4-c_1^2) + 2(d_1+d_2)c_1^2(4-c_1^2)|x| - \left\{(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2\right\}(4-c_1^2)|x|^2|. \tag{15}$$

Using the values of d_1, d_2, d_3 and d_4 from the relation (14), upon simplification, we obtain

$$\{(d_1+d_2) = \beta^2; d_1 = (\alpha+3\beta)^2\}; \tag{16}$$

$$\left\{(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2\right\} = \left\{\beta^2c_1^2 + 2(\alpha+3\beta)^2c_1 + 4(\alpha+2\beta)(\alpha+4\beta)\right\}. \tag{17}$$

Consider

$$\begin{aligned}
&\left\{\beta^2c_1^2 + 2(\alpha+3\beta)^2c_1 + 4(\alpha+2\beta)(\alpha+4\beta)\right\} \\
&\quad = \beta^2\left\{c_1^2 + \frac{2(\alpha+3\beta)^2}{\beta^2}c_1 + \frac{4(\alpha+2\beta)(\alpha+4\beta)}{\beta^2}\right\}. \\
&\quad = \beta^2\left[\left\{c_1 + \frac{(\alpha+3\beta)^2}{\beta^2}\right\}^2 - \frac{(\alpha+3\beta)^4}{\beta^4} + \frac{4(\alpha+2\beta)(\alpha+4\beta)}{\beta^2}\right]. \\
&\quad = \beta^2\left[\left\{c_1 + \frac{(\alpha+3\beta)^2}{\beta^2}\right\}^2 - \sqrt{\frac{\alpha^4 + 49\beta^4 + 50\alpha^2\beta^2 + 84\alpha\beta^3 + 12\alpha^3\beta}{\beta^4}}\right].
\end{aligned}$$

After simplifying, the above expression reduces to

$$\begin{aligned}
&\left\{\beta^2c_1^2 + 2(\alpha+3\beta)^2c_1 + 4(\alpha+2\beta)(\alpha+4\beta)\right\} \\
&\quad = \beta^2\left[c_1 + \left\{\frac{(\alpha+3\beta)^2}{\beta^2} + \sqrt{\frac{\alpha^4 + 49\beta^4 + 50\alpha^2\beta^2 + 84\alpha\beta^3 + 12\alpha^3\beta}{\beta^4}}\right\}\right] \times \\
&\qquad \left[c_1 + \left\{\frac{(\alpha+3\beta)^2}{\beta^2} - \sqrt{\frac{\alpha^4 + 49\beta^4 + 50\alpha^2\beta^2 + 84\alpha\beta^3 + 12\alpha^3\beta}{\beta^4}}\right\}\right].
\end{aligned} \tag{18}$$

Since $c_1 \in [0,2]$, using the result $(c_1+a)(c_1+b) \geq (c_1-a)(c_1-b)$, where $a, b \geq 0$ in the right hand side of (18), upon simplification, we obtain

$$\begin{aligned}&\left\{\beta^2 c_1^2 + 2(\alpha+3\beta)^2 c_1 + 4(\alpha+2\beta)(\alpha+4\beta)\right\}\\ &\qquad\qquad \geq \left\{\beta^2 c_1^2 - 2(\alpha+3\beta)^2 c_1 + 4(\alpha+2\beta)(\alpha+4\beta)\right\} \quad (19)\end{aligned}$$

From the relations (17) and (19), we get

$$-\left\{(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2\right\} \leq -\left\{\beta^2 c_1^2 - 2(\alpha+3\beta)^2 c_1 + 4(\alpha+2\beta)(\alpha+4\beta)\right\}. \quad (20)$$

Substituting the calculated values from (16) and (20) in the right hand side of (15), we obtain

$$\begin{aligned}4|d_1c_1c_3 + d_2c_2^2| \leq{}& |\beta^2 c_1^4 + 2(\alpha+3\beta)^2 c_1(4-c_1^2) + 2\beta^2 c_1^2(4-c_1^2)|x|\\ &- \left\{\beta^2 c_1^2 - 2(\alpha+3\beta)^2 c_1 + 4(\alpha+2\beta)(\alpha+4\beta)\right\}(4-c_1^2)|x|^2|.\end{aligned}$$

Choosing $c_1 = c \in [0,2]$, applying Triangle inequality and replacing $|\, x \,|$ by μ in the right hand side of the above inequality, it reduces to

$$\begin{aligned}4|d_1c_1c_3 + d_2c_2^2| \leq{}& [\beta^2 c^4 + 2(\alpha+3\beta)^2 c(4-c^2) + 2\beta^2 c^2(4-c^2)\mu\\ &+ \left\{\beta^2 c^2 - 2(\alpha+3\beta)^2 c + 4(\alpha+2\beta)(\alpha+4\beta)\right\}(4-c^2)\mu^2].\\ &= F(c,\mu), \quad \text{for} \quad 0 \leq \mu = |x| \leq 1. \quad (21)\end{aligned}$$

Where

$$\begin{aligned}F(c,\mu) ={}& [\beta^2 c^4 + 2(\alpha+3\beta)^2 c(4-c^2) + 2\beta^2 c^2(4-c^2)\mu\\ &+ \left\{\beta^2 c^2 - 2(\alpha+3\beta)^2 c + 4(\alpha+2\beta)(\alpha+4\beta)\right\}(4-c^2)\mu^2]. \quad (22)\end{aligned}$$

We next maximize the function $F(c,\mu)$ on the closed square $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ in (22) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = [2\beta^2 c^2(4-c^2) + 2\left\{\beta^2 c^2 - 2(\alpha+3\beta)^2 c + 4(\alpha+2\beta)(\alpha+4\beta)\right\}(4-c^2)\mu]. \quad (23)$$

For $0 < \mu < 1$, for fixed c with $0 < c < 2$ and $\alpha, \beta > 0$, from (23), we observe that $\frac{\partial F}{\partial \mu} > 0$. Consequently, $F(c,\mu)$ is an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed square $[0,2] \times [0,1]$. Further, for a fixed $c \in [0,2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c,\mu) = F(c,1) = G(c)(say). \quad (24)$$

From the relations (22) and (24), upon simplification, we obtain

$$G(c) = F(c,1) = \left\{-2\beta^2 c^4 - 4\beta(\alpha^2 + 5\beta^2 + 6\alpha\beta)c^2 + 16(\alpha+2\beta)(\alpha+4\beta)\right\}. \quad (25)$$

$$G'(c) = \left\{-8\beta^2 c^3 - 8\beta(\alpha^2 + 5\beta^2 + 6\alpha\beta)c\right\}. \tag{26}$$

$$G''(c) = \left\{-24\beta^2 c^2 - 8\beta(\alpha^2 + 5\beta^2 + 6\alpha\beta)\right\}. \tag{27}$$

From the expression (26), we observe that $G'(c) \leq 0$, for all values of $c \in [0,2]$ and for fixed values of α, $\beta > 0$, where $(0 < \alpha + \beta \leq 1)$.Therefore, $G(c)$ is a monotonically decreasing function of c in $0 \leq c \leq 2$, attains its maximum value at $c = 0$. From the expression (25), we have G-maximum value at $c = 0$ is given by

$$\max_{0 \leq c \leq 2} G(c) = G(0) = 16(\alpha + 2\beta)(\alpha + 4\beta). \tag{28}$$

Considering, only the maximum value of $G(c)$ at $c = 0$, from the relations (21) and (28), after simplifying, we get

$$|d_1 c_1 c_3 + d_2 c_2^2| \leq 4(\alpha + 2\beta)(\alpha + 4\beta). \tag{29}$$

From the expressions (13) and (29), upon simplification, we obtain

$$|a_2 a_4 - a_3^2| \leq \left[\frac{4(\alpha + \beta - \gamma)^2}{(\alpha + 3\beta)^2}\right]. \tag{30}$$

This completes the proof of our Theorem. □

Remark. For the choice of $\alpha = (1 - \sigma)$. $\beta = \sigma$ and $\gamma = 0$, we get$(\alpha, \beta, \gamma) = ((1 - \sigma), \sigma, 0)$, for which, from (30), upon simplification, we obtain $|a_2 a_4 - a_3^2| \leq \left[\frac{4}{(1+2\sigma)^2}\right]$, for $0 \leq \sigma \leq 1$. This result is a special case to that of Murugusundaramoorthy and Magesh [12].

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