Armenian Journal of Mathematics
Volume 4, Number 2, 2012, 49–79

On a convergence of the Fourier-Pade approximation

Arnak Poghosyan

Institute of Mathematics,
National Academy of Sciences of Armenia
Bagramian ave. 24b, 0019 Yerevan, Armenia
arnak@instmath.sci.am

Abstract

We consider convergence acceleration of the truncated Fourier series by sequential application of polynomial and rational corrections. Polynomial corrections are performed along the ideas of the Krylov-Lanczos approximation. Rational corrections contain unknown parameters which determination is a crucial problem for realization of the rational approximations. We consider approach connected with the Fourier-Pade approximations. This rational-trigonometric-polynomial approximation we continue calling the Fourier-Pade approximation. We investigate its convergence for smooth functions in different frameworks and derive the exact constants of asymptotic errors. Detailed analysis and comparisons of different rational-trigonometric-polynomial approximations are performed and the convergence properties of the Fourier-Pade approximation are outlined. In particular, fast convergence of the Fourier-Pade approximation is observed in the regions away from the endpoints.

Key Words: Convergence Acceleration, Fourier-Pade Approximation, Rational Approximation, Krylov-Lanczos Approximation
Mathematics Subject Classification 2000: 41A20, 65T40

Introduction

It is well known that approximation of a 2-periodic and smooth function by the truncated Fourier series

$$S_N(f;x) = \sum_{n=-N}^{N} f_n e^{i\pi nx},$$

$$f_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-i\pi nx}dx$$

is highly effective. When the approximated function has a point of discontinuity, approximation by the partial sum $S_N(f)$ is noneffective.

In this paper we consider convergence acceleration of the truncated Fourier series by sequential application of polynomial and rational corrections. Method of polynomial corrections is known as the Krylov-Lanczos approximation (see [1], [3]-[5], [13] with references therein). The approach was suggested by Krylov ([8]) and later (see also [16] and [19]) independently by Lanczos ([9]).

Additional convergence acceleration of the KL-approximation is achieved by application of rational functions (in terms of $e^{i\pi x}$) as corrections of the error along the ideas of the rational approximations ([2], [6], [7]). Rational approximations considered in this paper depend on parameter θ which determination is the principal problem for realization of the accelerating convergence.

Different approaches are known for parameter determination. One approach leads to the L_2-minimal approximation ([12], [15]), the other leads to the limit function minimal approximation ([11]) and the third approach is connected with the roots of the associated Laguerre polynomials ([14]). In this paper we investigate parameter determination according to the main stream which is considered in [2], [6] and [7] and which is known as the Fourier-Pade approximation.

In our approach we have additional acceleration of convergence of the Fourier-Pade approximation due to application of the polynomial corrections. We consider smooth functions and investigate convergence of the resultant rational-trigonometric-polynomial (RTP-) approximation deriving exact constants of the asymptotic errors in different frameworks: L_2-norm, and pointwise convergence in the regions away from the endpoints $x = \pm 1$. Comparison with other RTP-approximations reveals the convergence properties of the Fourier-Pade approximation. In particular, we observe the fast pointwise convergence in the regions away from the endpoints.

The paper is organized as follows:

Section 1 displays the idea of polynomial correction and with some numerical results outlines its properties.

Section 2 describes additional acceleration of convergence by the rational corrections. Different approaches for parameter determination are discussed. Theoretical and numerical results perform comparison of different RTP-approximations with each other and with the Krylov-Lanczos approximation.

Section 3.1 introduces RTP-approximation with parameter θ determined along the ideas of the Fourier-Pade approximation and investigates the pointwise convergence in the regions away from the endpoints. Comparison with other RTP-approximations reveals the fast pointwise convergence of the Fourier-Pade approximation.

Section 3.2 explores the convergence of the Fourier-Pade approximation in the frameworks of L_2 and uniform convergence.

1 The Krylov-Lanczos Approximation

We recap the main ideas of the Krylov-Lanczos approximation from [13]. Let $f \in C^{q-1}[-1,1]$. Denote

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1), \quad k = 0, \dots, q-1.$$

We limit our discussion to functions which are smooth on $[-1,1]$ with discontinuities only at the endpoints of the interval. Throughout the paper we suppose that the exact values of the jumps are known.

The KL-approximation is based on the following representation of the approximated function

$$f(x) = F(x) + \sum_{k=0}^{q-1} A_k(f) B_k(x), \tag{1}$$

where B_k are 2-periodic extensions of the Bernoulli polynomials with the Fourier coefficients

$$B_{k,n} = \begin{cases} 0, & n = 0 \\ \dfrac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, & n \neq 0 \end{cases}$$

and F is a 2-periodic and smooth function on the real line ($F \in C^{q-1}(R)$) with the Fourier coefficients

$$F_n = f_n - \sum_{k=0}^{q-1} A_k(f) B_{k,n}.$$

Approximation of F by the truncated Fourier series leads to the Krylov-Lanczos (KL-) approximation

$$S_{N,q}(f;x) = \sum_{n=-N}^{N} F_n e^{i\pi n x} + \sum_{k=0}^{q-1} A_k(f) B_k(x)$$

with the error

$$R_{N,q}(f;x) = f(x) - S_{N,q}(f;x).$$

Next results are for further comparisons. Theorem 1 describes the asymptotic behavior of $R_{N,q}(f)$ on the segment $[-1,1]$ in the L_2-norm.

Theorem 1. *[13] Suppose $f \in C^q[-1,1]$ and $f^{(q)} \in AC[-1,1]$ for some $q \geq 1$. Then the following estimate holds*

$$\lim_{N\to\infty} N^{q+\frac{1}{2}} \|R_{N,q}(f)\|_{L_2} = |A_q(f)| c(q),$$

where

$$c(q) = \frac{1}{\pi^{q+1}\sqrt{2q+1}}.$$

Table 1 shows the values of $c(q)$.

Theorem 2 describes the pointwise convergence of the KL-approximation in the regions away from the endpoints $x = \pm 1$.

q	1	2	3	4	5	6	7	8
$c(q)$	0.058	0.014	0.0039	0.0011	0.00031	0.000091	0.000027	$8.1 \cdot 10^{-6}$

Table 1: The values of $c(q)$ from Theorem 1.

Theorem 2. *[13] Suppose $f \in C^{q+1}[-1,1]$ and $f^{(q+1)} \in AC[-1,1]$ for some $q \geq 1$. Then the following estimates hold for $|x| < 1$*

$$R_{N,q}(f;x) = A_q(f)\frac{(-1)^{N+\frac{q}{2}}}{2\pi^{q+1}N^{q+1}}\frac{\sin\frac{\pi}{2}(2N+1)x}{\cos\frac{\pi x}{2}} + o(N^{-q-1}),\ N \to \infty$$

for even values of q, and

$$R_{N,q}(f;x) = A_q(f)\frac{(-1)^{N+\frac{q+1}{2}}}{2\pi^{q+1}N^{q+1}}\frac{\cos\frac{\pi}{2}(2N+1)x}{\cos\frac{\pi x}{2}} + o(N^{-q-1}),\ N \to \infty$$

for odd values of q.

Theorem 3 outlines the behavior of the error at the endpoints of the interval in terms of the limit function.

Theorem 3. *[11] Let $f \in C^q[-1,1]$ and $f^{(q)} \in AC[-1,1]$ for some $q \geq 1$. Then the following estimate holds for $h \geq 0$*

$$\lim_{N\to\infty} N^q R_{N,q}\left(f; 1 - \frac{h}{N}\right) = A_q(f)\ell_q(h),$$

where

$$\ell_q(h) = \frac{(-1)^q}{\pi^{q+1}}\int_1^\infty \frac{\sin\left(\pi h x - \frac{\pi q}{2}\right)}{x^{q+1}}dx.$$

The value of $\frac{1}{N^q}\max_{h\geq 0}|\ell_q(h)|$ describes (asymptotically) the uniform error at the endpoints of the interval. Taking into account that according to Theorem 2 the rate of convergence in the regions away from the endpoints is higher than at the endpoints then the value $\frac{1}{N^q}\max_{h\geq 0}|\ell_q(h)|$ is the uniform error of approximation on the entire interval. Table 2 presents the values of $\max_{h\geq 0}|\ell_q(h)|$ for different values of q.

q	1	2	3	4	5	6	7	8
$\max_{h\geq 0}\lvert\ell_q(h)\rvert$	0.10	0.012	0.0034	0.00074	0.00021	0.000053	0.000015	$4.1 \cdot 10^{-6}$

Table 2: The values of $\max_{h\geq 0}|\ell_q(h)|$ from Theorem 3.

Now consider the following simple testing function that we use for further comparisons

$$f(x) = \sin(x-1). \qquad (2)$$

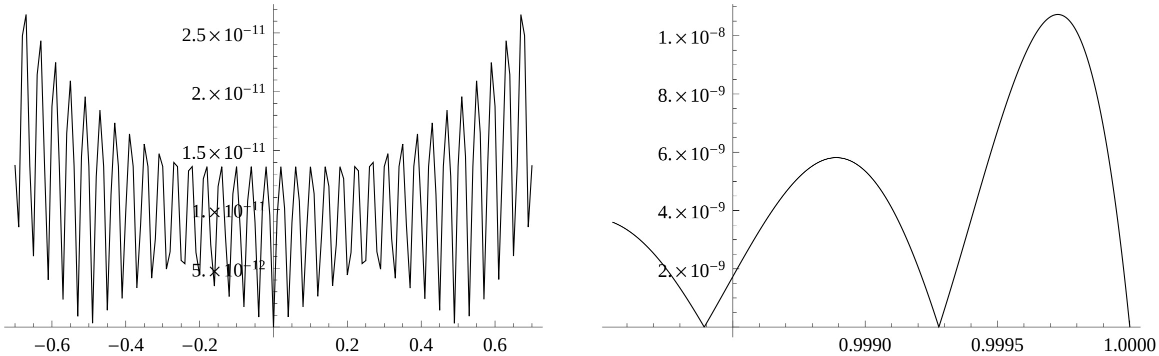

Figure 1: Graphs of $|R_{1024,2}(f;x)|$ on the interval $[-0.7, 0.7]$ (left) and at the point $x = 1$ (right) for the function (2).

Figure 1 shows the behavior of $|R_{N,q}(f;x)|$ on the interval $[-0.7, 0.7]$ (left figure) and at the point $x = 1$ (right figure) for $q = 2$ and $N = 1024$.

Also we calculated the L_2-norm of the error

$$\|R_{1024,2}(f)\|_{L_2} = 3.9 \cdot 10^{-10} \tag{3}$$

and the uniform error

$$\max_{x\in[-1,1]} |R_{1024,2}(f;x)| = 1.1 \cdot 10^{-8}. \tag{4}$$

Let us show how the actual errors in equations (3) and (4) coincide with the theoretical estimates of Tables 1 and 2. According to Theorem 1 and Table 1

$$\|R_{1024,2}(f)\|_{L_2} \approx |A_2(f)| \frac{c(2)}{1024^{2.5}} = 3.79 \cdot 10^{-10}$$

which almost equals to the value of (3) and according to Theorem 3 and Table 2

$$\max_{x\in[-1,1]} |R_{1024,2}(f;x)| \approx |A_2(f)| \frac{\max_h |\ell_2(h)|}{1024^2} = 1.04 \cdot 10^{-8}$$

which coincides with (4).

2 Rational-Trigonometric-Polynomial Approximations

Now we introduce the idea of rational corrections and recap the main ideas from [12], [14], and [15].

Consider a finite sequence of complex numbers $\theta = \{\theta_k\}_{|k|=1}^{p}$, $p \geq 1$ and by $\Delta_n^k(\theta, F_n)$ denote generalized finite differences

$$\Delta_n^0(\theta, F_n) = F_n,\ \Delta_n^k(\theta, F_n) = \Delta_n^{k-1}(\theta, F_n) + \theta_{k\,sgn(n)} \Delta_{(|n|-1)sgn(n)}^{k-1}(\theta, F_n),\ k \geq 1,$$

where $sgn(n) = 1$ if $n \geq 0$ and $sgn(n) = -1$ if $n < 0$. By $\Delta_n^k(F_n)$ we denote the classical finite differences which correspond to generalized differences $\Delta_n^k(\theta, F_n)$ with $\theta \equiv 1$.

We have

$$R_{N,q}(f) = R_N^+(F) + R_N^-(F),$$

where

$$R_N^+(F) = \sum_{n=N+1}^{\infty} F_n e^{i\pi nx}, \quad R_N^-(F) = \sum_{n=-\infty}^{-N-1} F_n e^{i\pi nx}.$$

The Abel transformation implies

$$R_N^+(F) = -\frac{\theta_1 F_N\, e^{i\pi(N+1)x}}{1+\theta_1 e^{i\pi x}} + \frac{1}{1+\theta_1 e^{i\pi x}} \sum_{n=N+1}^{\infty} \Delta_n^1(\theta, F_n) e^{i\pi nx}.$$

Reiteration of it up to p times leads to the following expansion

$$\begin{aligned} R_N^+(F) &= -e^{i\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_k \Delta_N^{k-1}(\theta, F_n)}{\prod_{s=1}^{k}(1+\theta_s e^{i\pi x})} \\ &+ \frac{1}{\prod_{k=1}^{p}(1+\theta_k e^{i\pi x})} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, F_n) e^{i\pi nx}, \end{aligned}$$

where the first term can be viewed as correction of the error and the last term is real error. Similar expansion for $R_N^-(F)$ reduces to the following rational-trigonometric-polynomial approximation

$$\begin{aligned} S_{N,q,p}(f;x) &= \sum_{k=0}^{q-1} A_k(f) B_k(x) + \sum_{n=-N}^{N} F_n e^{i\pi nx} \\ &- e^{i\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_k \Delta_N^{k-1}(\theta, F_n)}{\prod_{s=1}^{k}(1+\theta_s e^{i\pi x})} \\ &- e^{-i\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_{-k} \Delta_{-N}^{k-1}(\theta, F_n)}{\prod_{s=1}^{k}(1+\theta_{-s} e^{-i\pi x})} \end{aligned} \tag{5}$$

with the error

$$R_{N,q,p}(f;x) = f(x) - S_{N,q,p}(f;x) = R_{N,q,p}^+(f;x) + R_{N,q,p}^-(f;x),$$

where

$$R_{N,q,p}^{\pm}(f;x) = \frac{1}{\prod_{k=1}^{p}(1+\theta_{\pm k} e^{\pm i\pi x})} \sum_{n=N+1}^{\infty} \Delta_{\pm n}^p(\theta, F_n) e^{\pm i\pi nx}. \tag{6}$$

Approximation (5) is undetermined until the values of θ_k are undefined. Determination of parameter θ is crucial for realization of the RTP-approximation and different choices have been investigated in different frameworks. One approach is investigated for smooth functions in a series of papers ([11], [12], [14], and [15]) where the following determination of parameters θ_k is considered

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N}, \ k = 1, \cdots, p, \tag{7}$$

where τ_k are independent of N. Determination of new parameters τ_k will be discussed below. Now we present different theorems which outline the behavior of the RTP-approximation in different frameworks for such choice of parameters θ_k. Theorem 4 shows behavior of the RTP-approximation in the L_2-norm. Let

$$\prod_{k=1}^{p}(1+\tau_k x) = \sum_{k=0}^{p} \gamma_k(\tau) x^k. \tag{8}$$

Theorem 4. *[12] Suppose $f \in C^{q+p}[-1,1]$ and $f^{(q+p)} \in AC[-1,1]$ for some $q, p \geq 1$. If*

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N},\ k = 1, \cdots, p,\ \tau_k > 0,\ \tau_j \neq \tau_i,\ j \neq i;$$

then the following estimate holds

$$\lim_{N\to\infty} N^{q+\frac{1}{2}} ||R_{N,q,p}(f)||_{L_2} = |A_q(f)| c_p(q),$$

where

$$c_p(q) = \frac{1}{\pi^{q+1}} \left(\int_1^{\infty} |\phi_{p,q}(t)|^2\, dt \right)^{1/2},$$

and

$$\begin{aligned} \phi_{p,q}(t) &= \frac{(-1)^p}{t^{q+1}} - \frac{1}{q!} \sum_{j=1}^{p} \frac{e^{-\tau_j(t-1)}}{\prod_{\substack{i=1\\ i\neq j}}^{p}(\tau_i - \tau_j)} \sum_{k=0}^{p} \gamma_k(\tau)(-1)^{k+1} \\ &\times \sum_{m=0}^{p-k-1} (q+p-k-m-1)! \tau_j^m. \end{aligned}$$

Here $\gamma_k(\tau)$ are defined by (8).

Next theorem outlines the behavior of the RTP-approximation in the regions away from the endpoints for choice (7).

Theorem 5. *[15] Let $f \in C^{q+p+1}[-1,1]$ and $f^{(q+p+1)} \in AC[-1,1]$. If parameters θ_k are chosen as in (7) then the following estimates hold for $|x| < 1$*

$$\begin{aligned} R_{N,q,p}(f;x) &= A_q(f) \frac{(-1)^{N+p+\frac{q}{2}}}{2^{p+1}\pi^{q+1} q! N^{q+p+1}} \frac{\sin \frac{\pi x}{2}(2N-p+1)}{\cos^{p+1}\frac{\pi x}{2}} \sum_{k=0}^{p} (-1)^k (p-k+q)! \gamma_k(\tau) \\ &+ o(N^{-q-p-1}),\ N \to \infty, \end{aligned}$$

for even values of q and

$$\begin{aligned} R_{N,q,p}(f;x) &= A_q(f) \frac{(-1)^{N+p+\frac{q+1}{2}}}{2^{p+1}\pi^{q+1} q! N^{q+p+1}} \frac{\cos \frac{\pi x}{2}(2N-p+1)}{\cos^{p+1}\frac{\pi x}{2}} \sum_{k=0}^{p} (-1)^k (p-k+q)! \gamma_k(\tau) \\ &+ o(N^{-q-p-1}),\ N \to \infty, \end{aligned}$$

for odd values of q. Here $\gamma_k(\tau)$ are defined by (8).

Finally, Theorem 6 describes the behavior of the RTP-approximation at the endpoints of the interval.

Theorem 6. *[11] Let $f \in C^{q+p}[-1,1]$ and $f^{(q+p)} \in AC[-1,1]$ for some $p, q \geq 1$. If parameters θ_k are chosen as in (7) then the following estimate holds for $h \geq 0$*

$$\lim_{N\to\infty} N^q R_{N,q,p}\left(f; 1-\frac{h}{N}\right) = A_q(f)\ell_{q,p}(h)$$

where

$$\begin{aligned}\ell_{q,p}(h) &= \ell_q(h) + \frac{(-1)^{p+1}}{q!\,\pi^{q+1}} \\ &\times \mathbf{Re}\left(\frac{e^{-i\pi h}}{i^{q+1}\prod_{m=1}^{p}(\tau_m + i\pi h)} \sum_{k=0}^{p}\gamma_k(\tau)\sum_{s=0}^{p-k-1}(i\pi h)^s(-1)^{k+s}(q+p-k-s-1)!\right),\end{aligned}$$

and $\ell_q(h)$ is defined in Theorem 3. Here $\gamma_k(\tau)$ are defined by (8).

Theorems 4-6 are valid nonetheless parameters τ_k are still undefined but an important convergence property of the RTP-approximations can be seen from comparison of Theorems 2 and 5 without determination of parameters τ_k: in the regions away from the endpoints the RTP-approximations are more accurate than the KL-approximation and improvement in accuracy is $O(N^p)$ as $N \to \infty$.

Parameters τ_k can be determined differently and Theorems 4-6 will outline approaches towards determination of their values.

2.1 L_2-minimal and LF-minimal RTP-approximations

Estimate of Theorem 4 leads to the L_2-minimal RTP-approximations which are investigated in [12]. This approach is connected with appropriate choice of parameters τ_k which minimize the constant $c_p(q)$ and consequently the asymptotic L_2-error

$$\lim_{N\to\infty} N^{q+\frac{1}{2}}\|R_{N,q,p}(f)\|_{L_2} \to \min.$$

Tables 3 and 4 present the optimal values of τ_k, $k = 1, \cdots, p$ that realize the L_2-minimal RTP-approximation. We show also the values of $c_p(q)$ and the ratio $c(q)/c_p(q)$ that displays the efficiency of the L_2-minimal RTP-approximation compared to the KL-approximation. Results for $1 \leq p \leq 4$ can be found in [12].

The tables show that the L_2-minimal RTP-approximation is much more accurate by the L_2-norm (however asymptotically) compared to the KL-approximation. As larger are the values of parameters p and q as more precise is the RTP-approximation.

Table 5 displays the values of $\max_{h\geq 0}|\ell_{q,p}(h)|$. The values $\frac{1}{N^q}\max_{h\geq 0}|\ell_{q,p}(h)|$ coincide with the uniform errors (however asymptotically). We see that by uniform norm L_2-minimal RTP-approximation is also rather accurate compared to the KL-approximation.

q	1	2	3	4	5	6
$c_1(q)$	0.010	0.0016	0.00031	0.000070	0.000017	$4.2 \cdot 10^{-6}$
$c(q)/c_1(q)$	5.7	9.0	12.2	15.5	18.7	22.0
τ_1	1.3533	2.3194	3.3021	4.2916	5.2846	6.2796
$c_2(q)$	0.0028	0.00031	0.000047	$8.5 \cdot 10^{-6}$	$1.7 \cdot 10^{-6}$	$3.7 \cdot 10^{-7}$
$c(q)/c_2(q)$	21.0	46.7	82.3	128.1	183.8	249.7
τ_1	2.7595	4.0837	5.3580	6.6001	7.8190	9.0202
τ_2	0.5320	1.1360	1.8177	2.5460	3.3060	4.0890
$c_3(q)$	0.00095	0.000078	$9.4 \cdot 10^{-6}$	$1.4 \cdot 10^{-6}$	$2.4 \cdot 10^{-7}$	$4.6 \cdot 10^{-8}$
$c(q)/c_3(q)$	61.3	185.1	411.6	771.8	1296.7	2017.4
τ_1	0.2511	0.6382	1.1230	1.6731	2.2700	2.9023
τ_2	1.2855	2.2363	3.2067	4.1869	5.1725	6.1617
τ_3	4.2225	5.7813	7.2573	8.6782	10.0589	11.4090
$c_4(q)$	0.00037	0.000022	$2.3 \cdot 10^{-6}$	$2.9 \cdot 10^{-7}$	$4.3 \cdot 10^{-8}$	$7.0 \cdot 10^{-9}$
$c(q)/c_4(q)$	156.5	621.3	1704.4	3794.9	7377.9	13034.6
τ_1	0.6663	0.3861	0.7379	1.1602	1.6358	2.1534
τ_2	0.1305	1.3459	2.0908	2.8748	3.6852	4.5147
τ_3	2.2056	3.4131	4.5976	5.7649	6.9188	8.0620
τ_4	5.7355	7.4661	9.0951	10.6547	12.1630	13.6315
$c_5(q)$	0.00016	$7.8 \cdot 10^{-6}$	$6.3 \cdot 10^{-7}$	$6.8 \cdot 10^{-8}$	$8.7 \cdot 10^{-9}$	$1.3 \cdot 10^{-9}$
$c(q)/c_5(q)$	363.2	1852.0	6166.3	16110.9	35915.4	71530.5
τ_1	0.0727	0.2459	3.1216	4.0912	14.1902	6.0515
τ_2	0.3709	0.8560	10.9048	12.5769	8.6157	3.4312
τ_3	3.2540	4.6573	0.5057	2.0598	2.7304	1.6450
τ_4	7.2925	9.1544	1.4294	0.8341	1.2173	15.7573
τ_5	1.2353	2.1655	6.0097	7.3263	5.0688	9.8837

Table 3: Numerical values of $c_p(q)$ and $c(q)/c_p(q)$ with optimal values of τ_k that correspond to the L_2-minimal RTP-approximation.

Then, in the regions away from the endpoints the L_2-minimal RTP-approximation is $O(N^{-q-p-1})$ and we have improvement in convergence by factor $O(N^p)$ compared to the KL-approximation (compare also Figures 1 and 2).

Figure 2 confirms these observations for testing function (2). The left figure presents the behavior of the error on the interval $[-0.7, 0.7]$ and the right one at the point $x = 1$.

Also, we derived the following values for the L_2 and uniform norms

$$\|R_{1024,2,2}(f)\|_{L_2} = 8.4 \cdot 10^{-12}, \tag{9}$$

$$\max_{x\in[-1,1]} |R_{1024,2,2}(f;x)| = 4.3 \cdot 10^{-10}. \tag{10}$$

Both values are smaller than their counterparts (3) and (4).

Let us show, as above for the KL-approximation, that (9) and (10) almost coincide with

q	1	2	3	4	5	6
$c_6(q)$	0.000074	$2.9 \cdot 10^{-6}$	$5.8 \cdot 10^{-7}$	$1.8 \cdot 10^{-8}$	$2.0 \cdot 10^{-9}$	$2.7 \cdot 10^{-10}$
$c(q)/c_6(q)$	785.7	5047.3	6680.3	61110.1	155008.3	345876.7
τ_1	0.2172	10.8514	1.3972	14.4673	10.2897	11.6629
τ_2	0.7246	5.9586	50.0000	8.8870	16.1694	17.8206
τ_3	8.8848	1.4319	3.0504	3.0073	6.4538	4.6873
τ_4	4.4008	0.1627	0.4943	0.6160	2.0768	1.2839
τ_5	0.0426	3.0843	10.6365	5.3377	3.8383	2.6717
τ_6	1.9466	0.5661	5.8693	1.5192	0.9276	7.5637
$c_7(q)$	0.000036	$1.1 \cdot 10^{-6}$	$2.9 \cdot 10^{-7}$	$5.1 \cdot 10^{-9}$	$5.2 \cdot 10^{-10}$	$6.1 \cdot 10^{-11}$
$c(q)/c_7(q)$	1607.5	12813.5	13276.0	211823.9	607353.7	$2 \cdot 10^6$
τ_1	1.1979	12.5591	12.6330	10.4540	11.9533	13.4172
τ_2	0.4419	7.3084	15.9240	16.3382	18.1168	19.8412
τ_3	5.6258	0.9763	2.4915	2.2642	4.9757	3.7070
τ_4	0.0259	0.3858	1.1504	0.4646	0.7201	2.1181
τ_5	0.1323	4.0881	8.0502	6.6165	2.9717	9.0691
τ_6	2.7805	0.1109	0.4081	1.1452	1.6109	1.0190
τ_7	10.5071	2.1082	4.6940	4.0073	7.8504	5.9512
$c_8(q)$	0.000034	$1.0 \cdot 10^{-6}$	$2.3 \cdot 10^{-8}$	$1.6 \cdot 10^{-9}$	$1.4 \cdot 10^{-10}$	$1.5 \cdot 10^{-11}$
$c(q)/c_8(q)$	1699.1	14035.0	170322.7	681710.0	$2.2 \cdot 10^6$	$6.1 \cdot 10^6$
τ_1	5.4076	11.2354	6.5631	18.1970	6.1456	4.7688
τ_2	10.0609	6.7837	16.2819	7.9265	0.5676	0.8199
τ_3	0.0250	0.9453	0.5408	1.7362	13.6136	10.5758
τ_4	0.1276	0.3754	2.2544	3.0701	9.2623	15.1568
τ_5	0.4259	3.8836	3.9676	5.0576	20.0424	21.8313
τ_6	1.1544	0.1096	0.1915	0.3565	3.9105	7.2311
τ_7	2.6774	2.0321	1.1780	0.8786	1.2693	1.7033
τ_8	32.5859	19.2307	10.3970	12.0302	2.3396	2.9776

Table 4: Numerical values of $c_p(q)$ and $c(q)/c_p(q)$ with optimal values of τ_k that correspond to the L_2-minimal RTP-approximation.

the theoretical estimates of Tables 3, 4 and 5. From Theorem 4 and Table 3 we have

$$\|R_{1024,2,2}(f)\|_{L_2} \approx |A_2(f)|\frac{c_2(2)}{1024^{2.5}} = 8.13 \cdot 10^{-12}$$

and according to Theorem 6 and Table 5 we calculate

$$\max_{x\in[-1,1]} |R_{1024,2,2}(f;x)| \approx |A_2(f)|\frac{\max_h |\ell_{2,2}(h)|}{1024^2} = 4.34 \cdot 10^{-10}.$$

Both results coincide with (9) and (10) rather precisely.

One thing that worth noting is that all theorems concerning the RTP-approximations put additional smoothness requirements on the approximated functions so it is supposed that parameters p and q are chosen such (for infinitely differentiable functions no matter how they are chosen) that all theorems are valid and thus comparisons are reasonable.

$p\backslash q$	q=1	q=2	q=3	q=4	q=5	q=6
$p=1$	0.026	0.0018	0.00031	0.000054	0.000011	$2.5 \cdot 10^{-6}$
$p=2$	0.012	0.00050	0.000074	$9.0 \cdot 10^{-6}$	$1.8 \cdot 10^{-6}$	$3.1 \cdot 10^{-7}$
$p=3$	0.0052	0.00015	0.000014	$1.6 \cdot 10^{-6}$	$2.2 \cdot 10^{-7}$	$3.6 \cdot 10^{-8}$
$p=4$	0.0029	0.000030	$5.0 \cdot 10^{-6}$	$4.0 \cdot 10^{-7}$	$5.7 \cdot 10^{-8}$	$7.1 \cdot 10^{-9}$
$p=5$	0.0015	0.000013	$1.4 \cdot 10^{-6}$	$1.0 \cdot 10^{-7}$	$1.0 \cdot 10^{-8}$	$1.3 \cdot 10^{-9}$
$p=6$	0.00094	$5.5 \cdot 10^{-6}$	$1.3 \cdot 10^{-6}$	$2.1 \cdot 10^{-8}$	$3.3 \cdot 10^{-9}$	$3.1 \cdot 10^{-10}$
$p=7$	0.00055	$1.4 \cdot 10^{-6}$	$7.1 \cdot 10^{-7}$	$7.4 \cdot 10^{-9}$	$7.7 \cdot 10^{-10}$	$7.4 \cdot 10^{-11}$
$p=8$	0.00053	$1.5 \cdot 10^{-6}$	$8.4 \cdot 10^{-8}$	$2.4 \cdot 10^{-9}$	$2.8 \cdot 10^{-10}$	$2.1 \cdot 10^{-11}$

Table 5: The values of $\max_{h\geq 0} |\ell_{q,p}(h)|$ with parameters τ_k corresponding to the L_2-minimal RTP-approximation.

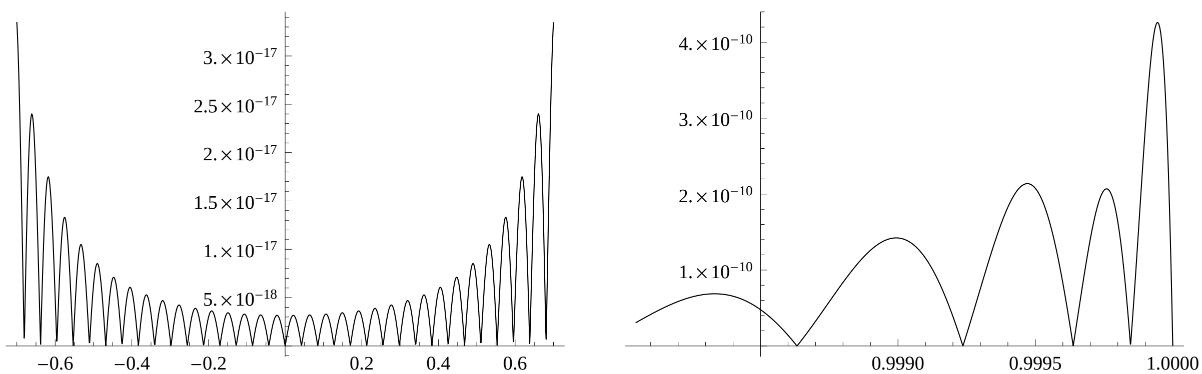

Figure 2: Graphs of $|R_{1024,2,2}(f;x)|$ on the interval $[-0.7, 0.7]$ (left) and at the point $x=1$ (right) that correspond to the L_2-minimal RTP-approximation.

Paper [15] performs another approach for comparison of the KL- and the L_2-minimal RTP-approximations. Suppose that approximated function has finite smoothness $f \in C^{M+1}[-1,1]$ and $f^{(M+1)} \in AC[-1,1]$ and all available jumps $A_k(f)$, $k = 0, \cdots, M+1$ are exactly known. If parameters p and q are chosen such that $p+q=M$ then Theorem 4 is valid for all values of p for $p = 0, \ldots, M-q$ and comparison of different RTP-approximations is valid. Note that $p=0$ coincides with the KL-approximation. The paper concluded that not always utilization of all available jumps by the KL-approximation leads to the best approximation. When the values of jumps are rapidly increasing then better accuracy can be achieved by utilization of smaller number of jumps (consequently with smaller $A_q(f)$ in the estimates of the corresponding theorems) and appropriately chosen rational corrections based on the smoothness of the approximated function. Comparisons showed that for functions with rapidly increasing jumps the choice $q=1$ and $p=M-1$ gave much more accuracy than the choice $p=0$ and $q=M$ which coincided with the KL-approximation. In practice, which choice of q and p is the best in such cases can be concluded from comparison of the corresponding estimates. Similar investigations can be carried out for different

RTP-approximations in different frameworks.

Estimate of Theorem 6 leads to another approach of parameter determination and hence to another type of RTP-approximations. In particular, parameters τ_k minimize the values of $\max_h |\ell_{q,p}(h)|$ getting more accurate RTP-approximations in the framework of the uniform norm. Such approximations are considered in [11] and are known as the limit function minimal (LF-minimal) RTP-approximations. We are not giving detailed analysis of this approximations as, in general, they mimic the behavior of the L_2-minimal RTP-approximations.

2.2 RTP-approximations by the roots of the Laguerre polynomials

Another approach for determination of parameters τ_k is connected with Theorem 5. Efforts towards minimization of the pointwise error in the regions away from the endpoints drive us to the RTP-approximations where parameters τ_k are the roots of the associated Laguerre polynomials $L_p^q(x)$

$$L_p^q(\tau_k) = 0, \ \ k = 1, \ldots, p.$$

It is well-known that the roots of the associated Laguerre polynomials are distinct and positive. Associated Laguerre polynomials have the well-known representation

$$L_p^q(x) = \sum_{k=0}^{p} (-1)^k \frac{(p+q)!}{k!(p-k)!(q+k)!} x^k. \tag{11}$$

This representation allows calculation of parameters τ_k explicitly for $p = 1, 2, 3$. For $p = 1$ we have

$$L_1^q(x) = -x + (q+1), \ \ \tau_1 = q + 1.$$

For $p = 2$

$$L_2^q(x) = \frac{x^2}{2} - x(q+2) + \frac{(q+3)(q+2)}{2}$$

and hence

$$\tau_1 = 2 + q + \sqrt{2+q}, \ \ \tau_2 = 2 + q - \sqrt{2+q}.$$

For $p = 3$

$$L_3^q(x) = -\frac{x^3}{6} + x^2 \frac{q+3}{2} - x\frac{(q+3)(q+2)}{2} + \frac{(q+3)(q+2)(q+1)}{6}$$

and by the Cardano formula, after some manipulations, we obtain

$$\tau_1 = 3 + q + 2\sqrt{3+q} \cos\left(\frac{1}{3} arctg \sqrt{q+2})\right),$$

$$\tau_2 = 3 + q - 2\sqrt{3+q} \cos\left(\frac{1}{3} arctg \sqrt{q+2})\right),$$

and

$$\tau_3 = 3 + q - \sqrt{q+3}\left(\cos\left(\frac{1}{3} arctg \sqrt{q+2}\right) + \sqrt{3} \sin\left(\frac{1}{3} arctg \sqrt{q+2}\right)\right).$$

	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$	$q=6$	$q=7$
τ_1	0.7433	1.2268	1.7555	2.3192	2.9108	3.5256	4.1599
τ_2	2.5716	3.4125	4.2656	5.1287	6.0000	6.8783	7.7626
τ_3	5.7312	6.9027	8.0579	9.2009	10.3341	11.4594	12.5780
τ_4	10.9539	12.4580	13.9209	15.3513	16.7551	18.1368	19.4996

Table 6: The roots of the associated Laguerre polynomial $L_4^q(x)$ for different values of q.

For other values of p the values of τ_k can be calculated numerically with any required precision. Table 6 shows that values for $p=4$.

Rtp-approximations by the roots of the associated Laguerre polynomials was investigated in [14]. We recap the main results of this paper. Next theorems present the pointwise convergence in the regions away from the endpoints. Theorem 7 investigates even values of p and Theorem 8 odd values.

Theorem 7. *[14] Let p be even, $f \in C^{q+p+\frac{p}{2}+1}[-1,1]$ and $f^{(q+p+\frac{p}{2}+1)} \in AC[-1,1]$ for some $q,p \geq 1$. If parameter θ is chosen as in (7) where τ_k are the roots of the associated Laguerre polynomial $L_p^q(x)$ then the following estimates hold for $|x|<1$*

$$\begin{aligned} R_{N,q,p}(f;x) &= A_q(f)\frac{(-1)^{N+\frac{q}{2}}}{2^{p+1}\pi^{q+1}N^{q+p+\frac{p}{2}+1}}\frac{\sin\frac{\pi x}{2}(2N-p+1)}{\cos^{p+1}\frac{\pi x}{2}}\delta_{p,q}\left(0,\frac{p}{2},\frac{p}{2}\right) \\ &+ o(N^{-q-p-\frac{p}{2}-1}),\ N\to\infty, \end{aligned}$$

for even values of q, and

$$\begin{aligned} R_{N,q,p}(f;x) &= A_q(f)\frac{(-1)^{N+\frac{q+1}{2}}}{2^{p+1}\pi^{q+1}N^{q+p+\frac{p}{2}+1}}\frac{\cos\frac{\pi x}{2}(2N-p+1)}{\cos^{p+1}\frac{\pi x}{2}}\delta_{p,q}\left(0,\frac{p}{2},\frac{p}{2}\right) \\ &+ o(N^{-q-p-\frac{p}{2}-1}),\ N\to\infty, \end{aligned}$$

for odd values of q, where

$$\delta_{p,q}(w,s,t) = \sum_{k=0}^{p}\gamma_k(\tau)\binom{t+p-k+q}{p-k+s}\alpha_{k,s+p-k}(w)$$

and $\alpha_{k,s}(w)$ is defined by (24).

Theorem 8. *[14] Let p be odd, $f \in C^{q+p+\frac{p+1}{2}+1}[-1,1]$ and $f^{(q+p+\frac{p+1}{2}+1)} \in AC[-1,1]$ for some $q,p \geq 1$. If parameter θ is chosen as in (7) where τ_k are the roots of the associated*

Laguerre polynomial $L_p^q(x)$ then the following estimates hold for $|x| < 1$

$$\begin{aligned}
R_{N,q,p}(f;x) &= A_q(f)\frac{(-1)^{N+\frac{q}{2}}}{\pi^{q+1}}\frac{\sin\frac{\pi x}{2}(2N-p+1)}{2^{p+1}\cos^{p+1}\frac{\pi x}{2}}\delta_{p,q}\left(0,\frac{p+1}{2},\frac{p+1}{2}\right)\\
&+ A_{q+1}(f)\frac{(-1)^{N+\frac{q}{2}+1}}{\pi^{q+2}}\frac{\cos\frac{\pi x}{2}(2N-p+1)}{2^{p+1}\cos^{p+1}\frac{\pi x}{2}}\delta_{p,q}\left(0,\frac{p-1}{2},\frac{p+1}{2}\right)\\
&+ A_q(f)\frac{(-1)^{N+\frac{q}{2}}}{\pi^{q+1}}\frac{\sin\frac{\pi x}{2}(2N-p)}{2^{p+2}\cos^{p+2}\frac{\pi x}{2}}\delta_{p,q}\left(1,\frac{p+1}{2},\frac{p+1}{2}\right)\\
&+ o(N^{-p-q-\frac{p+1}{2}-1}),\ N\to\infty
\end{aligned}$$

for even values of q, and

$$\begin{aligned}
R_{N,q,p}(f;x) &= A_q(f)\frac{(-1)^{N+\frac{q+1}{2}}}{\pi^{q+1}}\frac{\cos\frac{\pi x}{2}(2N-p+1)}{2^{p+1}\cos^{p+1}\frac{\pi x}{2}}\delta_{p,q}\left(0,\frac{p+1}{2},\frac{p+1}{2}\right)\\
&+ A_{q+1}(f)\frac{(-1)^{N+\frac{q+1}{2}}}{\pi^{q+2}}\frac{\sin\frac{\pi x}{2}(2N-p+1)}{2^{p+1}\cos^{p+1}\frac{\pi x}{2}}\delta_{p,q}\left(0,\frac{p-1}{2},\frac{p+1}{2}\right)\\
&+ A_q(f)\frac{(-1)^{N+\frac{q+1}{2}}}{\pi^{q+1}}\frac{\cos\frac{\pi x}{2}(2N-p)}{2^{p+2}\cos^{p+2}\frac{\pi x}{2}}\delta_{p,q}\left(1,\frac{p+1}{2},\frac{p+1}{2}\right)\\
&+ o(N^{-p-q-\frac{p+1}{2}-1}),\ N\to\infty
\end{aligned}$$

for odd values of q.

Comparison with Theorem 5 shows that RTP-approximation by the roots of the associated Laguerre polynomial has extra accuracy by factors $O(N^{\frac{p}{2}})$ and $O(N^{\frac{p+1}{2}})$ for even and odd values of p, respectively, in the regions away from the endpoints compared to the L_2-minimal RTP-approximation. Compared to the KL-approximation it has improvement in accuracy by factors $O(N^{p+\frac{p}{2}})$ and $O(N^{p+\frac{p+1}{2}})$.

Let us compare approximations by the L_2 and uniform norms.

$p\backslash q$	q=1	q=2	q=3	q=4	q=5	q=6
$p=1$	0.017	0.0029	0.00058	0.00058	0.00013	0.000032
$p=2$	0.0092	0.0012	0.00019	0.000036	$7.5\cdot10^{-6}$	$1.7\cdot10^{-6}$
$p=3$	0.0059	0.00061	0.000084	0.000014	$2.5\cdot10^{-6}$	$4.9\cdot10^{-7}$
$p=4$	0.0042	0.00037	0.000043	$6.2\cdot10^{-6}$	$1.0\cdot10^{-6}$	$1.8\cdot10^{-7}$
$p=5$	0.0032	0.00024	0.000025	$3.2\cdot10^{-6}$	$4.7\cdot10^{-7}$	$7.6\cdot10^{-8}$
$p=6$	0.0025	0.00017	0.000016	$1.8\cdot10^{-6}$	$2.4\cdot10^{-7}$	$3.5\cdot10^{-8}$

Table 7: The values of $c_p(q)$ corresponding to the RTP-approximations by the roots of the associated Laguerre polynomials.

Table 7 displays the values of $c_p(q)$ and Table 8 the values of $\max\limits_{h\geq 0}|\ell_{q,p}(h)|$ in this case. Comparison with the results of Tables 3, 4 and 5 shows worse L_2 and uniform convergence of

$p\backslash q$	q=1	q=2	q=3	q=4	q=5	q=6
$p=1$	0.051	0.0039	0.00086	0.00014	0.000035	$7.3\cdot10^{-6}$
$p=2$	0.034	0.0019	0.00034	0.000047	$9.9\cdot10^{-6}$	$1.8\cdot10^{-6}$
$p=3$	0.025	0.0012	0.00017	0.000020	$3.7\cdot10^{-6}$	$6.0\cdot10^{-7}$
$p=4$	0.020	0.00077	0.000098	$9.9\cdot10^{-6}$	$1.7\cdot10^{-6}$	$2.4\cdot10^{-7}$
$p=5$	0.017	0.00055	0.000061	$5.5\cdot10^{-6}$	$8.3\cdot10^{-7}$	$1.1\cdot10^{-7}$
$p=6$	0.014	0.00041	0.000041	$3.3\cdot10^{-6}$	$4.5\cdot10^{-7}$	$5.4\cdot10^{-8}$

Table 8: The values of $\max_{h\geq0}|\ell_{q,p}(h)|$ corresponding to the RTP-approximations by the roots of the associated Laguerre polynomials.

the RTP-approximation by the Laguerre polynomials compared to the L_2-minimal approximations.

Figure 3 shows the behavior of the error of the RTP-approximation $|R_{1024,2,2}(f;x)|$ by the roots of the associated Laguerre polynomials in the regions away from the endpoints and at the point $x=1$. We see the higher accuracy inside the interval of approximation and lower accuracy at the endpoint compared to the L_2-minimal RTP-approximation $|R_{1024,2,2}(f;x)|$.

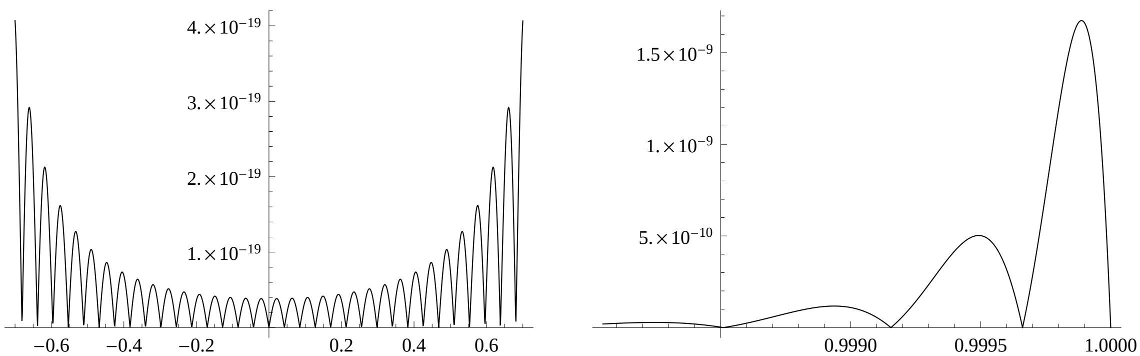

Figure 3: Graphs of $|R_{N,q,p}(f;x)|$ on the interval $[-0.7,0.7]$ (left) and at the point $x=1$ (right) for $q=2$, $p=2$ and $N=1024$ that correspond to the RTP-approximation by the roots of the associated Laguerre polynomial.

We also calculated L_2 and uniform norms of the errors

$$\|R_{1024,2}(f)\|_{L_2}=3.2\cdot10^{-11}, \tag{12}$$

$$\max_{x\in[-1,1]}|R_{1024,2}(f)|=1.7\cdot10^{-9} \tag{13}$$

which confirm the above statements. Low accuracy by L_2 and uniform norms is connected with low accuracy of the RTP-approximation by the roots of the associated Laguerre polynomials at the endpoints of the interval.

3 The Fourier-Pade approximations

Now we consider RTP-approximations where parameters θ_k are determined along the ideas of the Fourier-Pade approximation. More specifically we consider the following system

$$\Delta_n^p(\theta, F_n) = 0,\ n = N, N-1, \ldots, N-p+1 \tag{14}$$

for determination of θ_k, $k = 1, \ldots, p$, and

$$\Delta_n^p(\theta, F_n) = 0,\ n = -N, -N+1, \ldots, -N+p-1 \tag{15}$$

for determination of θ_{-k}, $k = 1, \ldots, p$.

By $\gamma_k^+(\theta)$ and $\gamma_k^-(\theta)$ we denote the coefficients of the polynomials

$$\prod_{k=1}^{p}(1+\theta_k x) = \sum_{k=0}^{p} \gamma_k^+(\theta) x^k. \tag{16}$$

and

$$\prod_{k=1}^{p}(1+\theta_{-k} x) = \sum_{k=0}^{p} \gamma_k^-(\theta) x^k, \tag{17}$$

respectively. It is well known (see [2]) that knowledge of $\gamma_k^{\pm}(\theta)$ is sufficient for construction of the Fourier-Pade approximation $S_{N,q,p}(f)$. We rewrite (14) and (15) in the form

$$\Delta_n^p(\theta, F_n) = F_{n-k} + \sum_{s=1}^{p} \gamma_s^+(\theta) F_{n-k-s} = 0,\ n = N, N-1, \ldots, N-p+1 \tag{18}$$

and

$$\Delta_n^p(\theta, F_n) = F_{n+k} + \sum_{s=1}^{p} \gamma_s^-(\theta) F_{n+k+s},\ n = -N, -N+1, \ldots, -N+p-1 \tag{19}$$

which give us systems of linear equations for determination of γ_k^+ and γ_k^-, respectively.

Such RTP-approximations we continue calling as the Fourier-Pade approximations. We investigate convergence of the Fourier-Pade approximation in different frameworks, derive exact constants of the asymptotic errors and perform comparisons with other RTP-approximations.

In subsection 3.1 we explore pointwise convergence of the Fourier-Pade approximation in the regions away from the endpoints and in subsection 3.2 the L_2 and uniform errors on the entire interval.

3.1 Pointwise convergence of the Fourier-Pade approximation in the regions away from the endpoints

First we need some lemmas revealing the behavior of the generalized finite differences when $\gamma_k^+(\theta)$ and $\gamma_k^-(\theta)$ are solutions of systems (18) and (19), respectively.

Lemma 1. *Let $f \in C^{q+2p+1}[-1,1]$ and $f^{(q+2p+1)} \in AC[-1,1]$. Let systems (18) and (19) have unique solutions. Then the following estimate holds*

$$\Delta_n^w(\Delta_n^p(\theta, F_n)) = O(n^{-q-w-1}) + o(n^{-q-2p-2}),\ |n| \geq N+1,\ N \to \infty.$$

Proof. According to definitions of the classical and generalized finite differences, we have

$$\Delta_n^w(\Delta_n^p(\theta, F_n)) = \sum_{s=0}^{p} \gamma_s^{\pm}(\theta)\Delta_n^w(F_{n\mp s}),$$

where the upper signs correspond to positive n.

In view of the smoothness of f and expansion (1), by means of integration by parts, we derive

$$F_n = \frac{(-1)^{n+1}}{2} \sum_{k=q}^{q+2p+1} A_k(f)B_{k,n} + o(n^{-q-2p-2}). \tag{20}$$

Thus

$$\Delta_n^w(\Delta_n^p(\theta, F_n)) = \frac{(-1)^{n+1}}{2} \sum_{s=0}^{p} \gamma_s^{\pm}(\theta) \sum_{k=q}^{q+2p+1} A_k(f)\Delta_n^w(B_{k,n\mp s}) + o(n^{-q-2p-2}).$$

This completes the proof as

$$\Delta_n^w(B_{k,n}) = \Delta_n^w\left(\frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}\right) = O(n^{-k-w-1})$$

and

$$\gamma_s^{\pm}(\theta) = O(1),\ N \to \infty.$$

The last estimate follows from

$$\lim_{N\to\infty} \gamma_s^{\pm}(\theta) = \binom{p}{s} \tag{21}$$

which can be proved in view of asymptotic expansion (20) and systems (18), (19). □

Lemma 2. *Let $f \in C^{q+2p+1}[-1,1]$ with $f^{(q+2p+1)} \in AC[-1,1]$ and $A_q(f) \neq 0$. Let systems (18) and (19) have unique solution $\gamma_s^+(\theta)$ and $\gamma_s^-(\theta)$, respectively. Then the following estimate holds for $w \geq 0$, $p > 0$*

$$\begin{aligned}\Delta_{\pm N}^w(\Delta_n^p(\theta, F_n)) &= A_q(f)\frac{(-1)^{N+w+1}(w+q)!}{2N^{w+p}(\pm i\pi N)^{q+1}q!} \sum_{t=0}^{p} \beta_t^{\pm}(p-t)\binom{t+w+q}{w+q} \\ &+ O(N^{-w-p-q-2}) + o(N^{-2p-q-2}),\ N \to \infty\end{aligned}$$

where

$$\beta_u^{\pm}(t) = \sum_{s=0}^{p}(-1)^s \gamma_{s,\pm}^{(t)} s^u$$

and $\gamma_{s,\pm}^{(t)}$ are the coefficients of the asymptotic expansion of $\gamma_s^{\pm}(\theta)$

$$\gamma_s^{\pm}(\theta) = \sum_{t=0}^{2p+1} \frac{\gamma_{s,\pm}^{(t)}}{N^t} + o(N^{-2p-1}),\ N \to \infty. \tag{22}$$

Proof. We will prove for "+" sign. The case of "-" sign can be handled similarly. Existence of the asymptotic expansion (22) follows from the smoothness of f, systems (18), (19) and from the Crammer rule for solution of system of linear equations.

Then, definition of the classical and generalized finite differences leads to the following representation

$$\Delta_N^w(\Delta_n^p(\theta,F_n)) = \sum_{s=0}^{p}\gamma_s^+(\theta)\Delta_N^w(F_{n-s}) = \sum_{k=0}^{w}\binom{w}{k}\sum_{s=0}^{p}\gamma_s^+(\theta)F_{N-k-s}, \tag{23}$$

where γ_s^+ are the solutions of system (18).

From (20) we derive

$$\begin{aligned} F_{N-k-s} &= \frac{(-1)^{N+k+s+1}}{2}\sum_{\ell=q}^{q+2p+1}\frac{A_\ell(f)}{(i\pi(N-k-s))^{\ell+1}} + o(N^{-q-2p-2}) \\ &= \frac{(-1)^{N+k+s+1}}{2}\sum_{\ell=q}^{q+2p+1}\frac{A_\ell(f)}{(i\pi)^{\ell+1}}\sum_{j=\ell}^{\infty}\binom{j}{\ell}\frac{(k+s)^{j-\ell}}{N^{j+1}} + o(N^{-q-2p-2}) \\ &= \frac{(-1)^{N+k+s+1}}{2(i\pi N)^{q+1}}\sum_{j=0}^{2p+1}\frac{1}{N^j}\sum_{\ell=0}^{j}\frac{A_{\ell+q}(f)}{(i\pi)^\ell}\binom{j+q}{\ell+q}(k+s)^{j-\ell} + o(N^{-q-2p-2}). \end{aligned}$$

Substituting this and (22) into (23) we obtain

$$\begin{aligned} \Delta_N^w(\Delta_n^p(\theta,F_n)) &= \frac{(-1)^{N+1}}{2(i\pi N)^{q+1}}\sum_{k=0}^{w}(-1)^k\binom{w}{k}\sum_{s=0}^{p}(-1)^s\left(\sum_{t=0}^{2p+1}\frac{\gamma_{s,+}^{(t)}}{N^t} + o(N^{-2p-1})\right) \\ &\times \left(\sum_{j=0}^{2p+1}\frac{1}{N^j}\sum_{\ell=0}^{j}\frac{A_{\ell+q}(f)}{(i\pi)^\ell}\binom{j+q}{\ell+q}(k+s)^{j-\ell} + o(N^{-q-2p-2})\right) \\ &= \frac{(-1)^{N+1}}{2(i\pi N)^{q+1}}\sum_{j=0}^{2p+1}\frac{1}{N^j}\sum_{t=0}^{j}\sum_{\ell=0}^{j-t}\frac{A_{j-t-\ell+q}(f)}{(i\pi)^{j-t-\ell}}\binom{j-t+q}{\ell} \\ &\times \sum_{u=0}^{\ell}\binom{\ell}{u}\alpha_{w,u}\beta_{\ell-u}^+(t) + o(N^{-q-2p-2}), \end{aligned}$$

where

$$\alpha_{r,s} = \sum_{\ell=0}^{r}(-1)^\ell\binom{r}{\ell}\ell^s. \tag{24}$$

Taking into account that (see [17])

$$\alpha_{r,s} = 0, \ \ s \le r-1$$

we get

$$\begin{aligned}\Delta_N^w(\Delta_n^p(\theta,f_n)) &= \frac{(-1)^{N+1}}{2(i\pi N)^{q+1}}\sum_{j=w}^{2p+1}\frac{1}{N^j}\sum_{t=0}^{j-w}\sum_{\ell=w}^{j-t}\frac{A_{j-t-\ell+q}(f)}{(i\pi)^{j-t-\ell}}\binom{j-t+q}{\ell}\\ &\times\sum_{u=w}^{\ell}\binom{\ell}{u}\alpha_{w,u}\beta^+_{\ell-u}(t)+o(N^{-q-2p-2})\\ &= \frac{(-1)^{N+1}}{2N^w(i\pi N)^{q+1}}\sum_{j=0}^{2p-w+1}\frac{1}{N^j}\sum_{t=0}^{j}\sum_{\ell=0}^{j-t}\frac{A_{j-t-\ell+q}(f)}{(i\pi)^{j-t-\ell}}\binom{j+w-t+q}{\ell+w}\\ &\times\sum_{u=0}^{\ell}\binom{\ell+w}{u+w}\alpha_{w,u+w}\beta^+_{\ell-u}(t)+o(N^{-q-2p-2}).\end{aligned}$$

From here we derive

$$\begin{aligned}\Delta_N^w(\Delta_n^p(\theta,F_n)) &= \frac{(-1)^{N+1}}{2N^w(i\pi N)^{q+1}}\sum_{j=0}^{2p-w+1}\frac{1}{N^j}\sum_{t=0}^{j}\sum_{u=0}^{t}\beta^+_u(j-t)\\ &\times\sum_{\ell=u}^{t}\frac{A_{t-\ell+q}}{(i\pi)^{t-\ell}}\binom{t+w+q}{\ell+w}\binom{\ell+w}{u}\alpha_{w,\ell-u+w}+o(N^{-2p-q-2}).\end{aligned}\tag{25}$$

This will complete the proof if we show that

$$\beta^+_u(j-t)=0,\ j=0,\dots,p-1,\ 0\le t\le j,\ 0\le u\le t. \tag{26}$$

System (14) can be rewritten in the form

$$\Delta_N^w(\Delta_n^p(\theta,F_n))=0,\ w=0,\dots,p-1$$

and from (25) we get

$$\begin{aligned}\sum_{t=0}^{j}\sum_{u=0}^{t}\beta^+_u(j-t)\sum_{\ell=u}^{t}\frac{A_{t-\ell+q}(f)}{(i\pi)^{t-\ell}}\binom{t+w+q}{\ell+w}\binom{\ell+w}{u}\alpha_{w,\ell-u+w}=0,&\\ 0\le j\le 2p-w+1,\ w=0,\dots,p-1.&\end{aligned}\tag{27}$$

We prove (26) from (27) by means of the mathematical induction. Let $j=0$ in (27). Thus

$$\beta^+_0(0)A_q(f)\binom{w+q}{w}\alpha_{w,w}=0$$

and therefore $\beta^+_0(0)=0$ as $A_q(f)\neq 0$. Suppose that identity (26) is true for $j=j_0-1$, $j_0\le p-1$

$$\beta^+_u(j_0-1-t)=0,\ 0\le u\le t,\ 0\le t\le j_0-1. \tag{28}$$

Let us prove that

$$\beta^+_u(j_0-t)=0.$$

For $j = j_0$ from (27) we have

$$\sum_{t=0}^{j_0}\sum_{u=0}^{t}\beta_u^+(j_0-t)X_{u,t}(w)=0, \tag{29}$$

where

$$X_{u,t}(w)=\sum_{\ell=u}^{t}\frac{A_{t-\ell+q}(f)}{(i\pi)^{t-\ell}}\binom{t+w+q}{\ell+w}\binom{\ell+w}{u}\alpha_{w,\ell-u+w}.$$

Then (29) can be rewritten in the form

$$\begin{aligned}\sum_{t=0}^{j_0}\sum_{u=0}^{t}\beta_u^+(j_0-t)X_{u,t}(w) &= \sum_{t=1}^{j_0}\sum_{u=0}^{t-1}\beta_u^+(j_0-t)X_{u,t}(w)\\ &+ \sum_{t=0}^{j_0}\beta_t^+(j_0-t)X_{t,t}(w)=0,\end{aligned}$$

where the first term vanishes according to (28). Hence

$$\sum_{t=0}^{j_0}\beta_t^+(j_0-t)X_{t,t}(w)=0,$$

where

$$X_{t,t}(w)=A_q(f)\binom{t+w+q}{t+w}\binom{t+w}{t}w!(-1)^w.$$

Taking into account that $A_q(f)\neq 0$ we get the following system of linear equations for determination of $\beta_t^+(j_0-t)$

$$\sum_{t=0}^{j_0}\beta_t^+(j_0-t)\binom{t+w+q}{t+w}\binom{t+w}{t}=0,\ \ w=0,\dots,j_0$$

which can be rewritten in the form

$$\sum_{t=0}^{j_0}\beta_t^+(j_0-t)\binom{t+w+q}{t}=0,\ \ w=0,\dots,j_0. \tag{30}$$

In the Appendix we show (see (42)) that matrix $\binom{t+w+q}{t}$ has nonzero determinant and as a consequence system (30) has unique zero-vector solution $\beta_t(j_0-t)=0,\ t=0,\dots,j_0\leq p-1$. Thus, from (25) we obtain

$$\begin{aligned}\Delta_N^w(\Delta_n^p(\theta,F_n)) &= \frac{(-1)^{N+1}}{2N^{w+p}(i\pi N)^{q+1}}\sum_{t=0}^{p}\sum_{u=0}^{t}\beta_u^+(p-t)\sum_{\ell=u}^{t}\frac{A_{t-\ell+q}(f)}{(i\pi)^{t-\ell}}\binom{t+w+q}{\ell+w}\\ &\times \binom{\ell+w}{u}\alpha_{w,\ell-u+w}+O(N^{-w-p-q-2})+o(N^{-2p-q-2}),\ N\to\infty\end{aligned}$$

It remains to notice that only term $u=t$ is nonzero and $\alpha_{w,w}=(-1)^w w!$. □

Now we prove one of the main results of this paper that reveals the pointwise convergence of the Fourier-Pade approximations in the regions away from the endpoints.

Theorem 9. *Let $f \in C^{q+2p+1}[-1,1]$ with $f^{(q+2p+1)} \in AC[-1,1]$ and $A_q(f) \neq 0$. Let systems (18) and (19) have unique solutions. Then the following estimate holds*

$$R_{N,q,p}(f;x) = A_q(f)\frac{(-1)^{N+\frac{q}{2}}}{2^{2p+1}\pi^{q+1}N^{q+2p+1}}\frac{(p+q)!p!}{q!}\frac{\sin\frac{\pi x}{2}(2N-2p+1)}{\cos^{2p+1}\frac{\pi x}{2}}+o(N^{-q-2p-1}),\ N\to\infty$$

for even values of q and

$$R_{N,q,p}(f;x) = A_q(f)\frac{(-1)^{N+\frac{q+1}{2}}}{2^{2p+1}\pi^{q+1}N^{q+2p+1}}\frac{(p+q)!p!}{q!}\frac{\cos\frac{\pi x}{2}(2N-2p+1)}{\cos^{2p+1}\frac{\pi x}{2}}+o(N^{-q-2p-1}),\ N\to\infty$$

for odd values of q.

Proof. Let us start with estimation of $R^{+}_{N,q,p}(f;x)$ (see (6))

$$R^{+}_{N,q,p}(f;x) = \frac{1}{\prod_{k=1}^{p}(1+\theta_k e^{i\pi x})}\sum_{n=N+1}^{\infty}\Delta_n^p(\theta,F_n)e^{i\pi nx}. \tag{31}$$

Recall (see (21)) that $\gamma_s^{+}(\theta)\to\binom{p}{s}$ as $N\to\infty$ and therefore

$$\prod_{k=1}^{p}(1+\theta_k e^{i\pi x})\to(1+e^{i\pi x})^p,\ N\to\infty.$$

Hence we need to estimate only the sum in the right hand side of (31). By application of the Abel transformation we derive

$$\begin{aligned}\sum_{n=N+1}^{\infty}\Delta_n^p(\theta,F_n)e^{i\pi nx} &= -e^{i\pi(N+1)x}\sum_{w=0}^{2p+1}\frac{\Delta_N^w(\Delta_n^p(\theta,F_n))}{(1+e^{i\pi x})^{w+1}}\\ &+ \frac{1}{(1+e^{i\pi x})^{2p+2}}\sum_{n=N+1}^{\infty}\Delta_n^{2p+2}(\Delta_n^p(\theta,F_n))e^{i\pi nx}.\end{aligned}$$

Taking into account that

$$\Delta_N^k(\Delta_n^p(\theta,F_n)) = \sum_{s=0}^{k}\binom{k}{s}\Delta_{N-s}^p(\theta,F_n)$$

we see from system (14) that

$$\Delta_N^k(\Delta_n^p(\theta,F_n)) = 0,\ k=0,\ldots,p-1.$$

Therefore

$$\begin{aligned}\sum_{n=N+1}^{\infty}\Delta_n^p(\theta,F_n)e^{i\pi nx} &= -e^{i\pi(N+1)x}\frac{\Delta_N^p(\Delta_n^p(\theta,F_n))}{(1+e^{i\pi x})^{p+1}} - e^{i\pi(N+1)x}\sum_{w=p+1}^{2p+1}\frac{\Delta_N^w(\Delta_n^p(\theta,F_n))}{(1+e^{i\pi x})^{w+1}}\\ &+ \frac{1}{(1+e^{i\pi x})^{2p+2}}\sum_{n=N+1}^{\infty}\Delta_n^{2p+2}(\Delta_n^p(\theta,F_n))e^{i\pi nx}.\end{aligned} \tag{32}$$

Lemma 1 shows that

$$\Delta_n^{2p+2}(\Delta_n^p(\theta, F_n)) = o(n^{-2p-q-2}),\ n \to \infty.$$

Hence the last term in the right hand side of (32) is $o(N^{-2p-q-1})$ as $N \to \infty$.

According to Lemma 2

$$\Delta_N^w(\Delta_n^p(\theta, F_n)) = O(N^{-w-p-q-1}) + o(N^{-q-2p-1}),\ N \to \infty.$$

As in the second term of the right hand side of (32) parameter w is ranging from $w = p+1$ to $w = 2p+1$ then this term is $O(N^{-q-2p-2})$. Hence

$$R_{N,q,p}^+(f) = -e^{i\pi(N+1)x} \frac{\Delta_N^p(\Delta_n^p(\theta, F_n))}{(1+e^{i\pi x})^{2p+1}} + o(N^{-q-2p-1}),\ N \to \infty. \tag{33}$$

Now we need to estimate $\Delta_N^p(\Delta_n^p(\theta, F_n))$. Again by Lemma 2 we have

$$\begin{aligned} \Delta_N^p(\Delta_n^p(\theta, F_n)) &= A_q(f)\frac{(-1)^{N+p+1}}{2N^{2p}(i\pi N)^{q+1}q!} \\ &\times \left(\sum_{t=0}^{p-1} \beta_t^+(p-t)\frac{(t+p+q)!}{t!} + \beta_p^+(0)\frac{(2p+q)!}{p!}\right) + o(N^{-q-2p-1}). \end{aligned} \tag{34}$$

Now we calculate $\beta_t^+(p-t)$ for $t = 0, \dots, p-1$. Using (27) for $j = p$ and $w = 0, \dots, p-1$ we derive

$$\beta_p^+(0)\binom{p+w+q}{p+w}\binom{p+w}{p} + \sum_{t=0}^{p-1}\beta_t^+(p-t)\binom{t+w+q}{t+w}\binom{t+w}{t} = 0,\ w = 0, \dots, p-1.$$

After some simplifications we get

$$\sum_{t=0}^{p-1}\beta_t^+(p-t)\binom{t+w+q}{w+q} = -\beta_p^+(0)\binom{p+w+q}{p},\ w = 0, \dots, p-1. \tag{35}$$

We know that determinant of matrix $\binom{t+w+q}{w+q}$ is nonzero, hence it is invertible matrix and its inverse we calculate in Appendix (see (44)). Now from (35) we get

$$\beta_t^+(p-t) = -\beta_p^+(0)\sum_{w=0}^{p-1}(-1)^{t+w}\sum_{s=0}^{p-1}\binom{s}{w}\binom{q+s}{q+t}\binom{p+w+q}{p}.$$

In view of identity ([17])

$$\sum_{w=0}^{p-1}(-1)^w\binom{s}{w}\binom{p+w+q}{p} = (-1)^s\binom{p+q}{q+s}.$$

we derive

$$\beta_t^+(p-t) = -\beta_p^+(0)(-1)^t\sum_{s=0}^{p-1}(-1)^s\binom{q+s}{q+t}\binom{p+q}{s+q}. \tag{36}$$

Then (see (34))

$$
\begin{aligned}
&\sum_{t=0}^{p-1}\beta_t^+(p-t)\frac{(t+p+q)!}{t!}+\beta_p^+(0)\frac{(2p+q)!}{p!}\\
&= (p+q)!\sum_{t=0}^{p-1}\beta_t^+(p-t)\binom{t+p+q}{p+q}+\beta_p^+(0)\frac{(2p+q)!}{p!}\\
&= \beta_p^+(0)\left[\frac{(2p+q)!}{p!}-(p+q)!\sum_{t=0}^{p-1}(-1)^t\binom{t+p+q}{p+q}\sum_{s=0}^{p-1}(-1)^s\binom{q+s}{q+t}\binom{p+q}{s+q}\right]\\
&= \beta_p^+(0)\left[\frac{(2p+q)!}{p!}+(p+q)!(-1)^p\sum_{t=0}^{p-1}(-1)^t\binom{t+p+q}{p}\binom{p}{t}\right],
\end{aligned}
$$

where we applied identity ([17])

$$
\sum_{s=k}^{p}(-1)^s\binom{p}{s}\binom{s}{k}=0. \tag{37}
$$

Finally, applying

$$
\begin{aligned}
\sum_{t=0}^{p-1}(-1)^t\binom{p}{t}\binom{t+p+q}{p} &= \sum_{t=0}^{p}(-1)^t\binom{p}{t}\binom{t+p+q}{p}-(-1)^p\binom{2p+q}{p}\\
&= (-1)^p-(-1)^p\binom{2p+q}{p}
\end{aligned}
$$

we get

$$
\sum_{t=0}^{p-1}\beta_t^+(p-t)\frac{(t+p+q)!}{t!}+\beta_p^+(0)\frac{(2p+q)!}{p!}=(p+q)!\beta_p^+(0).
$$

Then

$$
\beta_p^+(0)=\sum_{s=0}^{p}(-1)^s\gamma_{s,+}^{(0)}s^p=\sum_{s=0}^{p}(-1)^s\binom{p}{s}s^p=(-1)^p p! \tag{38}
$$

and from (34) we get

$$
\Delta_N^p(\Delta_n^p(\theta,F_n))=A_q(f)\frac{(-1)^{N+1}}{2N^{2p}(i\pi N)^{q+1}q!}p!(p+q)!+o(N^{-q-2p-1}).
$$

Substitution this into (33) implies

$$
R_{N,q,p}^+(f;x) = A_q(f)\frac{(-1)^N}{2N^{2p}(\pi N)^{q+1}}\frac{p!(p+q)!}{q!}\frac{e^{i\pi(N+1)x}}{i^{q+1}(1+e^{i\pi x})^{2p+1}}+o(N^{-q-2p-1}).
$$

Similarly

$$
R_{N,q,p}^-(f;x) = A_q(f)\frac{(-1)^N}{2N^{2p}(\pi N)^{q+1}}\frac{p!(p+q)!}{q!}\frac{e^{-i\pi(N+1)x}}{(-i)^{q+1}(1+e^{-i\pi x})^{2p+1}}+o(N^{-q-2p-1}).
$$

Thus

$$
R_{N,q,p}(f;x)=A_q(f)\frac{(-1)^N}{\pi^{q+1}N^{q+2p+1}}\frac{(p+q)!p!}{q!}Re\left[\frac{e^{i\pi(N+1)x}}{i^{q+1}(1+e^{i\pi x})^{2p+1}}\right]+o(N^{-q-2p-1})
$$

which completes the proof. □

Comparisons show that the Fourier-Pade approximation is much more precise in the regions away from the endpoints than the KL-approximation and other RTP-approximations presented here. Comparing with Theorem 2 we see that improvement in accuracy is $O(N^{2p})$, compared with Theorem 5 improvement is $O(N^P)$ and compared with Theorems 7 and 8 improvement is almost $O(N^{p/2})$.

3.2 Convergence of the Fourier-Pade Approximation on the Entire Interval: L_2 and uniform errors.

In this subsection we investigate convergence of the Fourier-Pade approximation in the frameworks of L_2 and uniform errors. For such investigations we need additional information concerning the behavior of parameters θ_k as $N \to \infty$. As was mentioned above it is easy to verify that $\gamma_k^{\pm}(\theta) \to \binom{p}{k}$ as $N \to \infty$. In view of (16) and (17) this means that $\theta_k \to 1$ as $N \to \infty$ but this information is not enough for our purposes and we need to estimate the second term in the asymptotic expansion of θ_k

$$\theta_k = 1 - \frac{\tau_k}{N} + o(N^{-1}),\ \theta_{-k} = 1 - \frac{\tau_{-k}}{N} + o(N^{-1}),\ k = 1, \dots, p. \tag{39}$$

For determination of parameters τ_k we compare two results that outline the behavior of $\Delta_n^p(\theta, F_n)$.

Lemma 3. *Let $f \in C^{q+p}[-1,1]$ with $f^{(q+p)} \in AC[-1,1]$ and $A_q(f) \neq 0$. Let systems (18) and (19) have unique solutions $\gamma_s^{\pm}(\theta)$. Then the following estimate holds*

$$\Delta_n^p(\theta, F_n) = A_q(f)\frac{(-1)^{n+1}}{2(i\pi n)^{q+1}N^p}\frac{(p+q)!}{q!}\left(1 - \frac{1}{|n|/N}\right)^p + o(N^{-p})\frac{1}{n^{q+1}},\ |n| > N,\ N \to \infty.$$

Proof. The proof, in general, mimics the one of Lemma 2, so we omit some details. We proceed as there and write for positive n

$$\begin{aligned}
\Delta_n^p(\theta, F_n) &= \sum_{s=0}^{p} \gamma_s^{+}(\theta) F_{n-s} = \frac{(-1)^{n+1}}{2(i\pi n)^{q+1}} \sum_{s=0}^{p} (-1)^s \left[\sum_{t=0}^{p} \frac{\gamma_{s,+}^{(t)}}{N^t} + o(N^{-p})\right] \\
&\times \left[\sum_{j=0}^{q+p} \frac{1}{n^j} \sum_{\ell=0}^{j} \frac{A_{\ell+q}(f)}{(i\pi)^{\ell}} \binom{j+q}{\ell+q} s^{j-\ell} + o(n^{-q-p-1})\right] \\
&= \frac{(-1)^{n+1}}{2(i\pi n)^{q+1}} \sum_{j=0}^{p} \frac{1}{N^j} \sum_{t=0}^{j} \frac{1}{\left(\frac{n}{N}\right)^t} \sum_{\ell=0}^{t} \beta_{\ell}^{+}(j-t) \frac{A_{t-\ell+q}(f)}{(i\pi)^{t-\ell}} \binom{t+q}{\ell} \\
&+ o(N^{-p})\frac{1}{n^{q+1}}.
\end{aligned}$$

From the proof of Lemma 2 we know that

$$\beta_{\ell}^{+}(j-t) = 0,\ \ell + j - t \leq p - 1.$$

Therefore

$$\begin{aligned}\Delta_n^p(\theta, F_n) &= \frac{(-1)^{n+1}}{2(i\pi n)^{q+1}} \frac{1}{N^p} \sum_{t=0}^{p} \frac{1}{\left(\frac{n}{N}\right)^t} \sum_{\ell=0}^{t} \beta_\ell^+(p-t) \frac{A_{t-\ell+q}(f)}{(i\pi)^{t-\ell}} \binom{t+q}{\ell} + o(N^{-p}) \frac{1}{n^{q+1}} \\ &= A_q(f) \frac{(-1)^{n+1}}{2(i\pi n)^{q+1} N^p} \sum_{t=0}^{p} \frac{\beta_t^+(p-t)}{\left(\frac{n}{N}\right)^t} \binom{t+q}{t} + o(N^{-p}) \frac{1}{n^{q+1}}.\end{aligned}$$

Application of (36), (38) and identity (37) implies

$$\begin{aligned}\Delta_n^p(\theta, F_n) &= A_q(f) \frac{(-1)^n}{2(i\pi n)^{q+1} N^p} \beta_p^+(0) \sum_{t=0}^{p} \frac{(-1)^t}{\left(\frac{n}{N}\right)^t} \binom{t+q}{t} \sum_{s=0}^{p-1} (-1)^s \binom{q+s}{q+t} \binom{p+q}{q+s} \\ &+ o(N^{-p}) \frac{1}{n^{q+1}} = A_q(f) \frac{(-1)^{n+p}}{2(i\pi n)^{q+1} N^p} \frac{(p+q)!}{p!} \sum_{t=0}^{p} \frac{(-1)^t}{\left(\frac{n}{N}\right)^t} \sum_{s=0}^{p-1} (-1)^s \binom{s}{t} \binom{p}{s} \\ &+ o(N^{-p}) \frac{1}{n^{q+1}} = A_q(f) \frac{(-1)^{n+1}}{2(i\pi n)^{q+1} N^p} \frac{(p+q)!}{p!} \sum_{t=0}^{p} \frac{(-1)^t}{\left(\frac{n}{N}\right)^t} \binom{p}{t} + o(N^{-p}) \frac{1}{n^{q+1}}.\end{aligned}$$

This completes the proof for positive n. Negative values can be explored similarly. □

Lemma 4. *[12] Suppose $f \in C^{q+p}[-1,1]$ and $f^{(q+p)} \in AC[-1,1]$. If*

$$\theta_k = 1 - \frac{\tau_k}{N}, \ \theta_{-k} = 1 - \frac{\tau_{-k}}{N}, \ k = 1, \ldots, p.$$

then the following estimate holds for $|n| > N$ as $N \to \infty$

$$\Delta_n^p(\theta) = A_q(f) \frac{(-1)^{n+p+1}}{2(i\pi n)^{q+1} q!} \sum_{k=0}^{p} \frac{(q+p-k)!(-1)^k \gamma_k^{\pm}(\tau)}{N^k |n|^{p-k}} + o(N^{-p}) \frac{1}{n^{q+1}},$$

where

$$\prod_{k=1}^{p} (1 + \tau_k x) = \sum_{k=0}^{p} \gamma_k^+(\tau) x^k, \ \prod_{k=1}^{p} (1 + \tau_{-k} x) = \sum_{k=0}^{p} \gamma_k^-(\tau) x^k.$$

Comparison of Lemmas 3 and 4 shows that

$$(-1)^p \sum_{k=0}^{p} \frac{(-1)^k (q+p-k)!}{(|n|/N)^{p-k}} \gamma_k^{\pm}(\tau) = \sum_{k=0}^{p} \frac{(-1)^k}{(|n|/N)^k} \binom{p}{k}$$

and hence

$$\gamma_k^{\pm}(\tau) = \frac{1}{(q+k)!} \binom{p}{k}.$$

It means that in the Fourier-Pade approximation parameters $\tau_k = \tau_{-k}$ are the roots of the associated Laguerre polynomials $L_p^q(x)$ (see (11))

$$L_p^q(\tau_k) = 0, \ k = 1, \ldots, p.$$

Now, we are ready to proceed with L_2 and uniform convergence investigation. First we estimate the L_2-error.

Let h_k be the complete homogeneous symmetric polynomial (see [10]) of degree k in p variables $\theta_1, \theta_2, \ldots, \theta_p$

$$h_k(\theta_1, \ldots, \theta_p) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le p} \theta_{i_1} \ldots \theta_{i_k}.$$

The complete homogeneous symmetric polynomials are characterized by the following identity of formal power series

$$\frac{1}{\prod_{k=1}^{p}(1+\theta_k x)} = \sum_{k=0}^{\infty}(-1)^k h_k(\theta_1, \cdots, \theta_p) x^k,$$

where

$$h_k(\theta_1, \cdots, \theta_p) = \sum_{i=1}^{p} \frac{\theta_i^{p+k-1}}{\prod\limits_{\substack{j=1\\ j\neq i}}^{p}(\theta_i - \theta_j)}. \tag{40}$$

We start with estimation of (31) and write

$$\begin{aligned}
R^{+}_{N,q,p}(f) &= \frac{1}{\prod_{k=1}^{p}(1+\theta_k e^{i\pi x})} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, F_n) e^{i\pi n x} \\
&= \sum_{k=0}^{\infty}(-1)^k h_k e^{i\pi k x} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, F_n) e^{i\pi n x} \\
&= \sum_{s=N+1}^{\infty}(-1)^s e^{i\pi s x} \sum_{n=N+1}^{s}(-1)^n \Delta_n^p(\theta, F_n) h_{s-n}.
\end{aligned}$$

Performing similar manipulations for $R^{-}_{N,q,p}(f)$ we derive

$$\|R_{N,q,p}(f)\|^2_{L_2} = 4 \sum_{s=N+1}^{\infty} \left| \sum_{n=N+1}^{s}(-1)^n h_{s-n} \Delta_n^p(\theta, F_n)\right|^2. \tag{41}$$

Theorem 10. *Let $f \in C^{q+p}[-1,1]$ with $f^{(q+p)} \in AC[-1,1]$ and $A_q(f) \neq 0$. If systems (18) and (19) have unique solutions then the following estimate holds for the Fourier-Pade approximation $S_{N,q,p}(f)$*

$$\lim_{N\to\infty} N^{q+\frac{1}{2}} ||R_{N,q,p}(f)|| = |A_q(f)| d_p(q),$$

where

$$d_p(q) = \frac{1}{\pi^{q+1}} \frac{(p+q)!}{q!} \left(\int_1^{\infty} dt \left| \int_1^t \frac{(1-x)^p}{x^{q+p+1}} \sum_{j=1}^{p} \frac{e^{-\tau_j(t-x)}}{\prod_{\substack{k=1\\ k\neq j}}^{p}(\tau_j - \tau_k)} dx \right|^2 \right)^{1/2}$$

and τ_k are the roots of the associated Laguerre polynomial $L_p^q(x)$.

Proof. We use equation (41). In view of (40) and (39) we have as $N \to \infty$

$$h_{s-n} = \sum_{i=1}^{p} \frac{\theta_i^{p+s-n-1}}{\prod_{\substack{j=1 \\ j\neq i}}^{p} (\theta_i - \theta_j)} = N^{p-1} \sum_{i=1}^{p} \frac{(1 - \frac{\tau_i}{N} + o(N^{-1}))^{p+s-n-1}}{\prod_{\substack{j=1 \\ j\neq i}}^{p} (\tau_j - \tau_i + o(1))}.$$

Substituting this and estimate of Lemma 3 into (41), tending N to infinity and replacing the integral sums by the corresponding integrals we complete the proof. $\square$

Now we calculate the limit function corresponding to the Fourier-Pade approximation.

Theorem 11. *Let $f \in C^{q+p}[-1,1]$ with $f^{(q+p)} \in AC[-1,1]$ and $A_q(f) \neq 0$. If systems (18) and (19) have unique solutions then the following estimate holds for the Fourier-Pade approximation $S_{N,q,p}(f;x)$ for $h \geq 0$*

$$\lim_{N\to\infty} N^q R_{N,q,p}\left(f; 1 - \frac{h}{N}\right) = A_q(f) \wp_{q,p}(h),$$

where

$$\wp_{q,p}(h) = -A_q(f) \frac{(p+q)!}{\pi^{q+1} q!} Re \left[\frac{1}{i^{q+1} \prod_{k=1}^{p} (\tau_k + i\pi h)} \int_1^{\infty} \frac{e^{-i\pi h x}(1-x)^p}{x^{q+p+1}} dx \right]$$

and τ_k are the roots of the associated Laguerre polynomial $L_p^q(x)$.

Proof. According to (39) for $x = 1 - h/N$ we write

$$\frac{1}{\prod_{k=1}^{p} (1 + \theta_k e^{i\pi x})} = \frac{N^p}{\prod_{k=1}^{p} (\tau_k + i\pi h)} + o(N^p).$$

Then in view of Lemma 3 we get

$$\begin{aligned} R_{N,q,p}^{+}\left(f; 1 - \frac{h}{N}\right) &= \left[\frac{N^p}{\prod_{k=1}^{p}(\tau_k + i\pi h)} + o(N^p) \right] \\ \times \sum_{n=N+1}^{\infty} &\left[A_q(f) \frac{(-1)^{n+1}}{2(i\pi n)^{q+1} N^p} \frac{(p+q)!}{q!} \left(1 - \frac{1}{n/N}\right)^p + o(N^{-p}) n^{-q-1} \right] (-1)^n e^{-i\pi h \frac{n}{N}} \\ = -A_q(f) \frac{(p+q)!}{q!} \frac{1}{2N^q (i\pi)^{q+1}} &\frac{1}{\prod_{k=1}^{p} (\tau_k + i\pi h)} \left(\frac{1}{N} \sum_{n=N+1}^{\infty} \frac{e^{-i\pi h \frac{n}{N}}}{(n/N)^{q+1}} \left(1 - \frac{1}{n/N}\right)^p \right) \\ &+ o(N^{-q}), \ N \to \infty. \end{aligned}$$

Similar observations are valid for $R_{N,q,p}^{-}\left(f; 1 - \frac{h}{N}\right)$. Then we complete the proof by tending N to infinity and replacing the integral sums by the corresponding integral. $\square$

It can be verified that estimates in Theorems 10 and 11 coincide with the ones in Theorems 4 and 5, respectively, if in the latest we put instead of parameters τ_k the roots of the associated Laguerre polynomials (see the corresponding values in Tables 7 and 8). From here we conclude that on the entire interval of approximation the Fourier-Pade approximation has the same L_2- and uniform errors as the RTP-approximations by the roots of the associated Laguerre polynomials. As a consequence it has worse accuracy compared with the L_2-minimal RTP-approximation on $[-1, 1]$. On the other hand as was mentioned above the Fourier-Pade approximation has the best pointwise convergence in the regions away from the endpoints.

Figure 4 shows the behavior of the Fourier-Pade approximation for the testing function (2) in the regions away from the endpoints (left figure) and at the right endpoint of the interval of approximation (right figure). Compare it with the results in Figures 1, 2, and 3.

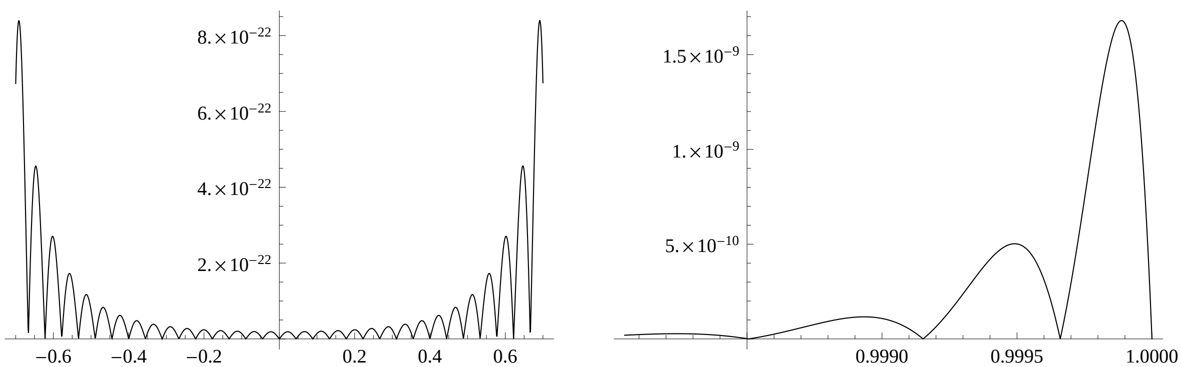

Figure 4: Graphs of $|R_{N,q,p}(f;x)|$ on the interval $[-0.7, 0.7]$ (left) and at the point $x = 1$ (right) for $q = 2$, $p = 2$ and $N = 1024$ that correspond to the Fourier-Pade approximation.

4 Appendix

Consider matrix $A = (a_{wt})$ with

$$a_{wt} = \binom{t+w+q}{t}, \quad w, t = 0, \dots, M, \ \ M \geq 0.$$

Our aim is calculation of $\det(A)$ and A^{-1}. We do it based on decomposition of A into three upper or lower diagonal matrices.

Consider the following identities ([17]) of binomial coefficients

$$\binom{w+q+t}{t} = \sum_{k=0}^{M} \binom{t}{k} \binom{w+q}{k}, \quad t, w = 0, \dots, M$$

and

$$\binom{w+q}{k} = \sum_{s=0}^{w} \binom{w}{s} \binom{q}{k-s}, \quad k, w = 0, \dots, M.$$

We get by sequential application of these identities

$$\begin{aligned}\binom{w+q+t}{t} &= \sum_{k=0}^{M}\binom{t}{k}\binom{w+q}{k} \\ &= \sum_{s=0}^{M}\binom{w}{s}\sum_{k=0}^{M}\binom{q}{k-s}\binom{t}{k},\ w,t=0,\dots,M\end{aligned}$$

which allows decomposition of matrix A in the form

$$A = BCD$$

where $B=(b_{ws})$, $C=(c_{sk})$, $D=(d_{kt})$ and

$$b_{ws}=\binom{w}{s},\ c_{sk}=\binom{q}{k-s},\ d_{kt}=\binom{t}{k}.$$

Note that B is lower and C, D are upper triangular matrices which simplifies calculation of $\det(A)$ and A^{-1}.

The value of determinant can be obtained immediately

$$\det(A)=\det(B)\det(C)\det(D)=1 \tag{42}$$

The inverse of A can be calculated by formula

$$A^{-1}=D^{-1}C^{-1}B^{-1}.$$

We denote by a^{-1}_{tw}, b^{-1}_{sw}, c^{-1}_{ks} and d^{-1}_{tk} the elements of A^{-1}, B^{-1}, C^{-1} and D^{-1}, respectively. We have

$$b^{-1}_{sw}=(-1)^{s+w}\binom{w}{s},\ c^{-1}_{ks}=(-1)^{k+s}\binom{q-1+s-k}{q-1},\ d^{-1}_{tk}=(-1)^{k+t}\binom{k}{t}. \tag{43}$$

The first and third formulae in (43) follow from identity (see [18])

$$\sum_{s=k}^{w}(-1)^s\binom{w}{s}\binom{s}{k}=(-1)^k\delta_{k,w}.$$

The second formula in (43) follows from identity (see [17])

$$\sum_{k=0}^{n}(-1)^k\binom{q}{k}\binom{q-1+n-k}{q-1}=\delta_{0,n}$$

as

$$\begin{aligned}\sum_{k=0}^{M}c_{sk}c^{-1}_{kw} &= (-1)^w\sum_{k=s}^{w}(-1)^k\binom{q}{k-s}\binom{q-1+w-k}{q-1} \\ &= (-1)^{w+s}\sum_{\ell=0}^{w-s}(-1)^\ell\binom{q}{\ell}\binom{q-1+w-s-k}{q-1} \\ &= (-1)^{w+s}\delta_{0,w-s}=\delta_{w,s}.\end{aligned}$$

Hence

$$
\begin{aligned}
a_{tw}^{-1} &= \sum_{k=0}^{p-1}(-1)^{k+t}\binom{k}{t}\sum_{s=0}^{p-1}(-1)^{k+s}\binom{q-1+s-k}{q-1}(-1)^{s+w}\binom{s}{w} \\
&= (-1)^{t+w}\sum_{s=0}^{p-1}\binom{s}{w}\sum_{k=0}^{p-1}\binom{k}{t}\binom{q-1+s-k}{q-1}.
\end{aligned}
$$

In view of identity

$$
\sum_{k=t}^{s}\binom{k}{t}\binom{q-1+s-k}{q-1} = \binom{q+s}{q+t}
$$

we rewrite a_{tw}^{-1} as follows

$$
a_{tw}^{-1} = (-1)^{t+w}\sum_{s=0}^{M}\binom{s}{w}\binom{q+s}{q+t}. \tag{44}
$$

References

[1] B. Adcock, Modified Fourier expansions: theory, construction and applications. PHD thesis, Trinity Hall, University of Cambridge, 2010.

[2] G.A. Baker, and P. Graves-Morris, Padė Approximants, Encyclopedia of mathematics and its applications. Vol. 59, 2nd ed., Cambridge Univ. Press, Cambridge, 1966.

[3] G. Baszenski, F.-J. Delvos, and M. Tasche. *A united approach to accelerating trigonometric expansions*, Comput. Math. Appl. **30**(36)(1995), 33–49.

[4] D. Batenkov, and Y. Yomdin. *Algebraic Fourier reconstruction of piecewise smooth functions,* Mathematics of Computation, **81**(277)(2012), 277–318.

[5] J. P. Boyd, *Acceleration of algebraically-converging Fourier series when the coefficients have series in powers of* $1/n$, J. Comp. Phys. **228**(5)(2009), 1404-1411.

[6] E.W. Cheney, Introduction to Approximation Theory, McGraw-Hill, New York, 1966.

[7] J. Geer, *Rational trigonometric approximations using Fourier series partial sums.* Journal of Scientific Computing, **10**(3)(1995), 325-356.

[8] A. Krylov, On approximate calculations, Lectures delivered in 1906 (in Russian), St. Petersburg 1907, Tipolitography of Birkenfeld.

[9] C. Lanczos, Discourse on Fourier series, Oliver and Boyd, Edinburgh, 1966.

[10] I. G. Macdonald, Symmetric Functions and Hall Polynomials, second ed. Oxford: Clarendon Press, 1995.

[11] A. Nersessian, and A. Poghosyan, *On a linear rational approximation on a finite interval*, Reports of NAS RA, **104**(3) (2004), 177–184.

[12] A. Nersessian, and A. Poghosyan, *On a rational linear approximation of Fourier series for smooth functions,* Journal of Scientific Computing, **26**(1)(2006), 111–125.

[13] A. Poghosyan, *On an autocorrection phenomenon of the Krylov-Gottlieb-Eckhoff method,* IMA Journal of Numerical Analysis, **31**(2)(2011), 512–527, doi:10.1093/imanum/drp043.

[14] A. Poghosyan, *On a Convergence of the rational-trigonometric-polynomial approximation realized by the roots of the associated Laguerre polynomials*, Submitted to the Journal of Contemporary Mathematical Analysis, 2012.

[15] A. Poghosyan, *On a convergence of the L_2-optimal rational approximation*, Submitted to Reports of NAS RA, 2012.

[16] I. I. Privalov, Fourier series (in Russian), Scientific-Technical State Publishing, Moscow-Leningrad, 1931.

[17] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series, Vol. 1: Elementary Functions. New York: Gordon and Breach, 1986.

[18] J. Riordan, Combinatorial Identities, Wiley, New York, 1979.

[19] G. P. Tolstov, Fourier series, (translated from the Russian (1950) by R. A. Silverman), Prentice-Hall, New Jersey, 1962.