Armenian Journal of Mathematics
Volume 14, Number 15, 2022, 1–16
https://doi.org/10.52737/18291163-2022.14.15-1-16

Existence of Solutions for a Fractional Boundary Value Problem at Resonance

A. S. Silva

Abstract. In this paper, we focus on the existence of solutions to a fractional boundary value problem at resonance. By constructing suitable operators, we establish an existence theorem upon the coincidence degree theory of Mawhin.

Key Words: Fractional Differential Equations, Boundary Value Problem, Caputo Derivative, Coincidence Degree Theory, Resonance
Mathematics Subject Classification 2010: 34A08, 34B15, 26A33, 47H11

Introduction

Fractional calculus theory has attracted many authors in the last decade. In particular, fractional differential equations, as a branch of fractional calculus, has proved to be better to describe certain problems than differential equations of integer order. The properties of genetics and memory make those equations, accompanied by initial or boundary conditions, important in many fields of engineering, physics, biology and mechanics (see [8,15,16]).

Many researchers have focused in the existence, uniqueness or multiplicity of solutions for such problems with fractional derivatives. In this regard, intersting results have been obtained by using some fixed point theorems, such as the Banach's fixed point theorem, the Schauder's fixed point theorem, the Krasnoselskii's fixed point theorem, etc. (see, e.g., [19–22, 25]). Other techniques can be used, in particular, for the study of solution of fractional boundary value problems at resonance. A boundary value problem is said to be resonance if the corresponding homogeneous boundary value problem has a nontrivial solution. One of the most important theory used in the study of those problems is the coincidence degree of Mawhin (see [9–11]).

In the last years, many researchers have studied the existence of solutions of resonant boundary value problems using coincidence degree theory (see, e.g., [1–7, 12–14, 17, 18, 23, 24]). In particular, in 2012, Jiang et al. [7]

investigated the existence of solutions for the fractional differential equation at resonance

$$\begin{cases} \mathcal{D}^{\alpha}_{0+}x(t) + f(t, x(t)) = 0, \quad t \in [0, 1], \\ x(0) = 0, \;\; x(1) = \beta x(\xi), \end{cases}$$

where $\mathcal{D}^{\alpha}_{0+}$ is the Riemann-Liouville fractional derivative, $1 < \alpha \leq 2$, $0 < \xi < 1$, $\beta\xi^{\alpha-1} = 1$, and $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathe̒odory conditions.

Hu et al. [6], in 2013, studied a two-point boundary value problem for fractional equation at resonance

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{0+}x(t) = f(t, x(t), x'(t)), \quad t \in [0, 1], \\ x(0) = 0, \;\; x'(0) = x'(1), \end{cases}$$

where $1 < \alpha \leq 2$, ${}^{C}\mathcal{D}^{\alpha}_{0+}$ is Caputo fractional derivative and $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory conditions.

In 2015, Guezane-Lakoud et al. [3] studied the existence of solutions for the two-point fractional boundary value problem at resonance

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{0+}x(t) = f(t, x(t), x'(t), x''(t)), \quad 0 < t < 1, \\ x(0) = x'(0) = 0, \;\; x''(0) = 2x(1), \end{cases}$$

where $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, $2 < \alpha < 3$ and ${}^{C}\mathcal{D}^{\alpha}_{0+}$ is the Caputo fractional derivative.

Benchohra et al. [2], in 2016, studied the nonlinear implicit value problem for Caputo fractional derivative of order $0 < \alpha < 1$

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{0+}x(t) = f(t, x(t), {}^{C}\mathcal{D}^{\alpha}_{0+}x(t)), \quad t \in [0, T], \; T > 0, \\ x(0) = x(T), \end{cases}$$

where f is a continuous function.

Hu and Zhang [5], in 2017, investigated the existence of positive solutions of the problem with Caputo fractional derivative

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{0+}x(t) = f(t, x(t)), \quad 0 < t < 1, \\ x(0) = x(1), x'(0) = x'(1), x''(0) = x''(1), \end{cases}$$

with $2 < \alpha < 3$, where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In 2020, Wang and Wu [18] studied the fractional differential equation with Riemann-Liouville fractional derivative and under resonant boundary conditions

$$\begin{cases} \mathcal{D}^{\alpha}_{0+}x(t) = f(t, x(t)), \quad 0 < t < 1, \\ x(0) = 0, \mathcal{D}^{\beta}_{0+}u(1) = \eta\mathcal{D}^{\beta}_{0+}u(\xi), \end{cases}$$

where $1 < \alpha < 2$, $0 < \beta < \alpha - 1$, $\eta > 0$, $0 < \xi < 1$ with $\eta\xi^{\alpha-\beta-1} = 1$.

Inspired by mentioned works, this paper is devoted to the study of the fractional boundary value problem (FBVP) with Caputo fractional derivative

$$\begin{cases} {}^C\mathcal{D}^{\alpha}_{a+}x(t) - f(t, x(t), x'(t), x''(t)) = 0, \quad t \in [a, b], \\ x(a) - \beta x'(a) = 0, \;\; x'(a) = x'(b), \;\; x''(a) = 0, \end{cases} \tag{1}$$

where $\beta \in \mathbb{R}$, $0 \leq a < b$, $2 < \alpha < 3$ and $f : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous. The problem will be transformed into the equation $Lx = Nx$, where L is a linear operator and N is a given operator between Banach spaces. It will be shown that, with the present choice of boundary conditions, L is non-invertible ($\mathrm{Ker}L \geq 1$), and thus, the problem is at resonance. Proving that L is a Fredholm operator with Fredholm index 0, it is applied coincidence degree due to Mawhin.

To the best of the author's knowledge, this problem, with the given conditions and in a general interval $[a, b]$, has not been studied before.

The paper is organized as follows: in the Section 1, essential results and definitions are presented as well as Mawhin's coincidence theory. In Section 2, the main theorem on the existence of solutions to the problem under study is obtained. Finally, in Section 3, an example is presented to illustrate the previous results.

1 Preliminaries

In this section, we introduce some notations, definitions and results which are used thorough this paper.

Definition 1 *The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of a function u is defined by*

$$I^{\alpha}_{a}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} u(s)\mathrm{d}s,$$

provided the right-hand side is pointwise defined on (a, ∞), where Γ is Euler Gamma function (given by $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}\mathrm{d}t,\ \alpha > 0$).

Definition 2 *The Caputo fractional derivative of order $\alpha > 0$ of a continuous function u is given by*

$${}^C\mathcal{D}^{\alpha}_{a}u(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{u^{(n)}(s)}{(t - s)^{\alpha-n+1}}\mathrm{d}s,$$

provided that the right-hand side is pointwise defined on (a, ∞), where $n \in \mathbb{N}$ is such that $n - 1 < \alpha < n$. If $\alpha \in \mathbb{N}$, then ${}^C\mathcal{D}^{\alpha}_{a}u(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha} u(t)$.

Lemma 1 *[8] Let $n-1<\alpha<n$, $n\in\mathbb{N}$. If $f\in C^n([a,b])$ or $f\in AC^n([a,b])$, then the following relation holds:*

$$(I_a^\alpha\, {}^C\mathcal{D}_a^\alpha f)(t)=f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k. \tag{2}$$

Lemma 2 *[8] Let $\alpha>0$ and $f\in C([a,b])$. Then*

$$({}^C\mathcal{D}_{a+}^\alpha I_a^\alpha f)(t)=f(t).$$

In what follows, some notions of operator theory are recalled. Let X and Y be two normed spaces.

Definition 3 *A linear mapping $L:\mathrm{dom}(L)\subset X\to Y$ is said to be a Fredholm operator with Fredholm index zero if*

i. $\mathrm{Im}L$ *is a closed subset of* Y,

ii. $\dim\mathrm{Ker}L=\mathrm{codimIm}L<\infty$.

Definition 4 *A linear operator $P:X\to X$ is said to be a projection if $P^2=P$. In this case, $I-P:X\to X$ is also a projection and* $\mathrm{Ker}P=\mathrm{Im}(I-P)$ *and* $\mathrm{Im}P=\mathrm{Ker}(I-P)$.

Finally, let us recall a useful and well-known theorem [8].

Theorem 1 (Arzelà-Ascoli) *Let (X,d) be a compact metric space. A set of functions F in $C(X)$ is relatively compact if and only if it is bounded and equicontinuous.*

In this paper, we denote $X=C^2([a,b])$ with the usual norm $\|x\|_X=\max_{t\in[a,b]}\{\|x\|_\infty+\|x'\|_\infty+\|x''\|_\infty\}$ and $Y=C([a,b])$ with the norm $\|y\|_Y=\|y\|_\infty$, where $\|x\|_\infty=\max_{t\in[a,b]}|x(t)|$. It is known that X and Y, endowed with such norms, are Banach spaces.

1.1 Mawhin's coincidence theory

Let X and Y be two Banach spaces, and consider a linear mapping $L:\mathrm{dom}(L)\subset X\to Y$. Assume that L is a Fredholm operator with index zero. This implies that there exist continuous projectors $P:X\to X$, $Q:Y\to Y$ such that

$$\mathrm{Im}P=\mathrm{Ker}L,\quad \mathrm{Ker}Q=\mathrm{Im}L,\quad X=\mathrm{Ker}L\oplus\mathrm{Ker}P,\quad Y=\mathrm{Im}L\oplus\mathrm{Im}Q.$$

Consider the restriction of L on $\mathrm{dom}L\cap\mathrm{Ker}P$. It follows that

$$L|_{\mathrm{dom}\,L\cap\ker P}:\mathrm{dom}\,L\cap\ker P\to\mathrm{Im}\,L$$

is an isomorphism. The inverse of $L|_{\mathrm{dom}\,L\cap\mathrm{Ker}P}$ is denoted by $K_P:\mathrm{Im}\,L\to\mathrm{dom}\,L\cap\mathrm{Ker}P$.

Definition 5 *Let Ω be an open bounded subset of X with* $\mathrm{dom}L \cap \Omega \neq \emptyset$. *A mapping N is said to be L-compact on $\overline{\Omega}$ if it satisfies*

- $QN(\overline{\Omega})$ *is bounded;*
- $K_p(I-Q)N : \overline{\Omega} \to X$ *is completely continuous.*

The existence of a solution of the equation $Lx = Nx$ will be shown using the following Mawhin's Theorem.

Theorem 2 *[10] Let $\Omega \subset X$ be open and bounded, L be a Fredholm operator of index zero and $N(\overline{\Omega})$ be L-compact. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ *for every* $x \in \partial\Omega \cap (\mathrm{dom}L \backslash \mathrm{Ker}L)$ *and* $\lambda \in (0,1)$;

(ii) $Nx \notin \mathrm{Im}\, L$ *for every* $x \in \mathrm{Ker}L \cap \partial\Omega$;

(iii) $\deg(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) \neq 0$, *where* $Q : Y \to Y$ *is a projection such that* $\mathrm{Im}L = \mathrm{Ker}Q$.

Then the equation $Lx = Nx$ has at least one solution in $\mathrm{dom}\, L \cap \overline{\Omega}$.

2 Main results

In this section, we apply Mawhin's coincidence theory to prove the existence of solution to the fractional boundary value problem (1).

Define the operator $L : \mathrm{dom}L \subset X \to Y$ by

$$(Lx)(t) = ({}^C\mathcal{D}^{\alpha}_{a+}x)(t), \quad t \in [a,b], \tag{3}$$

where

$$\mathrm{dom}L = \{x \in X : ({}^C\mathcal{D}^{\alpha}_{a+}x)(t) \in Y,\ x(a) = \beta x'(a),\ x'(a) = x'(b),\ x''(a) = 0\}.$$

Let $N : X \to Y$ be the operator

$$(Nx)(t) = f(t, x(t), x'(t), x''(t)), \quad t \in [a,b].$$

Thus, we can rewrite the fractional boundary value problem (1) in the form

$$Lx = Nx, \quad x \in \mathrm{dom}L.$$

Lemma 3 *Let operator L defined by* (3). *Then*

$$\mathrm{Ker}L = \{x \in X : x(t) = c_1(t - a + \beta),\ \ t \in [a,b]\}, \tag{4}$$

$$\mathrm{Im}L = \left\{ y \in Y : \int_a^b (b-s)^{\alpha-2} y(s)\mathrm{d}s = 0 \right\}. \tag{5}$$

Proof. It follows from Lemma 1 and $Lx = 0$ that

$$x(t) = c_0 + c_1(t-a) + c_2(t-a)^2.$$

According to initial condition $x''(a) = 0$, we obtain that $c_2 = 0$. From $x(a) = \beta x'(a)$, it follows $c_0 = \beta c_1$, and we conclude that

$$x(t) = \beta c_1 + c_1(t-a) = c_1(t-a+\beta),$$

where condition $x'(a) = x'(b)$ is verified. Thus, we get (4).

Suppose that $y \in \mathrm{Im}L$. Then, there exists $x \in \mathrm{dom}L$ such that $y = Lx \in Y$. Thus, we have

$$x(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\mathrm{d}s.$$

Considering the initial conditions $x''(a) = 0$ and $x(a) = \beta x'(a)$, we obtain

$$x(t) = \beta c_1 + c_1(t-a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\mathrm{d}s.$$

Since $x'(a) = x'(b)$, we conclude that

$$\int_a^b (b-s)^{\alpha-2} y(s)\mathrm{d}s = 0$$

and get

$$\mathrm{Im}L \subset \left\{ y \in Y : \int_a^b (b-s)^{\alpha-2} y(s)\mathrm{d}s = 0 \right\}.$$

On the other hand, suppose $y \in Y$ satisfies $\int_a^b (b-s)^{\alpha-2} y(s)\mathrm{d}s = 0$. Then $x(t) = (I_a^\alpha y)(t) \in \mathrm{dom}L$ and $({}^C\mathcal{D}_{a+}^\alpha x)(t) = y(t)$, thus $y \in \mathrm{Im}L$, and (5) follows. $\square$

Lemma 4 *Let L be defined as* (3)*. Then L is a Fredholm operator of Fredholm index zero, and the linear continuous projection operators $P : X \to X$ and $Q : Y \to Y$ can be defined as*

$$\begin{aligned} (Px)(t) &= x'(a)(t-a+\beta), \\ (Qy)(t) &= \frac{\alpha-1}{(b-a)^{\alpha-1}} \int_a^b (b-s)^{\alpha-2} y(s)\mathrm{d}s, \quad t \in [a,b]. \end{aligned}$$

Furthermore, the operator $K_p : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ written by

$$(K_p y)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\mathrm{d}s, \ \ t \in [a,b]$$

is the inverse of $L|_{\mathrm{dom}L \cap \mathrm{Ker}P}$.

Proof. Obviously, $\mathrm{Im}P = \mathrm{Ker}L$ and $P^2x = Px$. In fact,

$$(P^2x)(t) \quad = \quad [P(Px)](t) = x'(a)(t-a+\beta) = (Px)(t).$$

For $v \in \mathrm{Ker}L$, one has $(Pv)(t) = v(t)$, which shows that $v \in \mathrm{Im}P$. Conversely, for every $v \in \mathrm{Im}P$, there is $x \in X$ such that $v(t) = (Px)(t)$, and we conclude that $\mathrm{Ker}L = \mathrm{Im}P$. Moreover, $\ker L \cap \mathrm{Ker}P = \{0\}$, and thus,

$$X = \mathrm{Ker}P \oplus \mathrm{Ker}L.$$

For $y \in Y$, it holds

$$\begin{aligned}
(Q^2y)(t) &= Q\left(\frac{\alpha-1}{(b-a)^{\alpha-1}}\int_a^b (b-s)^{\alpha-2}y(s)\mathrm{d}s\right)\\
&= \frac{\alpha-1}{(b-a)^{\alpha-1}}\int_a^b (b-u)^{\alpha-2}\left(\frac{\alpha-1}{(b-a)^{\alpha-1}}\int_a^b (b-s)^{\alpha-2}y(s)\mathrm{d}s\right)\mathrm{d}u\\
&= \frac{\alpha-1}{(b-a)^{\alpha-1}}\int_a^b (b-s)^{\alpha-2}y(s)\mathrm{d}s\cdot\frac{\alpha-1}{(b-a)^{\alpha-1}}\int_a^b (b-u)^{\alpha-2}\mathrm{d}u\\
&= \ (Qy)(t).
\end{aligned}$$

Thus, the linear operator Q is a continuous projector. For $y \in \mathrm{Im}L$, one has $Qy = 0$, which shows that $y \in \mathrm{Ker}Q$, in fact, $\mathrm{Im}L = \mathrm{Ker}Q$. Moreover, $y - Qy \in \mathrm{Im}L$. It follows that $Y = \mathrm{Im}L + \mathrm{Im}Q$, and since $\mathrm{Im}Q \cap \mathrm{Im}L = \{0\}$, we have

$$Y = \mathrm{Im}L \oplus \mathrm{Im}Q.$$

Thus, $\mathrm{codim}\mathrm{Im}L = \dim \mathrm{Im}Q = \dim \mathrm{Ker}L = 1$, which shows that L is a Fredholm operator of index zero.

Let us prove that

$$K_p : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$$

is the inverse of $L|_{\mathrm{dom}L\cap\mathrm{Ker}P}$. Let $y \in \mathrm{Im}L$. We have that $(K_py)(a) = 0 = (K_py)'(a)$, and since $y \in \mathrm{Im}L$, we obtain $(K_py)'(b) = 0$, and thus, $(K_py)'(a) = (K_py)'(b)$. Moreover, $(K_py)''(a) = 0$. Hence, $K_p \in \mathrm{dom}L$. Furthermore,

$$P(K_py)(t) = (K_py)'(a)(t-a+\beta) = 0,$$

which shows that $K_py \in \mathrm{Ker}P$. Thus, the definition of K_p is reasonable.

For $y \in \mathrm{Im}L$, we have

$$(LK_py)(t) = ({}^C\mathcal{D}^\alpha_{a+}I^\alpha_a y)(t) = y(t). \tag{6}$$

For $x \in \mathrm{dom}L \cap \mathrm{Ker}P$, we have $x'(a) = x(a) = x'(b) = x''(a)$. By Lemma 1,

$$(K_pLx)(t) = (I^{\alpha}_a {}^C\mathcal{D}^\alpha_{a+}x)(t) = x(t) + c_1 + c_2(t-a) + c_3(t-a)^2,\ c_0, c_1, c_2 \in \mathbb{R},$$

which yields

$$(K_pLx)(t) = (I_a^{\alpha C}\mathcal{D}_{a+}^{\alpha}x)(t) = x(t). \tag{7}$$

Combining (6) and (7), we conclude that $K_p = (L|_{\mathrm{dom}L\cap\mathrm{Ker}P})^{-1}$, and the proof is complete. □

Lemma 5 *Let*

$$\eta = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(b-a)^{\alpha-2}}{\Gamma(\alpha-1)}. \tag{8}$$

Then

$$\|K_py\|_X \leq \eta\|y\|_\infty$$

for any $y \in \mathrm{Im}L$.

Proof. For each $y \in \mathrm{Im}L$ and $t \in [a,b]$,

$$\begin{aligned}\|K_py\|_X &= \|K_py\|_\infty + \|(K_py)'\|_\infty + \|(K_py)''\|_\infty \\ &= \|I_a^{\alpha}y\|_\infty + \|I_a^{\alpha-1}y\|_\infty + \|I_a^{\alpha-2}y\|_\infty \\ &\leq \left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(b-a)^{\alpha-2}}{\Gamma(\alpha-1)}\right)\|y\|_\infty.\end{aligned}$$

□

2.1 Existence of solutions

In order to prove the existence of solutions of the FBVP (1), consider the following conditions:

(H1) There exist nonnegative functions $\gamma_1, \gamma_2, \gamma_3$ and δ such that for all $(u,v,w) \in \mathbb{R}^3$,

$$|f(t,u,v,w)| \leq \gamma_1(t)|u(t)| + \gamma_2(t)|v(t)| + \gamma_3(t)|w(t)| + \delta(t), \quad t \in [a,b]$$

with $p_1 = \|\gamma_1\|_\infty$, $p_2 = \|\gamma_2\|_\infty$, $p_3 = \|\gamma_3\|_\infty$, $q = \|\delta\|_\infty$ such that $\eta \cdot \max_{t\in[a,b]}\{p_1,p_2,p_3\} < 1$ (with η as defined in (8)).

(H2) There exists a constant $R > 0$ such that for $x \in \mathrm{dom}L$, if $|x'(t)| > R$ for all $t \in [a,b]$, then

$$\int_a^b (b-s)^{\alpha-2} f(s,x(s),x'(s),x''(s))\mathrm{d}s \neq 0.$$

(H3) There exists a positive constant R^* such that for $c_1 \in \mathbb{R}$, if $|c_1| > R^*$ for $t \in [a,b]$, either

$$c_1 f(t, c_1(t-a+\beta), c_1, 0) > 0, \ t \in [a,b], \tag{9}$$

or

$$c_1 f(t, c_1(t-a+\beta), c_1, 0) < 0, \ t \in [a,b]. \tag{10}$$

Lemma 6 *Assume* $\Omega \subset X$ *is an open bounded subset such that* $\mathrm{dom}L \cap \overline{\Omega} \neq \emptyset$*, then* N *is* L*-compact on* $\overline{\Omega}$.

Proof. By the continuity of f, $QN(\overline{\Omega})$ and $K_p(I-Q)N(\overline{\Omega})$ are bounded. According to Arzelà-Ascoli theorem, it is sufficient to prove that $K_p(I-Q)N(\overline{\Omega}) \subset X$ is equicontinuous.

Since f is continuous, there exists a constant $M > 0$ such that

$$|((I-Q)Nx)(t)| \leq M$$

for $x \in \overline{\Omega}$, $t \in [a,b]$. Denote $K_{P,Q} = K_p(I-Q)N$. For $a \leq t_1 < t_2 \leq b$, $x \in \overline{\Omega}$, we have

$$\begin{aligned}
&|(K_{P,Q}x)(t_2) - (K_{P,Q}x)(t_1)| \\
= \ & \frac{1}{\Gamma(\alpha)} \left| \int_a^{t_2} (t_2 - s)^{\alpha-1}(I-Q)Nx(s)\mathrm{d}s - \right. \\
& \left. \int_a^{t_1} (t_1 - s)^{\alpha-1}(I-Q)Nx(s)\mathrm{d}s \right| \\
\leq \ & \frac{M}{\Gamma(\alpha)} \left[\int_a^{t_1} (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\mathrm{d}s + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\mathrm{d}s \right] \\
= \ & \frac{M}{\Gamma(\alpha+1)} \left[(t_2 - a)^{\alpha} - (t_1 - a)^{\alpha} \right],
\end{aligned}$$

$$\begin{aligned}
&|(K_{P,Q}x)'(t_2) - (K_{P,Q}x)'(t_1)| \\
= \ & \frac{1}{\Gamma(\alpha-1)} \left| \int_a^{t_2} (t_2 - s)^{\alpha-2}(I-Q)Nx(s)\mathrm{d}s \right. \\
& \left. - \int_a^{t_1} (t_1 - s)^{\alpha-2}(I-Q)Nx(s)\mathrm{d}s \right| \\
\leq \ & \frac{M}{\Gamma(\alpha-1)} \left[\int_a^{t_1} (t_2 - s)^{\alpha-2} - (t_1 - s)^{\alpha-2}\mathrm{d}s + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-2}\mathrm{d}s \right] \\
= \ & \frac{M}{\Gamma(\alpha)} \left[(t_2 - a)^{\alpha-1} - (t_1 - a)^{\alpha-1} \right],
\end{aligned}$$

and

$$\begin{aligned}
&|(K_{P,Q}x)''(t_2) - (K_{P,Q}x)''(t_1)| \\
= \ & \frac{1}{\Gamma(\alpha-2)} \left| \int_a^{t_2} (t_2 - s)^{\alpha-3}(I-Q)Nx(s)\mathrm{d}s \right. \\
& \left. - \int_a^{t_1} (t_1 - s)^{\alpha-3}(I-Q)Nx(s)\mathrm{d}s \right| \\
\leq \ & \frac{M}{\Gamma(\alpha-2)} \left[\int_a^{t_1} (t_1 - s)^{\alpha-3} - (t_2 - s)^{\alpha-3}\mathrm{d}s + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-3}\mathrm{d}s \right]
\end{aligned}$$

$$\leq \frac{M}{\Gamma(\alpha-1)}\left[(t_2-a)^{\alpha-2}-(t_1-a)^{\alpha-2}+2(t_2-t_1)^{\alpha-2}\right].$$

Thus, when $t_1 \to t_1$, we have

$$\begin{aligned}
&|(K_{P,Q}x)(t_2)-(K_{P,Q}x)(t_1)| \to 0,\\
&|(K_{P,Q}x)'(t_2)-(K_{P,Q}x)'(t_1)| \to 0,\\
&|(K_{P,Q}x)''(t_2)-(K_{P,Q}x)''(t_1)| \to 0.
\end{aligned}$$

Consequently, it follows that $(K_{P,Q}x)(\overline{\Omega}) \subset C([a,b])$, $(K_{P,Q}x)'(\overline{\Omega}) \subset C([a,b])$ and $(K_{P,Q}x)''(\overline{\Omega}) \subset C([a,b])$ are equicontinuous. Thus, $K_{P,Q}x : \overline{\Omega} \to X$ is compact, and we conclude that N is L-compact. $\square$

Lemma 7 *Let $\Omega_1 = \{x \in \mathrm{dom}L\backslash\mathrm{Ker}L : Lx = \lambda Nx, \quad \lambda \in (0,1)\}$. If conditions (H1) and (H2) hold, then Ω_1 is bounded.*

Proof. Let $x \in \Omega_1$, then $x \in \mathrm{dom}L\backslash\mathrm{Ker}L$ and $Lx = \lambda Nx$, hence $\lambda \neq 0$ and $Nx \in \mathrm{Im}L = \mathrm{Ker}Q \subset Y$. Thus, $Q(Nx)(t) = 0$, which means that

$$\int_a^b (b-s)^{\alpha-2} f(s, x(s), x'(s), x''(s))\mathrm{d}s = 0.$$

From hypothesis (H2), we conclude that $|x'(t)| \leq R$ for all $t \in [a,b]$, and thus, in particular, $|x'(a)| \leq R$. Furthermore, for $x \in \Omega_1$, we have

$$\begin{aligned}
\|Px\|_X &= \max_{t\in[a,b]} \{|Px(t)| + |(Px)'(t)| + |(Px)''(t)|\}\\
&\leq |x'(a)(t-a+\beta)| + |x'(a)|\\
&\leq |x'(a)|(b-a+|\beta|) + |x'(a)|\\
&\leq (b-a+|\beta|+1)R
\end{aligned}$$

and

$$\begin{aligned}
\|(I-P)x\|_X &= \|K_pL(I-P)x\|_X\\
&\leq \eta\|L(I-P)x\|_\infty\\
&= \eta\|Lx\|_\infty\\
&\leq \eta\|Nx\|_\infty.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|x\|_X &= \|x-Px+Px\|_X\\
&\leq \|Px\|_X + \|(I-P)x\|_X\\
&\leq (b-a+|\beta|+1)R + \eta\|Nx\|_\infty. \qquad (11)
\end{aligned}$$

with η defined in (8). Under condition (H1), for each $x \in \Omega_1$, it follows that

$$\begin{aligned} \|Nx\|_\infty &= \max_{t\in[a,b]} |f(t,x(t),x'(t),x''(t))| \\ &\le p(|x(t)|+|x'(t)|+|x''(t)|)+q \\ &= p\|x\|_X + q, \end{aligned} \tag{12}$$

where $p = \max_{t\in[a,b]}\{p_1,p_2,p_3\}$. From (11) and (12), recalling that $\eta p < 1$ (cf. (H1)), one obtains

$$\|x\|_X \le \frac{(b-a+|\beta|+1)R+\eta q}{1-\eta p},$$

which shows that Ω_1 is bounded. $\square$

Lemma 8 *Suppose that (H2) holds. Then the set* $\Omega_2 = \{x \in \mathrm{Ker}L : Nx \in \mathrm{Im}L\}$ *is bounded.*

Proof. Assume that $x \in \Omega_2$. Then, $Nx \in \mathrm{Im}L$ and $x(t) = c_1(t-a+\beta)$. Therefore,

$$\int_a^b (b-s)^{\alpha-2}(Nx)(s)\mathrm{d}s = \int_a^b (b-s)^{\alpha-2} f(s,c_1(s-a+\beta),c_1,0)\mathrm{d}s = 0$$

and thus, from hypothesis (H2), $|x'(t)| = |c_1| \le R$ for $t\in[a,b]$. This leads to

$$\|x\|_X = \sup_{t\in[a,b]}\{|c_1(t-a+\beta)|+|c_1|\} \le (b-a+|\beta|+1)R,$$

and we conclude that Ω_2 is bounded. $\square$

Lemma 9 *Suppose that (H2) and* (9) *hold. Then the set* $\Omega_3 = \{x \in \mathrm{Ker}L : \lambda x + (1-\lambda)QNx = 0,\ \lambda\in[0,1]\}$ *is bounded.*

Proof. For $x \in \Omega_3$, we have $x(t) = c_1(t-a+\beta)$ and

$$\lambda c_1(t-a+\beta)+(1-\lambda)\frac{\alpha-1}{(b-a)^{\alpha-1}}\int_a^b (b-s)^{\alpha-2} f(s,c_1(s-a+\beta),c_1,0)\mathrm{d}s = 0. \tag{13}$$

If $\lambda = 0$, then $|c_1| \le R^*$ according to (H2). If $\lambda \in (0,1]$, we also obtains $|c_1| < R^*$. Otherwise, according to the first part of (H3), we have

$$c_1^2(t-a+\beta)+(1-\lambda)\frac{\alpha-1}{(b-a)^{\alpha-1}}\int_a^b (b-s)^{\alpha-2} c_1 f(s,c_1(s-a+\beta),c_1,0)\mathrm{d}s > 0$$

which contradicts (13). Therefore, Ω_3 is bounded. $\square$

With identical arguments, we obtain the following result.

Lemma 10 *Suppose that (H2) and* (10) *hold. Then the set* $\Omega_3' = \{x \in \mathrm{Ker}L : -\lambda x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}$ *is bounded.*

Once we have gathered the essential results, we can prove the following theorem, which is the main goal of this paper.

Theorem 3 *Let* $f : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ *be continuous, and suppose conditions (H1), (H2) and (H3) are verified. Then the fractional boundary value problem* (1) *has at least one solution in* X.

Proof. Let Ω be a bounded open subset of Y such that $\bigcup_{i=1}^{3} \overline{\Omega}_i \subset \Omega$. It follows from Lemma 6 that N is L-compact on $\overline{\Omega}$, and by Lemma 4, L is a Fredholm operator with index 0. By Lemmas 7, 8, 9 and 10, we get the following:

1. $Lx \neq \lambda Nx$ for every $x \in \partial\Omega \cap (\mathrm{dom}L \backslash \mathrm{Ker}L)$ and $\lambda \in (0,1)$;

2. $Nx \notin \mathrm{Im}L$ for every $x \in \mathrm{Ker}L \cap \partial\Omega$;

3. Let $H(x,\lambda) = \pm\lambda x + (1-\lambda)QNx$. We know that $H(x,\lambda) \neq 0$ for $x \in \mathrm{Ker}L \cap \partial\Omega$. Therefore, by homotopy property of degree, we obtain

$$\begin{aligned} \deg(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) &= \deg(H(\cdot,0), \mathrm{Ker}L \cap \partial\Omega, 0) \\ &= \deg(H(\cdot,1), \mathrm{Ker}L \cap \partial\Omega, 0) \\ &= \deg(\pm I, \mathrm{Ker}L \cap \partial\Omega, 0) \neq 0, \end{aligned}$$

   where I is the identity operator.

From 1–3, according to Theorem 2, it follows that $Lx = Nx$ has at least one solution in $\mathrm{dom}L \cap \overline{\Omega}$. Therefore, fractional boundary value problem (1) has at least one solution in X, and the proof is complete. $\square$

3 Example

Consider the following fractional boundary value

$$\begin{cases} {}^{C}\mathcal{D}_{1^+}^{\frac{5}{2}} x(t) = \dfrac{1}{4}\sin x(t) + \dfrac{1}{5}x'(t) + \dfrac{1}{10}e^{-|x''(t)|}, \quad t \in [1,2], \\ x(1) = \frac{1}{2}x'(1),\ x'(1) = x'(2),\ x''(1) = 0, \end{cases} \tag{14}$$

where $\alpha = 5/2$, $\beta = 1/2$ and

$$f(t, x(t), x'(t), x''(t)) = \frac{1}{4}\sin x(t) + \frac{1}{5}x'(t) + \frac{1}{10}e^{-|x''(t)|}$$

is a continuous function. It follows that

$$|f(t,u,v,w)| \leq \frac{1}{4}|u(t)| + \frac{1}{5}|v(t)| + \frac{1}{10}|w(t)|,$$

with $p_1 = 1/4$, $p_2 = 1/5$, $p_3 = 1/10$ and $q = 0$. It follows that $p = \max\{p_1, p_2, p_2\} = 1/4$. Moreover, $\eta = 58/(15\sqrt{\pi})$ and consequently, $p\eta < 1$. Thus, condition (H1) is verified.

Let $R = 2$, and for any $x \in \mathrm{dom}L$, assume $|x'(t)| > R$ holds for $t \in [1,2]$. From the continuity of x', either $x'(t) > R$ or $x'(t) < -R$ for $t \in [1,2]$.

If $x'(t) > 2$, one has

$$\begin{aligned} &\int_1^2 (2-s)^{3/2}\left[\frac{1}{4}\sin x(t) + \frac{1}{5}x'(t) + \frac{1}{10}e^{-|x''(t)|}\right]\mathrm{d}s \\ > \ &\left(-\frac{1}{4}+\frac{2}{5}\right)\int_1^2 (2-s)^{3/2}\mathrm{d}s = \frac{3}{50} > 0. \end{aligned}$$

If $x'(t) < -2$, one has

$$\begin{aligned} &\int_1^2 (2-s)^{3/2}\left[\frac{1}{4}\sin x(t) + \frac{1}{5}x'(t) + \frac{1}{10}e^{-|x''(t)|}\right]\mathrm{d}s \\ < \ &\left(\frac{1}{4}-\frac{2}{5}+\frac{1}{10}\right)\int_1^2 (2-s)^{3/2}\mathrm{d}s = -\frac{1}{50} < 0 \end{aligned}$$

Thus, for $|x'(t)| > 2$,

$$\int_1^2 (2-s)^{3/2} f(s, u(s), u'(s), u''(s))\mathrm{d}s \neq 0$$

and condition (H2) is verified.

Finally, we observe that

$$f\left(t, c_1\left(t-\frac{1}{2}\right), c_1, 0\right) = \frac{1}{4}\sin\left(c_1\left(t-\frac{1}{2}\right)\right) + \frac{1}{5}c_1 + \frac{1}{10}.$$

Take $R^* = 1$ and assume $|c_1| > 1$. Thus, if $c_1 > 1$,

$$c_1 f\left(t, c_1\left(t-\frac{1}{2}\right), c_1, 0\right) > -\frac{1}{4} + \frac{1}{5} + \frac{1}{10} = \frac{1}{20} > 0$$

and if $c_1 < -1$, one has

$$c_1 f\left(t, c_1\left(t-\frac{1}{2}\right), c_1, 0\right) < -\left(\frac{1}{4} - \frac{1}{5} + \frac{1}{10}\right) = -\frac{3}{20} < 0.$$

Therefore, condition (H3) is verified.

It follows from Theorem 3 that fractional boundary value problem (14) has at least one solution.

Acknowledgments

The author thanks the referee for constructive comments and recommendations which helped to improve the readability and quality of the paper.

This work is supported by the Center for Research and Development in Mathematics and Applications (CIDMA) through the Portuguese Foundation for Science and Technology (FCT - Fundação para a Ciência e a Tecnologia), reference UIDB/04106/2020, and by national funds (OE), through FCT, I.P., in the scope of the framework contract foreseen in the numbers 4, 5 and 6 of the article 23, of the Decree-Law 57/2016, of August 29, changed by Law 57/2017, of July 19.

References

[1] Z. Bai, W. Li and W. Ge, Existence and multiplicity of solutions for four-point boundary value problems at resonance. Nonlinear Anal., **60** (2005), no. 6, pp. 1151–1162. https://doi.org/10.1016/j.na.2004.10.013

[2] M. Benchohra, S. Bouriah and J. Graef, Nonlinear implicit differential equations of fractional order at resonance. Electron. J. Differ. Equ., **324** (2016), pp. 1–10.

[3] A. Guezane-Lakoud, S. Kouachi and F. Ellaggoune, Two point fractional boundary value problem at resonance. Int. J. Appl. Math., **33** (2015), no. 3-4, pp. 425–434. https://doi.org/10.14317/jami.2015.425

[4] B.B. He, Existence of solutions to fractional differential equations with three-point boundary conditions at resonance with general conditions. Fract. Calc. Appl., **9** (2018), pp. 120–136.

[5] L. Hu and S. Zhang, Existence of positive solutions to a periodic boundary value problems for nonlinear fractional differential equations at resonance. J. Fract. Calc. Appl., **8** (2017), no. 2, pp. 19–31.

[6] Z. Hu, W. Liu and T. Chen, Two-point boundary value problems for fractional differential equations at resonance. Bull. Malaysian Math. Sci. Soc. 2, **36** (2013), pp. 747–755.

[7] W. Jiang, X. Huang and B. Wang, Boundary value problems of fractional differential equations at resonance. Phys. Procedia, **25** (2012), pp. 965–972. https://doi.org/10.1016/j.phpro.2012.03.185

[8] A.A. Kilbas, H.M. Srivastava and J.J Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2016.

[9] J. Mawhin, Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces. J. Diff. Equations, **12** (1972), no. 3, pp. 610–636. https://doi.org/10.1016/0022-0396(72)90028-9

[10] J. Mawhin, Topological degree methods in nonlinear boundary value problems. In *NSF-CBMS Regional Conference Series in Mathematics*, **40**, Amer. Math. Soc., Providence, RI, 1979. https://doi.org/10.1090/cbms/040

[11] J. Mawhin, Topological degree and boundary value problems for nonlinear differential equations. In M. Furi, P. Zecca (eds) *Topological Methods for Ordinary Differential Equations, Lecture Notes in Mathematics*, **1537**, Springer, Berlin, Heidelberg, 1993. https://doi.org/10.1007/bfb0085076

[12] P.D. Phung and H.B. Minh, Existence of solutions to fractional boundary value problems at resonance in Hilbert spaces. Bound. Value Probl., **105** (2017), pp. 1–20. https://doi.org/10.1186/s13661-017-0836-3

[13] P.D. Phung and L.X. Truong, On the existence of a three point boundary value problem at resonance in $\mathbb{R}^n$. J. Math. Anal. Appl., **416** (2014), pp. 522–533.

[14] P.D. Phung and L.X. Truong, Existence of solutions to three-point boundary-value problems at resonance. Electron. J. Diff. Eqns., **115** (2016), pp. 1–13.

[15] I. Podlubny, *Fractional Differential Equations, Mathematics in Sciences and Applications*, Academic Press, New York, 1999.

[16] S.G. Samko, A.A. Kilbas and O.I. Mirichev, *Fractional Integral and Derivatives (Theory and Applications)*. Gordon and Breach, Switzerland, 1993.

[17] S. Song and Y. Cui, Existence of solutions for integral boundary value problems of mixed fractional differential equations under resonance. Bound. Value Probl., **23** (2020), pp. 1–12. https://doi.org/10.1186/s13661-020-01332-5

[18] Y. Wang and Y. Wu, Positive solutions of fractional differential equation boundary value problems at resonance, J. Appl. Anal. Comput., **10** (2020), pp. 2459–2475.

[19] C. Yang and C. Zhai, Uniqueness of positive solutions for a fractional differential equation via a fixed point theorem of a sum operator. Electron J. Differ. Equ., **70** (2021), pp. 1–8.

[20] C. Zhai and M. Hao, Fixed point theorems for mixed monotone operators with perturbation and applications to fractional differential equation boundary value problems. Nonlinear Anal., **75** (2012), no. 4, pp. 2542–2551. https://doi.org/10.1016/j.na.2011.10.048

[21] C. Zhai and L. Xu, Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter. Commun. Nonlinear Sci. Numer. Simul., **19** (2014), no. 8, pp. 2820–2827. https://doi.org/10.1016/j.cnsns.2014.01.003

[22] C. Zhai, W. Yan and C. Yang, A sum operator method for the existence and uniqueness of positive solutions to Riemann–Liouville fractional differential equation boundary value problems. Commun. Nonlinear Sci. Numer. Simul., **18** (2013), no. 4, pp. 858–866. https://doi.org/10.1016/j.cnsns.2012.08.037

[23] Y.H. Zhang, Z.B. Bai and T.T. Feng, Existence results for a coupled system of nonlinear fractional three-point boundary value problems at resonance. Comput. Math. Appl., **61** (2011), no. 4, pp. 1032–1047. https://doi.org/10.1016/j.camwa.2010.12.053

[24] H.C. Zhou, F.D. Ge and C.H. Kou, Existence of solutions to fractional differential equations with multi-point boundary conditions at resonance in Hilbert spaces. Electron. J. Differ. Equ., **61** (2016), pp. 1–16.

[25] Y. Zhou, F. Jiao and J. Li, Existence and uniqueness for fractional neutral differential equations with infinite delay. Nonlinear Anal., **71** (2009), no. 7-8, pp. 3249–3256. https://doi.org/10.1016/j.na.2009.01.202

Anabela S. Silva
Center for Research and Development
in Mathematics and Applications (CIDMA),
University of Aveiro
Campus Universitário de Santiago,
3810-193 Aveiro, Portugal.
anabela.silva@ua.pt

Please, cite to this paper as published in
Armen. J. Math., V. **14**, N. 15(2022), pp. 1–16
https://doi.org/10.52737/18291163-2022.14.15-1-16