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Generalized Rational Evaluation Subgroups of the Inclusion between Complex Projective Spaces

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Abstract. We use a model of mapping spaces to compute the generalized rational Gottlieb groups of the inclusion $i_{n,k} : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+k}$ between complex projective spaces.

Key Words: Mapping Space, L_∞ Algebra, Gottlieb Groups
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Introduction

One of the main problems in Topology is the classification of topological spaces up to homotopy. A more tractable approach is to forget torsion in the homotopy groups. Such an approach has underlying theories which lead to algebraic models of nilpotent spaces, namely, Sullivan and Quillen theories [13,14]. Our interest is the determination of the rational homotopy type of mapping spaces between topological spaces. In this paper, we will study the connected component of the inclusion $i_{n,k} : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+k}$ between complex projective spaces. We will use Sullivan models and L_∞ models, which we briefly recall and for which details can be found in [4,7].

We assume that all vector spaces are over the field $\mathbb{Q}$ of rational numbers. The dual of a graded vector space V will be denoted by $V^\#$. Let $A = \oplus_{n\geq 0} A^n$ be a graded algebra. The degree of a homogeneous element $a \in A^n$ will be denoted by $|a|$. A graded algebra is called commutative if $ab = (-1)^{|a||b|} ba$, where a and b are homogeneous. A differential graded algebra is a graded algebra $A = \oplus_{\geq 0} A^n$ together with an algebra differential $d : A^n \to A^{n+1}$ such that $d^2 = 0$. We call (A, d) a cochain algebra. Let $V = \oplus_{\geq 1} V^n$ be a graded vector space. A Sullivan algebra $(\wedge V, d)$ is the free graded commutative algebra generated by V together with a filtration $V(0) \subset V(1) \subset \cdots \subset V$ such that $dV(i) \subset \wedge V(i-1)$. It is called minimal if $dV \subset \wedge^{\geq 2} V$. If (A, d) is a commutative differential graded algebra which is simply connected, that is,

$H^0(A,d) = \mathbb{Q}$ and $H^1(A,d) = 0$, then there exists a minimal Sullivan algebra $(\wedge V, d)$ together with a quasi-isomorphism $(\wedge V, d) \to (A,d)$. It is called the minimal Sullivan model of (A,d) and it is unique up to isomorphism [7, §12].

The minimal Sullivan model $(\wedge V, d)$ of a simply connected space X is the minimal Sullivan model of the commutative differential graded algebra of piecewise linear forms $A_{PL}(X)$ on X [14]. Moreover, if X is of finite type, that is, $H^i(X,\mathbb{Q})$ is a finite-dimensional vector space, then $V^n \cong \mathrm{Hom}_{\mathbb{Z}}(\pi_n(X), \mathbb{Q})$ [7, Theorem 15.11].

Let $f : X \to Y$ be a map between simply connected CW-complexes of finite type. We denote by $\mathrm{map}(X,Y;f)$ the set of continuous mappings from X to Y which are freely homotopic to f. Sullivan's model of $\mathrm{map}(X,Y;f)$ was first given by Haefliger [10], and more recently L_∞ models were developed in [2–5].

We denote by $\mathbb{C}P^n$ the complex projective space which is the smooth manifold of lines in $\mathbb{C}^{n+1}$. Its minimal Sullivan model is given by $(\wedge(x_2, x_{2n+1}), d)$, where subscripts indicate the degrees with $dx_2 = 0$ and $dx_{2n+1} = x_2^{n+1}$. Moreover, the projection

$$(\wedge(x_2, x_{2n+1}), d) \to \wedge((x_2)/(x_2^{n+1}), 0)$$

is a quasi-isomorphism.

Consider the natural inclusion $i_{n,k} : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+k}$ between complex projective spaces. An L_∞ model of $\mathrm{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ was described in [8], from which the following is derived.

Theorem 1 ([8], Theorem 11) *The mapping space* $\mathrm{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ *has the rational homotopy type of* $\mathbb{C}P^k \times S^{2k+3} \times \cdots \times S^{2(n+k)+1}$.

For $k = 0$, the theorem agrees with [14, §11] where a model of $B\,\mathrm{aut}_1\,\mathbb{C}P^n$ is computed. Here $\mathrm{aut}_1\,X$ denotes $\mathrm{map}(X, X, 1_X)$, the monoid of self homotopy equivalences of X.

1 Generalized evaluation subgroups of the inclusion $\mathbb{C}P^n \to \mathbb{C}P^{n+k}$

Let $\phi : (A,d) \to (B,d)$ be a map of cochain algebras. A ϕ-derivation of degree k is a linear mapping $\theta : A^* \to B^{*-k}$ such that $\theta(ab) = \theta(a)\phi(b) + (-1)^{k|a|}\phi(a)\theta(b)$. We denote by $\mathrm{Der}_k(A,B;\phi)$ the vector space of all derivations of degree k. There is a differential

$$D : \mathrm{Der}_k(A,B;\phi) \to \mathrm{Der}_{k-1}(A,B;\phi)$$

defined by $D\theta = d\theta - (-1)^k\theta d$. Define $\mathrm{Der}(A,B;\phi) = \oplus_{k\geq 1}\mathrm{Der}_k(A,B;\phi)$, where in degree 1, we restrict to those derivations which are cycles. Hence,

$(\mathrm{Der}(A,B;\phi),D)$ is a chain complex. Moreover, if $A=(\wedge V,d)$ and (B,d) are commutative differential graded algebras where A is a Sullivan algebra, then $s^{-1}(\mathrm{Der}(A,B;\phi),D)$ has an L_∞ structure [4, 5].

Let $\phi:(\wedge V,d)\to(B,d)$ be a morphism between commutative differential graded algebras. For $v\in V$ and $b\in B$, we denote by (v,b) the unique ϕ-derivation θ such that $\theta(v)=b$ and θ vanishes on the remaining generators of $\wedge V$. Let $f:X\to Y$ be a map between pointed CW-complexes of finite type and $\mathrm{ev}:\mathrm{map}(X,Y;f)\to Y$ be the evaluation at the base point of X. The generalized evaluation subgroup $G_*(Y,X;f)$ of f is the image of $\pi_*(\mathrm{ev}):\pi_*(\mathrm{map}(X,Y;f))\to\pi_*(Y)$. If $Y=X$ and f is the identity map, then one gets the usual Gottlieb group of X [9].

If Y has the homotopy type of a finite CW complex and $\phi:(\wedge V,d)\to(B,d)$ is a model of f, then $\pi_n(\mathrm{map}(X,Y;f))\otimes\mathbb{Q}\cong H_n(\mathrm{Der}(\wedge V,B;\phi),D)$ [6]. Moreover, $s^{-1}(\mathrm{Der}(\wedge V,B;\phi),D)$ is an L_∞ model of the universal cover of $\mathrm{map}(X,Y;f)$ [3–5].

In [1], Block and Lazarev showed that the chain complex $\mathrm{Der}(\wedge V,B;\phi)$ computes the André-Quillen cohomology $H^*_{AQ}(A;B)$ whenever there is a quasi-isomorphism $(\wedge V,d)\to(A,d)$. Therefore, if $\varphi:(B,d)\to(B,d')$ is a quasi-isomorphism, so is the induced map

$$\varphi_*:(\mathrm{Der}(\wedge V,B;\phi),D)\to(\mathrm{Der}(\wedge V,B';\varphi\circ\phi),D)$$

obtained by post composition with φ.

If $\rho:Y\to Y_{\mathbb{Q}}$ is the rationalization of Y, then $G_*(Y_{\mathbb{Q}},X;\rho\circ f)$ can be computed using Sullivan models. Let $\phi:(\wedge V,d)\to(B,d)$ be the minimal Sullivan model of f. The post composition with the augmentation $\epsilon:(B,d)\to(\mathbb{Q},0)$ yields a map of chain complexes

$$\epsilon_*:(\mathrm{Der}(\wedge V,B;\phi),D)\to(\mathrm{Der}(\wedge V,\mathbb{Q};\epsilon\circ\phi),0)=(V^{\#},0).$$

The generalized evaluation subgroups of $\rho\circ f$ are given by $\mathrm{im}\,H_*(\epsilon_*)$ [11]. In short, given $v\in V^n$, its dual $v^{\#}\in V_n^{\#}$ represents a generalized Gottlieb element in $\pi_n(Y)\otimes\mathbb{Q}$ if there is a ϕ-derivation $\theta\in\mathrm{Der}(\wedge V,B;\phi)$ such that $\theta(v)=1$ and $D\theta=0$. In this case, $H_n(\epsilon_*)([\theta])=v^{\#}$.

We assume that $k\geq 1$. The inclusion $i_{n,k}:\mathbb{C}P^n\to\mathbb{C}P^{n+k}$ is modelled by

$$\phi:(\wedge(y_2,y_{2n+2k+1}),d)\to(\wedge(x_2,x_{2n+1}),d)$$

where $\phi(y_2)=x_2$ and $\phi(y_{2n+2k+1})=x_2^kx_{2n+1}$. The quasi-isomorphism

$$\varphi:(\wedge(x_2,x_{2n+1}),d)\to(\wedge x_2/(x_2^{n+1}),0)=B$$

induces a quasi-isomorphism

$$\varphi_*:\mathrm{Der}(\wedge(y_2,y_{2n+2k+1}),\wedge(x_2,x_{2n+1});\phi)\to\mathrm{Der}(\wedge(y_2,y_{2n+2k+1}),B;\varphi\circ\phi).$$

Theorem 2 *The generalized Gottlieb group $G_*(\wedge(y_2, y_{2n+2k+1}), B; \varphi \circ \phi)$ is isomorphic to $< y_2^\#, y_{2n+2k+1}^\# >\cong \pi_*(\mathbb{C}P^{n+k}) \otimes \mathbb{Q}$.*

Proof. Consider the derivations $\beta_2 = (y_2, 1)$ and $\beta_{2n+2k+1} = (y_{2n+2k+1}, 1)$ in $\mathrm{Der}_*(\wedge(y_2, y_{2n+2k+1}), B; \varphi \circ \phi)$. The derivation $\beta_{2n+2k+1}$ cannot be a boundary for degree reasons. For $k \geq 2$, $\mathrm{Der}_3(\wedge(y_2, y_{2n+2k+1}), B; \varphi\circ\phi) = 0$. Hence, β_2 cannot be a boundary. If $k = 1$, the vector space of derivations of degree 3 is spanned by $\beta_3 = (y_{2n+2k+1}, x_2^n)$, which is a cycle. Hence, β_2 cannot be a boundary. Therefore, β_2 and $\beta_{2n+2k+1}$ represent non-zero cohomology classes in $\mathrm{Der}(\wedge(y_2, y_{2n+2k+1}), B; \psi)$. Moreover,

$$H_*(\epsilon_*)([\beta_2]) = y_2^\# \in \mathrm{Der}(\wedge(y_2, y_{2n+2k+1}), \mathbb{Q}) = V^\#.$$

In the same way, $H_*(\epsilon_*)([\beta_{2n+2k+1}]) = y_{2n+2k+1}^\#$. □

Corollary 1 *The generalized Gottlieb group*

$$G_*(\wedge(y_2, y_{2n+2k+1}), \wedge(x_2, x_{2n+1}); \phi)$$

is isomorphic to $< y_2^\#, y_{2n+2k+1}^\# >$.

Proof. This is a consequence of the above theorem and the quasi-isomorphism

$$\varphi_* : \mathrm{Der}(\wedge(y_2, y_{2n+2k+1}), \wedge(x_2, x_{2n+1}); \phi) \to \mathrm{Der}(\wedge(y_2, y_{2n+2k+1}), B; \varphi \circ \phi).$$

However, we will give a separate proof. Consider derivations α_2 and $\alpha_{2n+2k+1}$ in $\mathrm{Der}_*(\wedge(y_2, y_{2n+2k+1}), \wedge(x_2, x_{2n+1}); \phi)$ defined by $\alpha_2(y_2) = 1$, $\alpha_2(y_{2n+2k+1}) = (n + k + 1)x_2^{k-1}x_{2n+1}$ and $\alpha_{2n+2k+1} = (y_{2n+2k+1}, 1)$. A straightforward computation shows that α_2 and $\alpha_{2n+2k+1}$ are cycles. We show that they cannot be boundaries. The subspace of derivations of degree 3 is spanned by α_3, where $\alpha_3(y_2) = 0$ and $\alpha_3(y_{2n+2k+1}) = x_2^{2n+2k-2}$. As $D\alpha_3 = 0$, then α_2 is not a boundary. Moreover,

$$\mathrm{Der}_i(\wedge(y_2, y_{2n+2k+1}), \wedge(x_2, x_{2n+1}); \phi) = 0 \quad \text{for } i > 2n + 2k + 1.$$

Hence, $\alpha_{2n+2k+1}$ cannot be a boundary as well. As $H_*(\epsilon_*)([\alpha_2]) = y_2^\#$ and $H_*(\epsilon_*)([\alpha_{2n+2k+1}]) = y_{2n+2k+1}^\#$, we conclude that

$$G_*(\wedge(y_2, y_{2n+2k+1}), \wedge(x_2, x_{2n+1}); \phi) =< y_2^\#, y_{2n+2k+1}^\# >= \pi_*(\mathbb{C}P^{n+k}) \otimes \mathbb{Q}.$$

□

Remark 1 *The above result corrects Theorem 2.2 in [12], where it is stated that $G_2(\wedge(y_2, y_{2n+2k+1}), \wedge(x_2, x_{2n+1}); \phi) = 0$.*

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