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A Note on Location of the Zeros of Quaternionic Polynomials

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Abstract. The purpose of this paper is to investigate the extensions of the classical Eneström-Kakeya theorem and its various generalizations concerning the distribution of zeros of polynomials from the complex to the quaternionic setting. Using a maximum modulus theorem and the zero set structure in the recently published theory of regular functions and polynomials of a quaternionic variable, we construct new bounds of the Eneström-Kakeya type for the zeros of these polynomials. The obtained results for this subclass of polynomials and slice regular functions give generalizations of a number of results previously reported in the relevant literature.

Key Words: Quaternionic Polynomial, Zeros, Eneström-Kakeya theorem
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Introduction

Mathematical models are created by translating various scientific findings and investigations into mathematical terms. Solving these models can lead to some difficulties in solving algebraic polynomial equations. The study of zeros of these algebraic complex polynomials is an old topic in the analytic theory of polynomials spawning a vast amount of research over the past millennium including its applications both within and outside of mathematics. In addition to having numerous applications, this study inspired many theoretical research (including being the initial motivation for modern algebra).

Algebraic and analytic methods for finding zeros of a polynomial can in general, be quite complicated, thus it is desirable to put some restrictions on polynomials. The exact computation of zeros of polynomials of degree at most four is made possible by algebraic formulas designed for such polynomials. This accomplishment is impossible, or to put it another way, polynomial equations of degree five or higher cannot be solved by radicals. This crucial mathematical milestone was due to the revolutionary algebraic discoveries of N. H. Abel and E. Galois in the first quarter of the nineteenth century. This and substantial uses of zero bounds in fields of science, including stability theory, mathematical biology, communication theory, and computer engineering made it intriguing to pinpoint the appropriate areas in the complex plane where the zeros of a given polynomial fall.

The classical result due to Cauchy [3] on the distribution of zeros of a polynomial may be stated as follows:

Theorem 1 *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then all the zeros of p lie in*

$$|z| < 1 + \max_{1\leq v\leq n-1} \left|\frac{a_v}{a_n}\right|.$$

Although various results concerning the bounds for zeros of polynomials are available in the literature (see, for example, [11, 12, 15]), the remarkable property of the bound in Theorem 1 which distinguishes it from other such bounds, is its simplicity of computations. However, this simplicity comes at the cost of precision. The following elegant result on the location of zeros of a polynomial with restricted coefficients is known as the Eneström-Kakeya theorem (see, for example, [4, 11, 12]) which states that:

Theorem 2 *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n such that $0 < a_0 \leq a_1 \leq ... \leq a_n$, then all the zeros of p lie in $|z| \leq 1$.*

There exists various extensions and generalizations of the Eneström-Kakeya theorem, see, for example, [1,7–10,12,15]. By removing non-negative restriction over the coefficients of polynomial $p(z)$, Joyal et al. [10] proved the following result:

Theorem 3 *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n such that $a_0 \leq a_1 \leq ... \leq a_n$, then all the zeros of p lie in $|z| \leq (|a_0| + a_n - a_0)/|a_n|$.*

Govil and Rahman [7] presented the following result, applicable to polynomials with complex coefficients

Theorem 4 *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n with complex coefficients such that for some real β, $|arg\ a_v - \beta| \leq \alpha \leq \pi/2$, $v = 0, 1, 2, \dots, n$, and $|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|$, then all the zeros of $P(z)$ lie in*

$$\Big|z\Big| \leq \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{v=0}^{n-1}|a_v|.$$

Aziz and Zargar [1] extended Theorem 3 in the sense as they relaxed the hypothesis of the Eneström-Kakeya theorem

Theorem 5 *If $P(z) = \sum_{v=0} a_v z^v$ is a polynomial of degree n such that for some $k \geq 1$, $ka_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0$, then all the zeros of $P(z)$ lie in*

$$|z + k - 1| \leq \frac{ka_n - a_0 + |a_0|}{|a_n|}.$$

Further, Shah and Liman [13] extended Theorem 5 to the polynomials with complex coefficients

Theorem 6 *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n with complex coefficients such that for some real β, $|arg\ a_v - \beta| \leq \alpha \leq \pi/2$, $v = 0, 1, 2, \ldots, n$ and for $k \geq 1$, $k|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|$, then all the zeros of $P(z)$ lie in*

$$\Big|z - k - 1\Big| \leq \frac{1}{|a_n|}\Bigg\{(k|a_n| - |a_0|)(\cos\alpha + \sin\alpha) + |a_0| + 2\sin\alpha \sum_{v=0}^{n-1} |a_v|\Bigg\}.$$

There has been extensive research on where the zeros of complex polynomials lie, with a stronger emphasis on polynomials with constrained coefficients. Illustrative and notable examples of this type include the Enestr?m-Kakeya theorem and its several generalizations noted above. A natural question: "What kind of results in the quaternionic context may be obtained given such richness of the complex setting?" The purpose of this paper is to present certain classical Eneström-Kakeya type results that were addressed previously as extensions to the quaternionic setting.

Let us introduce some background information on quaternions and quaternionic polynomials before we state our results. In honour of Hamilton, this quaternion number system is represented by the letter $\mathbb{H}$ and are generally represented in the form $q = \alpha + i\beta + j\gamma + k\delta \in \mathbb{H}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and i, j, k are the fundamental quaternion units, such that $i^2 = j^2 = k^2 = ijk = -1$. Each quaternion q has a conjugate. The conjugate of a quaternion $q = \alpha + i\beta + j\gamma + k\delta$ is denoted by q^* and is defined as $q^* = \alpha - i\beta - j\gamma - k\delta$. Moreover, the norm (or length) of a quaternion q is given by

$$|q| = \sqrt{qq^*} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}.$$

The quaternions are the standard example of a noncommutative division ring and also forms a four dimensional vector space over $\mathbb{R}$ with $\{1, i, j, k\}$ as a basis.

In 2020, Carney et al. [2] proved the following extension of Theorems 2, 3 and 4 for the quaternionic polynomial $p(q)$.

Theorem 7 *If* $p(q) = \sum_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with real coefficients satisfying* $0 < a_0 \leq a_1 \leq ... \leq a_n$*, then all the zeros of* p *lie in* $|q| \leq 1$*.*

Theorem 8 *If* $p(q) = \sum\limits_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with quaternionic coefficients* $a_v = \alpha_v + i\beta_v + j\gamma_v + k\delta_v,\ v = 0, 1, 2, ..., n$*, satisfying*

$$\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_0, \qquad \beta_n \geq \beta_{n-1} \geq ... \geq \beta_0,$$

$$\gamma_n \geq \gamma_{n-1} \geq ... \geq \gamma_0, \qquad \delta_n \geq \delta_{n-1} \geq ... \geq \delta_0,$$

then all the zeros of p *lie in*

$$|q| \leq 1/|a_n|\left\{(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)\right\}.$$

Theorem 9 *Let* $p(q) = \sum\limits_{v=0}^{n} q^v a_v$ *be a quaternionic polynomial of degree* n *with quaternionic coefficients. Let* β *be a non-zero quaternion and suppose* $\angle(a_\nu, \beta) \leq \alpha \leq \pi/2$ *for some* α*,* $v = 0, 1, 2, \ldots, n$*, and* $|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|$*, then all the zeros of* p *lie in*

$$\big|q\big| \leq \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{\nu=0}^{n-1}|a_\nu|.$$

For more results in this direction, one can see [8, 9].

1 Results

We begin with the following result, which gives a generalization of Theorem 6, and hence, of Theorem 5 as well.

Theorem 10 *If* $p(q) = \sum\limits_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with real coefficients such that for some* $k_j \geq 1$*,* $1 \leq j \leq r$ *where* $1 \leq r \leq n$*, it holds*

$$k_1 a_n \geq k_2 a_{n-1} \geq k_3 a_{n-2} \geq \cdots \geq k_r a_{n-r+1} \geq a_{n-r} \geq \cdots \geq a_1 \geq a_0,$$

then all the zeros of p *lie in*

$$|q| \leq \frac{1}{|a_n|}\left\{k_1 a_n - (k_1 - 1)|a_n| + 2\sum_{v=1}^{r}(k_v - 1)|a_{n-v+1}| - a_0 + |a_0|\right\}. \quad (1)$$

If we take $r = 1$ and $a_0 > 0$ in Theorem 10, we get the following result:

Corollary 1 *If* $p(q) = \sum_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with real coefficients such that for some* $k_1 \geq 1$,

$$k_1 a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_{n-r+1} \geq a_{n-r} \geq \cdots \geq a_1 \geq a_0 > 0,$$

then all the zeros of p *lie in*

$$|q| \leq 2k_1 - 1. \tag{2}$$

Since the results discussed above apply to a small class of quaternionic polynomials, it is interesting to look for the results that apply to the large class of polynomials. Next, we extend Theorem 10 to the polynomials with quaternionic coefficients and thereby obtain a result with relaxed hypothesis that gives zero bounds of the polynomials with quaternionic coefficients.

Theorem 11 *Let* $p(q) = \sum_{v=0}^{n} q^v a_v$ *be a quaternionic polynomial of degree* n *with quaternionic coefficients. Let* β *be a non-zero quaternion and suppose* $\angle(a_\nu, \beta) \leq \alpha \leq \pi/2$ *for some* α, $v = 0, 1, 2, \ldots, n$, *and for* $k_j \geq 1$, $0 \leq j \leq r$ *where* $0 \leq r \leq n$, *it holds*

$$k_0|a_n| \geq k_1|a_{n-1}| \geq k_2|a_{n-2}| \geq \cdots \geq k_{r-1}|a_{n-r+1}| \geq k_r|a_{n-r}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of p *lie in*

$$|q| \leq \frac{1}{|a_n|}\Bigg\{\Big(k_0|a_n| - |a_0|\Big)\Big(\cos\alpha + \sin\alpha\Big) + 2\sin\alpha\Big(\sum_{v=1}^{r} k_v|a_{n-v}| + \sum_{v=r+1}^{n}|a_{n-v}|\Big) - (k_0-1)|a_n| + 2\sum_{v=0}^{r}(k_v-1)|a_{n-v}| + |a_0|\Bigg\}.$$

If we take $r = 0$ in Theorem 11, we get the following result:

Corollary 2 *Let* $p(q) = \sum_{v=0}^{n} q^v a_v$ *be a quaternionic polynomial of degree* n *with quaternionic coefficients. Let* β *be a non-zero quaternion and suppose* $\angle(a_\nu, \beta) \leq \alpha \leq \pi/2$ *for some* α, $v = 0, 1, 2, \ldots, n$, *and for* $k_0 \geq 1$,

$$k_0|a_n| \geq |a_{n-1}| \geq |a_{n-2}| \geq \cdots \geq |a_{n-r+1}| \geq |a_{n-r}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of p *lie in*

$$|q| \leq \frac{1}{|a_n|}\Bigg\{\Big(k_0|a_n| - |a_0|\Big)\Big(\cos\alpha + \sin\alpha\Big) + 2\sin\alpha\Big(\sum_{v=1}^{n}|a_{n-v}|\Big) + (k_0-1)|a_n| + |a_0|\Bigg\}.$$

Remark 1 *If we take* $k_0 = 1$ *in Theorem 11, we easily get Theorem 9.*

2 Proof of Theorems

Let $f(q) = \sum_{v=0}^{\infty} q^v a_v$ and $g(q) = \sum_{v=0}^{\infty} q^v b_v$ be two given quaternionic power series. The regular product of $f(q)$ and $g(q)$ is defined as $(f\star g)(q) = \sum_{v=0}^{\infty} q^v c_v$, where $c_v = \sum_{l=0}^{\infty} a_l b_{v-l}$.

For the proofs of our results, we need the following two lemmas due to Gentili and Stoppato [5] and Carney et al. [2], respectively.

Lemma 1 *Let $f(q) = \sum_{v=0}^{\infty} q^v a_v$ and $g(q) = \sum_{v=0}^{\infty} q^v b_v$ be two given quaternionic power series with radii of convergence greater than R. Let $|q_0| < R$, then $(f \star g)(q_0) = 0$ if and only if either $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g(f(q_0)^{-1} q_0 f(q_0)) = 0$.*

Lemma 2 *Let $q_1, q_2 \in \mathbb{H}$, $q_1 = \alpha_1 + \beta_1 i + \gamma_1 j + \delta_1 k$ and $q_2 = \alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k$ be such that $\angle(q_1, q_2) = 2\alpha' \leq 2\alpha$ and $|q_1| \leq |q_2|$. Then*

$$|q_2 - q_1| \leq (|q_2| - |q_1|)\cos\alpha + (|q_2| + |q_1|)\sin\alpha.$$

Proof of Theorem 10 Consider the polynomial

$$\begin{aligned}
f(q) &= \sum_{v=1}^{n} q^v (a_v - a_{v-1}) + a_0 \\
&= (a_n - a_{n-1})q^n + \cdots + (a_{n-r} - a_{n-r-1})q^{n-r} + \cdots + (a_1 - a_0)q + a_0 \\
&= (k_1 a_n - k_2 a_{n-1} - (k_1 - 1)a_n + (k_2 - 1)a_{n-1})q^n \\
&+ (k_2 a_{n-1} - k_3 a_{n-2} - (k_2 - 1)a_{n-1} + (k_3 - 1)a_{n-2})q^{n-1} + \ldots \\
&+ (k_{r-1} a_{n-r+2} - k_r a_{n-r+1} - (k_{r-1} - 1)a_{n-r+2} \\
&+ (k_r - 1)a_{n-r+1})q^{n-r+2} + (k_r a_{n-r+1} - a_{n-r} - (k_r - 1)a_{n-r+1})q^{n-r+1} \\
&+ (a_{n-r} - a_{n-r-1})q^{n-r} + \cdots + (a_1 - a_0)q + a_0.
\end{aligned}$$

Let $p(q) \star (1 - q) = f(q) - q^{n+1} a_n$. By Lemma 1, $p(q) \star (1 - q) = 0$ if and only if either $p(q) = 0$ or $p(q) \neq 0$ implies $p(q)^{-1} q p(q) - 1 = 0$, that is, $p(q)^{-1} q p(q) = 1$. If $p(q) \neq 0$, then $q = 1$. Therefore, the only zeros of $p(q) \star (1 - q)$ are $q = 1$ and the zeros of $p(q)$.

For $|q| = 1$, we have

$$\begin{aligned}
|f(q)| &\leq |a_0| + \sum_{v=1}^{n} |a_v - a_{v-1}| \\
&= |k_1 a_n - k_2 a_{n-1}| + (k_1 - 1)|a_n| + (k_2 - 1)|a_{n-1}| \\
&\quad + |k_2 a_{n-1} - k_3 a_{n-2}| - (k_2 - 1)|a_{n-1}| + (k_3 - 1)|a_{n-2}| + \ldots
\end{aligned}$$

$$\begin{aligned}
&+|k_{r-1}a_{n-r+2} - k_r a_{n-r+1}| + (k_{r-1}-1)|a_{n-r+2}| \\
&+(k_r-1)|a_{n-r+1}| + |k_r a_{n-r+1} - a_{n-r}| + (k_r-1)|a_{n-r+1}| \\
&+|a_{n-r} - a_{n-r-1}| + \cdots + |a_1 - a_0| + |a_0| \\
\leq\ & k_1 a_n - k_2 a_{n-1} + (k_1-1)|a_n| + (k_2-1)|a_{n-1}| \\
&+k_2 a_{n-1} - k_3 a_{n-2} + (k_2-1)|a_{n-1}| + (k_3-1)|a_{n-2}| + \dots \\
&+k_{r-1}a_{n-r+2} - k_r a_{n-r+1} + (k_{r-1}-1)|a_{n-r+2}| \\
&+(k_r-1)|a_{n-r+1}| + k_r a_{n-r+1} - a_{n-r} + (k_r-1)|a_{n-r+1}| \\
&+a_{n-r} - a_{n-r-1} + \cdots + a_1 - a_0 + |a_0| \\
=\ & k_1 a_n - (k_1-1)|a_n| + 2\sum_{v=1}^{r}(k_v-1)|a_{n-v+1}| - a_0 + |a_0|.
\end{aligned}$$

Since

$$\max_{|q|=1}\left|q^n \star f\Big(\frac{1}{q}\Big)\right| = \max_{|q|=1}\left|f\Big(\frac{1}{q}\Big)\right| = \max_{|q|=1}|f(q)|,$$

$q^n \star f\Big(1/q\Big)$ has the same bound on $|q|=1$ as $f(q)$, that is,

$$\left|q^n \star f\Big(\frac{1}{q}\Big)\right| \leq k_1 a_n - (k_1-1)|a_n| + 2\sum_{v=1}^{r}(k_v-1)|a_{n-v+1}| - a_0 + |a_0|.$$

Applying maximum modulus theorem ([6], Theorem 3.4), for $|q| \leq 1$, we obtain

$$\left|q^n \star f\Big(\frac{1}{q}\Big)\right| \leq k_1 a_n - (k_1-1)|a_n| + 2\sum_{v=1}^{r}(k_v-1)|a_{n-v+1}| - a_0 + |a_0|.$$

Replacing q by $1/q$, we get

$$|f(q)| \leq \left\{k_1 a_n - (k_1-1)|a_n| + 2\sum_{v=1}^{r}(k_v-1)|a_{n-v+1}| - a_0 + |a_0|\right\}|q|^n. \quad (3)$$

But $|p(q)\star(1-q)| = |f(q) - q^{n+1}a_n| \geq |a_n||q|^{n+1} - |f(q)|$.

Using (3), for $|q| \geq 1$, we obtain

$$\begin{aligned}
|p(q)\star(1-q)| \geq\ & |a_n||q|^{n+1} \\
& - \left\{k_1 a_n - (k_1-1)|a_n| + 2\sum_{v=1}^{r}(k_v-1)|a_{n-v+1}| - a_0 + |a_0|\right\}|q|^n.
\end{aligned}$$

This implies that $|p(q)\star(1-q)| > 0$, i.e., $p(q)\star(1-q) \neq 0$ if

$$|q| > \frac{1}{|a_n|}\left\{k_1 a_n - (k_1-1)|a_n| + 2\sum_{v=1}^{r}(k_v-1)|a_{n-v+1}| - a_0 + |a_0|\right\}.$$

Note that the only zeros of $p(q)\star(1-q)$ are $q=1$ and the zeros of $p(q)$. Therefore, $p(q)\neq 0$ for

$$|q|>\frac{1}{|a_n|}\left\{k_1a_n-(k_1-1)|a_n|+2\sum_{v=1}^{r}(k_v-1)|a_{n-v+1}|-a_0+|a_0|\right\}.$$

Hence, all the zeros of $p(q)$ lie in

$$|q|\leq\frac{1}{|a_n|}\left\{k_1a_n-(k_1-1)|a_n|+2\sum_{v=1}^{r}(k_v-1)|a_{n-v+1}|-a_0+|a_0|\right\}.$$

This completes the proof of Theorem 10. □

Proof of Theorem 11 Consider the polynomial

$$f(q)=\sum_{v=1}^{n}q^v(a_v-a_{v-1})+a_0.$$

Let $p(q)\star(1-q)=f(q)-q^{n+1}a_n$. By Lemma 1, $p(q)\star(1-q)=0$ if and only if either $p(q)=0$ or $p(q)\neq 0$ implies $p(q)^{-1}qp(q)-1=0$, that is, $p(q)^{-1}qp(q)=1$. If $p(q)\neq 0$, then $q=1$. Therefore, the only zeros of $p(q)\star(1-q)$ are $q=1$ and the zeros of $p(q)$.

For $|q|=1$, we have

$$\begin{aligned}|f(q)| &\leq |a_0|+\sum_{v=1}^{n}|a_v-a_{v-1}|\\ &\leq |k_0a_n-k_1a_{n-1}|+(k_0-1)|a_n|+(k_1-1)|a_{n-1}|\\ &\quad+|k_1a_{n-1}-k_2a_{n-2}|-(k_1-1)|a_{n-1}|+(k_2-1)|a_{n-2}|+\dots\\ &\quad+|k_{r-1}a_{n-r+1}-k_ra_{n-r}|+(k_{r-1}-1)|a_{n-r+1}|+(k_r-1)|a_{n-r}|\\ &\quad+|k_ra_{n-r}-a_{n-r-1}|-(k_r-1)|a_{n-r}|\\ &\quad+|a_{n-r-1}-a_{n-r-2}|+\dots+|a_1-a_0|+|a_0|.\end{aligned}$$

Using Lemma 2, we obtain

$$\begin{aligned}|f(q)| \leq\; & \Big(\big|k_0|a_n|-k_1|a_{n-1}|\big|+\big|k_1|a_{n-1}|-k_2|a_{n-2}|\big|+\dots\\ +\; & \big|k_{r-1}|a_{n-r+1}|-k_r|a_{n-r}|\big|+\big|k_r|a_{n-r}|-|a_{n-r-1}|\big|\\ +\; & \big||a_{n-r-1}|-|a_{n-r-2}|\big|\dots\big||a_1|-|a_0|\big|\Big)\cos\alpha\\ +\; & \Big(k_0|a_n|+k_1|a_{n-1}|+k_1|a_{n-1}|+k_2|a_{n-2}|+\dots+k_{r-1}|a_{n-r+1}|\\ +\; & k_r|a_{n-r}|+k_r|a_{n-r}|+|a_{n-r-1}|+|a_{n-r-2}|+\dots+|a_1|+|a_0|\Big)\sin\alpha\end{aligned}$$

$$- \quad (k_0-1)|a_{n-1}| + 2\sum_{v=0}^{r}(k_v-1)|a_{n-v}| + |a_0|,$$

which, in view of given hypothesis, yields

$$\begin{aligned}|f(q)| \quad &\leq \quad \Big(k_0|a_n| - |a_0|\Big)\Big(\cos\alpha + \sin\alpha\Big) + 2\sin\alpha\Big(\sum_{v=1}^{r} k_v|a_{n-v}| \\ &+ \sum_{v=r+1}^{n}|a_{n-v}|\Big) - (k_0-1)|a_n| + 2\sum_{v=0}^{r}(k_v-1)|a_{n-v}| + |a_0|.\end{aligned}$$

Since

$$\max_{|q|=1}\Big|q^n \star f\Big(\frac{1}{q}\Big)\Big| = \max_{|q|=1}\Big|f\Big(\frac{1}{q}\Big)\Big| = \max_{|q|=1}|f(q)|,$$

$q^n \star f(1/q)$ has the same bound on $|q|=1$ as $f(q)$, that is,

$$\begin{aligned}&|q^n \star f\Big(\frac{1}{q}\Big)| \leq \Big(k_0|a_n| - |a_0|\Big)\Big(\cos\alpha + \sin\alpha\Big) + 2\sin\alpha\Big(\sum_{v=1}^{r} k_v|a_{n-v}| \\ &+ \sum_{v=r+1}^{n}|a_{n-v}|\Big) - (k_0-1)|a_n| + 2\sum_{v=0}^{r}(k_v-1)|a_{n-v}| + |a_0|.\end{aligned}$$

Applying maximum modulus theorem ([6], Theorem 3.4), we obtain

$$\begin{aligned}&|q^n \star f\Big(\frac{1}{q}\Big)| \leq \Big(k_0|a_n| - |a_0|\Big)\Big(\cos\alpha + \sin\alpha\Big) + 2\sin\alpha\Big(\sum_{v=1}^{r} k_v|a_{n-v}| \\ &+ \sum_{v=r+1}^{n}|a_{n-v}|\Big) - (k_0-1)|a_n| + 2\sum_{v=0}^{r}(k_v-1)|a_{n-v}| + |a_0|.\end{aligned}$$

Replacing q by $1/q$, for $|q| \geq 1$, we get

$$\begin{aligned}&|f(q)| \leq \Big\{\Big(k_0|a_n| - |a_0|\Big)\Big(\cos\alpha + \sin\alpha\Big) + 2\sin\alpha\Big(\sum_{v=1}^{r} k_v|a_{n-v}| \\ &+ \sum_{v=r+1}^{n}|a_{n-v}|\Big) - (k_0-1)|a_n| + 2\sum_{v=0}^{r}(k_v-1)|a_{n-v}| + |a_0|\Big\}|q|^n. \qquad (4)\end{aligned}$$

But $|p(q) \star (1-q)| = |f(q) - q^{n+1}a_n| \geq |a_n||q|^{n+1} - |f(q)|$.

Using (4), for $|q| \geq 1$, we can write

$$|p(q) \star (1-q)| \geq |a_n||q|^{n+1} - \Big\{\Big(k_0|a_n| - |a_0|\Big)\Big(\cos\alpha + \sin\alpha\Big)$$

$$+ 2\sin\alpha\Big(\sum_{v=1}^{r} k_v|a_{n-v}| + \sum_{v=r+1}^{n}|a_{n-v}|\Big) - (k_0-1)|a_n|$$
$$+ 2\sum_{v=0}^{r}(k_v-1)|a_{n-v}| + |a_0|\Bigg\}|q|^n.$$

This implies that $|p(q)\star(1-q)| > 0$, i.e., $p(q)\star(1-q)\neq 0$ if

$$|q| > \frac{1}{|a_n|}\Bigg\{\Big(k_0|a_n|-|a_0|\Big)\Big(\cos\alpha+\sin\alpha\Big) - (k_0-1)|a_n| +2\sin\alpha\Big(\sum_{v=1}^{r} k_v|a_{n-v}| + \sum_{v=r+1}^{n}|a_{n-v}|\Big) + 2\sum_{v=0}^{r}(k_v-1)|a_{n-v}| + |a_0|\Bigg\}.$$

Note that the only zeros of $p(q)\star(1-q)$ are $q=1$ and the zeros of $p(q)$. Therefore, $p(q)\neq 0$ for

$$|q| > \frac{1}{|a_n|}\Bigg\{\Big(k_0|a_n|-|a_0|\Big)\Big(\cos\alpha+\sin\alpha\Big) - (k_0-1)|a_n| +2\sin\alpha\Big(\sum_{v=1}^{r} k_v|a_{n-v}| + \sum_{v=r+1}^{n}|a_{n-v}|\Big) + 2\sum_{v=0}^{r}(k_v-1)|a_{n-v}| + |a_0|\Bigg\}.$$

Hence, all the zeros of $p(q)$ lie in

$$|q| \le \frac{1}{|a_n|}\Bigg\{\Big(k_0|a_n|-|a_0|\Big)\Big(\cos\alpha+\sin\alpha\Big) - (k_0-1)|a_n| +2\sin\alpha\Big(\sum_{v=1}^{r} k_v|a_{n-v}| + \sum_{v=r+1}^{n}|a_{n-v}|\Big) + 2\sum_{v=0}^{r}(k_v-1)|a_{n-v}| + |a_0|\Bigg\}.$$

This completes the proof of Theorem 11. □

3 Conclusion

Some new results on Eneström-Kakeya theorem for quaternionic polynomials has been established that are beneficial in determining regions containing all zeros of a polynomial.

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