Armenian Journal of Mathematics
Volume 15, Number 1, 2023, 1–15
https://doi.org/10.52737/18291163-2023.15.1-1-15

On the Minimal Annulus of Triangles and Parallelograms

S. Vassallo

Abstract. Sharp upper and lower bounds for the isoperimetric deficit of triangles or parallelograms with the minimal annulus of radii R and r are given.

Key Words: Isoperimetric Inequality, Minimal Annulus, Bonnesen Inequality, Favard Inequality
Mathematics Subject Classification 2010: 52B60, 51M25, 52A10, 52A40, 52A38

Introduction and Notations

The classical isoperimetric inequality for a simple closed plane curve C with perimeter L enclosing a domain of area A states that

$$L^2 - 4\pi A \geq 0 \tag{1}$$

with equality if and only if C is a circle.

If we define the isoperimetric deficit by

$$\Delta(C) = L^2 - 4\pi A,$$

the isoperimetric inequality becomes $\Delta(C) \geq 0$ with equality if and only if C is a circle. There are inequalities stronger than (1). For example, Bonnesen [4] (see also [5]) proved that

$$L^2 - 4\pi A \geq \pi^2 (R_c - r_i)^2 \tag{2}$$

where R_c and r_i are, respectively, the radius of the circumscribed circle (i.e., the circle of minimum radius that contains C) and of the inscribed circle (i.e., one of the circles of maximum radius contained in C).

In the twenties of the past century, Bonnesen found many inequalities like (2), i.e., inequalities of the form

$$L^2 - 4\pi A \geq B, \tag{3}$$

where the quantity B at the right hand side has the following properties:

1. $B \geq 0$;
2. $B = 0$ only if the convex body is a circle;
3. B has a geometric significance.

Such inequalities are called *Bonnesen-style inequalities*. Many Bonnesen type inequalities have been proved, see, for example, [1,8,36]. For a detailed discussion on this subject, refer to Osserman's paper [26].

If C is a closed, convex curve in the plane, a circular annulus, bounded by two concentric circles with radii r and R, where $r \leq R$, is said to *enclose* C if no point of C is out of the circle with radius R, while no point of C is inside the circle with radius r. The annulus is said to *bi-enclose* C if C passes at least four times between the outer circle and the inner circle of the annulus, i.e., there are, at least, four points P_1, P_2, P_3, P_4 on C in this order (clockwise or counterclockwise) such that

- the inner circumference touches the curve in, at least, the two points P_1 and P_3;
- the outer circumference touches the curve in, at least, the two points P_2 and P_4.

In [4], Bonnesen proved that there exists a unique circular annulus enclosing C with minimal difference of the radii; this annulus, called the *minimal circular annulus* of C, is also the unique annulus that bi-encloses C. Also he proved the following inequality, which is sharper than (2)

$$L^2 - 4\pi A \geq 4\pi(R - r)^2, \tag{4}$$

where R and r (with $r \leq R$) are the radii of the minimal annulus, sharper than (2), with equality only for circles. In the same paper, Bonnesen found the curve with minimal deficit. Moreover, Nagy [24] proved that the minimal circular annulus of C has also minimal area.

There are also papers concerning the upper bound of the isoperimetric deficit. Favard in [9] (see also [10]) shows that for a curve C with minimal annulus of radii (r, R) the following inequality holds:

$$L^2 - 4\pi A \leq 4\pi^2 R(R - r) \tag{5}$$

with equality if and only if C is a circle. In the same paper, Favard shows also that for a given minimal annulus, the maximum of the deficit is attained for a polygon circumscribed to the small circle and with all the vertices, except at most one, on the big circle (if $\arccos(r/R) = \pi/k$ with $k \in \mathbb{N},\ k \neq 0$, the maximum deficit is attained by the regular polygon of k sides circumscribed to the small circle and inscribed in the big circle).

Bottema [6] shows that if C has continuous curvature radius (i.e., it is smooth and does not contain segments),

$$L^2 - 4\pi A \leq \pi^2(\rho_M - \rho_m)^2$$

where ρ_M and ρ_m are the maximum and the minimum of the curvature radius, and the equality holds if and only if C is a circle.

In [35], many upper bounds for the isoperimetric deficit have been found, for example,

$$L^2 - 4\pi A \leq 2\pi L(R_c - r_i),$$

that is shown also in [3].

Many authors consider particular classes of convex bodies, and they obtain stronger results than inequality (1). For example, for n-sided polygons, the following isoperimetric inequality holds:

$$L^2 - 4n \tan\frac{\pi}{n} A \geq 0 \tag{6}$$

with equality if and only if the polygon is regular (see [19, 20, 22, 25]). Improvements to the above inequality are shown, for example, in [7, 18, 34].

For triangles, Rabinowitz [31] obtained many Bonnessen-style inequalities, improving the general ones given in [26].

Let us note that isoperimetric problems have a very extensive literature, and there are results also in the space $\mathbb{R}^n$, in Minkowski spaces, for star bodies, and so on (see, for example, [3, 11–14, 17, 21, 23, 29, 32, 33]). On the minimal annulus and generalizations we refer also to [2, 15–17, 27, 28, 30].

In this paper, upper and lower bounds for the isoperimetric deficit in the form of inequality (6) for parallelograms,

$$\Delta(C) = L^2 - 16A$$

and for triangles,

$$\Delta(C) = L^2 - 12\sqrt{3}A$$

are found, using the minimal annulus. The bounds obtained are sharp, but only the upper bound has an explicit formula.

1 Parallelograms

Due to the symmetry, the center of the minimal annulus is the center of the parallelogram, and the two circles of the annulus are the inscribed and the circumscribed circles.

The isoperimetric inequality for quadrilaterals is

$$L^2 - 16A \geq 0$$

with equality for squares.

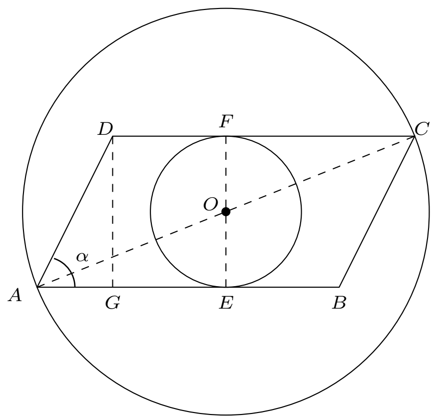

Figure 1: A parallelogram with minimal annulus of radii (r, R)

In Fig. 1, $\overline{OC} = \overline{OA} = R$, $\overline{OE} = \overline{OF} = r$. It is obvious that every parallelogram in this family is uniquely determined by the angle α, or, equivalently, by the length x of the segment AG.

We can suppose that $\overline{AB} \geq \overline{BC}$. We have

$$x := \overline{AG} = 2r\cot(\alpha), \quad a(x) := \overline{AB} = 2\sqrt{R^2 - r^2} - x$$
$$b(x) := \overline{AD} = \sqrt{4r^2 + x^2},$$

and, since $\overline{AB} \geq \overline{BC}$, x satisfies

$$0 \leq x \leq \frac{R^2 - 2r^2}{\sqrt{R^2 - r^2}}.$$

Let us write for simplicity $\overline{x} = (R^2 - 2r^2)/\sqrt{R^2 - r^2}$. For $x = 0$, the parallelogram is a rectangle, the points D and B are on the circumscribed circle and $\alpha = \pi/2$. For $x = \overline{x}$ the parallelogram is a rhombus and the segments AD and BC are tangent to the inscribed circle. Let us note that $r \leq R/\sqrt{2}$. In the following, we denote by Q the expression $\sqrt{R^2 - r^2}$.

Proposition 1 *Among all the parallelograms with minimal annulus of radii (r, R) (see Fig. 1), the parallelogram with maximal isoperimetric deficit is the rhombus with side AB of length $\overline{AB} = R^2/\sqrt{R^2 - r^2}$; the parallelogram with minimal isoperimetric deficit is the parallelogram with $\overline{AB} = 2\sqrt{R^2 - r^2} - \lambda$ and $\overline{AC} = \sqrt{4r^2 + \lambda^2}$, where λ is the unique real solution to the third degree equation*

$$t^3 + (r - Q)\,t^2 + r\,(4r - Q)\,t + r\left(R^2 + 2r^2 - 4rQ\right) = 0 \tag{7}$$

satisfying

$$0 \leq \lambda \leq \overline{x}.$$

The extreme cases are shown in Fig. 2.

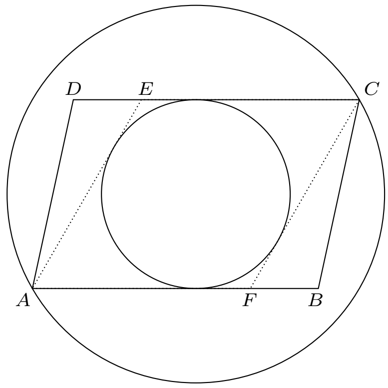

Figure 2: The parallelogram $ABCD$ with minimal deficit and the rhombus $AFCE$ with maximal deficit.

Proof. Let us write the isoperimetric deficit as a function of x:

$$\Delta(x) = L^2(x) - 16A(x) = 4\left(\sqrt{4r^2 + x^2} + 2Q - x\right)^2 - 32r\,(2Q - x)\,.$$

Since

$$\Delta''(x) = 2(L'(x))^2 + 2L(x)L''(x) \quad \text{and} \quad L''(x) = \frac{8r^2}{(4r^2 + x^2)^{3/2}} > 0,$$

the function $\Delta(x)$ is convex.

Since

$$\Delta'(0) = 16\,(r - Q) < 0 \quad \text{and} \quad \Delta'(\overline{x}) = \frac{32r\left(\sqrt{R^2 - r^2} - r\right)}{\sqrt{R^2 - r^2}} > 0,$$

the isoperimetric deficit has two local maxima in $x = 16R^2\left(R^2 - 2r^2\right)^2$ and in $x = 0$ and a unique global minimum where $\Delta'(x)$ vanishes.

Now

$$\Delta(0) = 16\left(R^2 - 2rQ\right) < \frac{16R^2}{Q^2}\left(R^2 - 2rQ\right) = \Delta\left(\overline{x}\right),$$

therefore, the global maximum for the isoperimetric deficit is assumed for the rhombus.

It is possible to show with elementary arguments that equations (7) and $\Delta'(x) = 0$ are equivalent for $0 \leq x \leq \overline{x}$, and the solution of equation (7) is then the minimum point of the isoperimetric deficit. $\square$

Using the previous result, it is easy to give an exact upper bound for the isoperimetric deficit in terms of the radii R and r of the minimal annulus:

$$\Delta(x) \leq \Delta\left(\overline{x}\right) = \frac{16R^2\left(R^2 - 2r^2\right)^2}{\left(R^2 - r^2\right)\left(R^2 + 2r\sqrt{R^2 - r^2}\right)}.$$

On the contrary, it is not so easy (but it is possible) to write the exact lower bound.

Example 1 *If $R = 1$ and $r = 0.5$ the minimal deficit is around* 1.53155166 *and it is attained for $x \approx 0.2170120446$; the maximum deficit is approximately* 2.85812471*.*

If $R = 1$ and $r = 0.2$ the minimal deficit is around 8.02821815 *and it is attained for $x \approx 0.3228270469$; the maximum deficit is approximately* 10.13469401*.*

In Fig. 3, the maximum and the minimum isoperimetric deficit Δ are drawn as functions of r/R.

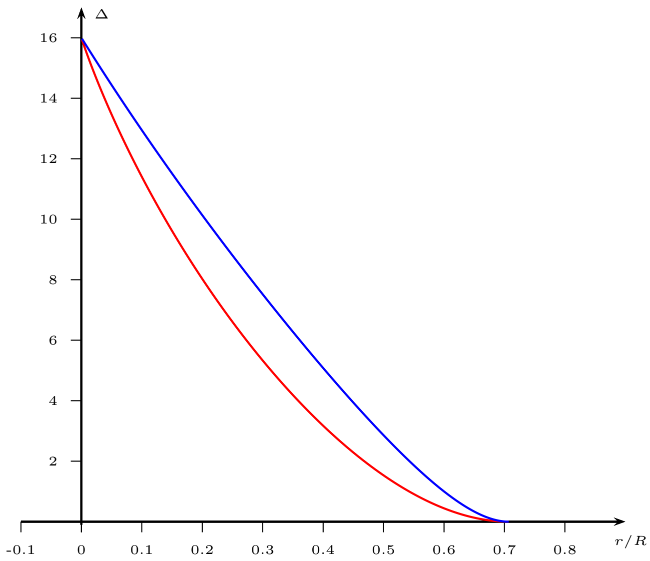

Figure 3: The maximum and the minimum isoperimetric deficit

The following result gives a lower and an upper bounds that are not sharp but more meaningful.

Corollary 1 *Let us consider a parallelogram, and let (r, R) be the radii of its minimal annulus. Then*

$$\begin{aligned} L^2 - 16A &\geq \frac{16\left(R^2 - 2r^2\right)^3}{\sqrt{R^2 - r^2}\left((R^2 + 4r^2)\sqrt{R^2 - r^2} + 2r(2R^2 - r^2)\right)} \\ &\geq \frac{16\left(R^2 - 2r^2\right)^3}{3\sqrt{2}R^3\sqrt{R^2 - r^2}} \\ &\geq \frac{16\left(R^2 - 2r^2\right)^3}{3\sqrt{2}R^4} \\ &\geq \frac{16\left(R - r\sqrt{2}\right)^3}{3R\sqrt{2}} \end{aligned} \tag{8}$$

and

$$\begin{aligned} L^2 - 16A &\leq 16\frac{\left(R^2-2r^2\right)^2}{R^2} \\ &\leq 64\left(R - r\sqrt{2}\right)^2. \end{aligned} \tag{9}$$

In the above inequalities, the equality sign holds only if the parallelogram is a square.

Proof. As proved in Proposition 1, the function $\Delta(x)$ is convex, and thus its graph lies above the tangent lines at $x = 0$ and $x = \overline{x}$, which equations are, respectively,

$$\begin{aligned} y &= 16\,(r-Q)\,x + 16\left(R^2 - 2rQ\right), \\ y &= \frac{32r\,(Q-r)}{Q}\left(x - \frac{R^2-2r^2}{Q}\right) + \frac{16R^2\,(R^2-2rQ)}{Q^2}. \end{aligned}$$

The ordinate of the intersection point of these two tangent lines is

$$\frac{16\left(R^2-2r^2\right)^3}{Q\left((R^2+4r^2)Q + 2r(2R^2-r^2)\right)},$$

and then the first inequality in (8) follows. All other inequalities in (8) are simple consequences of the first one.

To prove inequalities (9) we note that the denominator of $\Delta(\overline{x})$ is $h(t) = (1-t^2)\left(1+2t\sqrt{1-t^2}\right)$ where $t = r/R$. The function $h(t)$ takes its minima for $t = 0$ and $t = 1/\sqrt{2}$, and then $h(t) \geq 1$ which proves the first inequality. The following are similar. □

2 Triangles

Bonnessen [4] has shown that a triangle ABC with $\overline{BC} \geq \overline{AC} \geq \overline{AB}$ has a minimal annulus of radii (r, R) if the sides BC and AC are tangent to the interior circle and the vertices B and C are on the exterior circle (see Fig. 4). In Fig. 4, $\overline{OC} = \overline{OB} = R$, $\overline{OG} = \overline{OF} = r$, $\alpha = \arcsin(r/R)$, and $\overline{BC} = 2\sqrt{R^2 - r^2}$. It is obvious that every triangle in this family is uniquely determined by the length of the side AC or, equivalently, of the segment AE (E is the intersection of the tangent line to the inner circle from B and the line AC). Let us remark that for Euler's inequality $R \geq 2r$, and then $\alpha \leq \pi/6$.

The isoperimetric inequality for triangles is

$$L^2 - 12\sqrt{3}A \geq 0$$

with equality for the equilateral triangle.

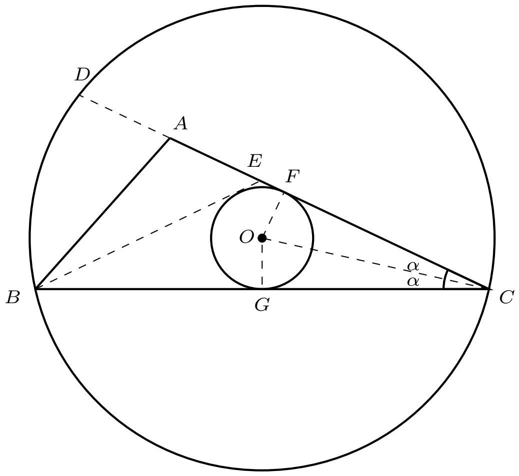

Figure 4: A triangle with minimal annulus of radii (R, r)

Proposition 2 *Among all the triangles with minimal annulus of radii (r, R) (see Fig. 4), the triangle with maximal isoperimetric deficit is the isosceles one with sides AB and AC with $\overline{AC} = \overline{AB} = \left(R^2\sqrt{R^2 - r^2}\right)/(R^2 - 2r^2)$ and basis BC; the triangle with minimal isoperimetric deficit is the triangle with $\overline{AC} = \left(R^2\sqrt{R^2 - r^2}\right)/(R^2 - 2r^2) + 2R\lambda\cos\alpha$, where λ is the unique real solution to the third degree equation in the unknown y*

$$\begin{aligned}&\big[-4y^2\cos^2(2\alpha) + 2\cos(2\alpha)(3\cos(2\alpha)+2)(\cos(2\alpha)-1)y\\ &+2\cos^3(2\alpha) + \cos^2(2\alpha) - \cos(2\alpha) - 1\big]^2\\ &= \big(1 + 4y\cos(2\alpha)4y^2\cos^2(2\alpha) - 8\cos^3(2\alpha)y\big)\\ &\times\Big(3\sqrt{3}\sin(2\alpha)\cos(2\alpha) - 1 - 2y\cos(2\alpha) - 2\sin^2\alpha\cos(2\alpha)\Big)^2 \qquad (10)\end{aligned}$$

satisfying $0 \le \lambda \le 1 - 1/(2\cos(2\alpha))$.

The extreme cases are shown in Fig. 5.

Proof. Let

$$\overline{BC} = a = 2R\cos\alpha, \quad \overline{AE} = x \quad \text{and} \quad x_F = a\left(1 - \frac{1}{2\cos(2\alpha)}\right).$$

Thus

$$b = b(x) = \overline{AC} = \frac{a}{2\cos(2\alpha)} + x, \quad c = c(x) = \overline{AB} = \sqrt{a^2 + b^2 - 2ab\cos(2\alpha)}.$$

The point A must be on the segment DE. If $A = E$, $x = 0$, and the triangle ABC is isosceles on the basis BC; if $A = D$, the triangle ABC is isosceles on the basis AB, and therefore $b(x) = a$ and $x = x_F$. Then $0 \le x \le x_F$.

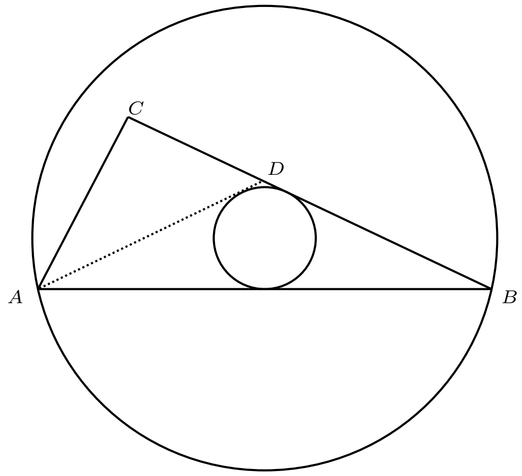

Figure 5: The triangle ABC with minimal deficit and the isosceles triangle ABD with maximal deficit.

The isoperimetric deficit of the triangle ABC is

$$\Delta(x) = L^2(x) - 12\sqrt{3}A(x) = [a + b(x) + c(x)]^2 - 6\sqrt{3}ab(x)\sin(2\alpha).$$

It is easy to see that $\Delta''(x) = 2\left(L'(x)\right)^2 + 2L(x)L''(x)$, and, since $L''(x) = a^2\sin^2(2\alpha)/[c(x)]^3 > 0$, the function $\Delta(x)$ is convex.

The derivative of $\Delta(x)$ in $x = 0$ is

$$\Delta'(0) = \frac{2a\sin(2\alpha)}{2\cos^2\alpha - 1}\left(3\sqrt{3} - 6\sqrt{3}\cos^2\alpha + 8\sin\alpha\cos^3\alpha\right),$$

and, since

$$3\sqrt{3} - 6\sqrt{3}\cos^2\alpha + 8\sin\alpha\cos^3\alpha < 3\sqrt{3} - 6\sqrt{3} + 8\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^3 = 0,$$

we obtain $\Delta'(0) < 0$.

The derivative of $\Delta(x)$ in $x = x_F$ is

$$\Delta'(x_F) = 4a(2 - \cos^2\alpha + 2\sin\alpha - 3\sqrt{3}\sin\alpha\cos\alpha).$$

The function $g(\alpha) = 2 - \cos^2\alpha + 2\sin\alpha - 3\sqrt{3}\sin\alpha\cos\alpha$ is decreasing on $[0, \pi/6]$, and then $g(\alpha) > g(\pi/6) = 0$ and $\Delta'(x_F) > 0$.

Therefore the isoperimetric deficit $\Delta(x)$ takes its local maxima at $x = 0$ and $x = x_F$ and its global minimum at the unique x in $[0, x_F]$ where $\Delta'(x)$ vanishes.

If $0 \leq x \leq x_F$, denoting $x = 2y\sqrt{R^2 - r^2}$, the equation $\Delta'(x) = 0$ is equivalent to equation (10), and then it has a unique solution.

It is possible, as in Proposition 1, to prove that $\Delta(0) > \Delta(x_F)$, and thus the maximum of the isoperimetric deficit is attained at $x = 0$. $\square$

Using the previous result it is easy to give an exact upper bound for the isoperimetric deficit in terms of the radii R and r of the minimal annulus, i.e., $\Delta(0)$:

$$\Delta(x) \leq \Delta(0) = \frac{6(R^2-r^2)^{3/2}(4R^2-7r^2)(R+2r)^2(R-2r)^2}{(R^2-2r^2)^2\left[2(R^2-r^2)^{3/2}+3\sqrt{3}r(R^2-2r^2)\right]}.$$

On the contrary, it is not so easy (but it is possible) to write the exact lower bound.

Example 2 *If $R=1$ and $r=0.2$, the minimal deficit is about* 6.09284610*, and it is attained for $x \approx 0.6574593176$; the maximum deficit is around* 8.224649710*. If $R=1$ and $r=0.4$, the minimal deficit is approximately* 0.73167813*, and it is attained for $x \approx 0.4440140348$; the maximum deficit is around* 1.68349019*.*

In Fig. 6, the maximum and the minimum of the isoperimetric deficit Δ are drawn as functions of the ratio r/R. The following result gives a lower

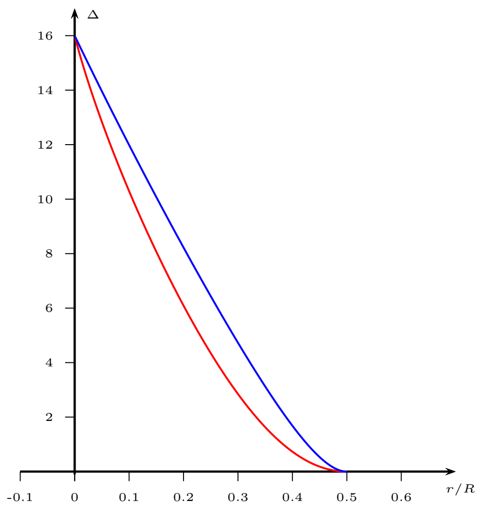

Figure 6: The maximum and the minimum isoperimetric deficit

and an upper bounds that are not sharp, but more meaningful.

Corollary 2 *Let T be a triangle, and let (r,R) be the radii of its minimal annulus. Then*

$$\begin{aligned}&L^2-12\sqrt{3}A \geq \left[16(R+2r)(R+r)^{3/2}(R-r)^3(R+7r)(R-2r)^2\right]\\&\left[(R^2-2r^2)(R^3+2R^2r-6Rr^2+4r^3)((R+r)^{3/2}\right.\\&\left.+3\sqrt{3}r\sqrt{R-r})\right]^{-1},\end{aligned} \tag{11}$$

$$L^2 - 12\sqrt{3}A \geq 16\frac{(R-r)^3(R-2r)^2}{R^3}, \tag{12}$$

$$L^2 - 12\sqrt{3}A \geq k(R-2r)^2, \tag{13}$$

and

$$\begin{aligned} L^2 - 12\sqrt{3}A &\leq 12\sqrt{3}\frac{(R^2-r^2)^{3/2}(R^2-4r^2)^2}{R^5} \\ &\leq 48\sqrt{3}\frac{(R^2-r^2)^{3/2}(R-2r)^2}{R^3} \\ &\leq 54(R-2r)^2. \end{aligned} \tag{14}$$

In the above inequalities, the equality sign holds only if the triangle is equilateral.

Proof. As proved in Proposition 2, the function $\Delta(x)$ is convex, and then its graph lies above the tangent lines at $x = 0$ and $x = x_F$. The ordinate of the intersection point of these two tangent lines is

$$\frac{16(R+2r)(R+r)^{3/2}(R-r)^3(R+7r)(R-2r)^2}{(R^2-2r^2)(R^3+2R^2r-6Rr^2+4r^3)((R+r)^{3/2}+3\sqrt{3}r\sqrt{R-r})},$$

and the result follows.

Inequalities (12) and (13) follow by studying (possibly by using some algebraic software) the functions

$$\frac{16(1+2z)(1+z)^{3/2}(1+7z)}{(1-2z^2)(1+2z-6z^2+4z^3)((1+z)^{3/2}+3\sqrt{3}z\sqrt{1-z})}$$

and

$$\frac{16(1+2z)(1+z)^{3/2}(1-z)^3(1+7z)}{(1-2z^2)(1+2z-6z^2+4z^3)((1+z)^{3/2}+3\sqrt{3}z\sqrt{1-z})}$$

where $z = r/R$.

Finally, denoting by $z = r/R$, it is easy to find the maxima of the function

$$h(z) = \frac{6(4-7z^2)}{(1-2z^2)^2\left[2(1-z^2)^{3/2}+3\sqrt{3}z(1-2z^2)\right]},$$

and since

$$\Delta(0) = h(z)\frac{(R^2-r^2)^{3/2}(R^2-4r^2)^2}{R^5},$$

the first inequality in (14) follows. The following are similar. $\square$

In [31], Rabinowitz shows many Bonnesen-style inequalities for triangles in which the second member depends on the radii of the circumscribed and the inscribed circle, on the area and the perimeter of the triangle. Let us explicitly note that these inequalities do not hold if we take the radii of the minimal annulus instead of the radii of the circumscribed and the inscribed circle. Consider, for example, Rabinowitz's inequality

$$L^2 - 12\sqrt{3}A \geq (L - 6\sqrt{3}\rho_i)^2$$

where ρ_i is the inradius of the triangle. If in the above inequality we take the radius r instead of the inradius, the inequality holds for the isosceles triangle with sides $\big(R^2\sqrt{R^2 - r^2}\big)/(R^2 - 2r^2)$ and basis $2\sqrt{R^2 - r^2}$ (since in this case the inner circle of the annulus is also the inscribed circle of the triangle), but it does not hold for the isosceles triangle with sides $2\sqrt{R^2 - r^2}$ and basis $4r\sqrt{R^2 - r^2}/R$.

References

[1] A. Alvino, V. Ferone, and C. Nitsch, A sharp isoperimetric inequality in the plane. J. Eur. Math. Soc., **13** (2011), pp. 185–206. https://doi.org/10.4171/jems/248

[2] I. Bárány, On the minimal ring containing the boundary of a convex body. Acta Sci. Math. (Szeged), **52** (1988), pp. 93–100.

[3] J. Bokowski and E. Heil, Integral representations of quermassintegrals and Bonnesen-style inequalities. Annales scientifiques de l'École Normale Supérieure, Série 3, **47** (1986), pp. 79–89. https://doi.org/10.1007/bf01202503

[4] T. Bonnesen, *Les problemes des isopérimetres et des isépiphanes*, Gauthier-Villars, Paris, 1929.

[5] T. Bonnesen and W. Fenchel, *Theory of convex bodies*, BCS Associates, Moscow, 1987.

[6] O. Bottema, Eine obere grenze für das isoperimetrische defizit ebener kurven. Ned. Akad. Wet. Proc., **66** (1933), pp. 442–446.

[7] A.R. Chouikha, Bonnesen-type inequalities and applications. In A. Eberhard, N. Hadjisavvas and D.T. Luc (eds.) *Generalized convexity, generalized monotonicity and applications*, **77**, Springer, New York, 2005, pp. 173–181. https://doi.org/10.1007/0-387-23639-2_10

[8] J. Cufí and A. Reventós, A lower bound for the isoperimetric deficit. Elem. Math., **71** (2016), pp. 156–167. https://doi.org/10.4171/em/312

[9] J. Favard, Problèmes d'extremums relatifs aux courbes convexes (premier mémoire). Annales scientifiques de l'École Normale Supérieure, Série 3, **46** (1929), pp. 345–369. https://doi.org/10.24033/asens.797

[10] J. Favard, Sur le déficit isopérimétrique maximum dans une couronne circulaire. Mat. Tidsskr. B (1929), pp. 62–68.

[11] B. Fuglede, Bonnesen's inequality for the isoperimetric deficiency of closed curves in the plane. Geom. Dedicata, **38** (1991), pp. 283–300. https://doi.org/10.1007/bf00181191

[12] R.J. Gardner and S. Vassallo, Inequalities for dual isoperimetric deficits. Mathematika, **45** (1998), pp. 269–285. https://doi.org/10.1112/s0025579300014200

[13] R.J. Gardner and S. Vassallo, Stability of inequalities in the dual Brunn-Minkowski theory. J. Math. Anal. Appl., **231** (1999), no. 2, pp. 568–587. https://doi.org/10.1006/jmaa.1998.6254

[14] G. He and W.X. Xu, Notes on Bonnesen-style inequality in the Euclidean plane. J. Math. (Wuhan), **32** (2012), pp. 1115–1120.

[15] M.A. Hernández Cifre and P.J. Herrero Piñeiro, Some optimization problems for the minimal annulus of a convex set. Math. Inequal. Appl., **9** (2006), pp. 359–374. https://doi.org/10.7153/mia-09-36

[16] M.A. Hernández Cifre and P.J. Herrero Piñeiro, Optimizing geometric measures for fixed minimal annulus and inradius. Rev. Mat. Iberoam., **23** (2007), no. 3, pp. 953–971. https://doi.org/10.4171/rmi/520

[17] M.A. Hernández Cifre and P.J. Herrero Piñeiro, Relating the minimal annulus with the circumradius of a convex set. J. Math. Inequal., **4** (2010), pp. 133–149. https://doi.org/10.7153/jmi-04-14

[18] E. Indrei, A sharp lower bound on the polygonal isoperimetric deficit. Proc. Amer. Math. Soc., **144** (2016), pp. 3115–3122. https://doi.org/10.1090/proc/12947

[19] N.D. Kazarinoff, *Geometric inequalities.* New Math Library, Mathematical Association of America, 1961.

[20] H.T. Ku, M.C. Ku, and X.M. Zhang, Analytic and geometric isoperimetric inequalities. J. Geom., **53** (1995), pp. 100–121. https://doi.org/10.1007/bf01224044

[21] Z.Q. Li, L.B. Hou, and W.X. Xu, A Bonnesen-style inequality in the Euclidean space $\mathbb{R}^n$. Math. Pract. Theory, **45** (2015), no. 21, pp. 210–214.

[22] D.S. Macnab, Cyclic polygons and related questions. Math. Gaz., **65** (1981), no. 431, pp. 22–28. https://doi.org/10.2307/3617929

[23] H. Martini and Z. Mustafaev, Extensions of a Bonnesen-style inequality to Minkowski spaces. Math. Inequal. Appl., **11** (2008), pp. 739–748. https://doi.org/10.7153/mia-11-63

[24] G.v.S. Nagy, Über konvexe kurven und einschließende kreisringe. Acta Sci. Math. Szeged, **10** (1941), pp. 174–184.

[25] R. Osserman, The isoperimetric inequality. Bull. Am. Math. Soc., **84** (1978), pp. 1182–1239.

[26] R. Osserman, Bonnesen-style isoperimetric inequalities. Am. Math. Mon., **86** (1979), no. 1, pp. 1–29. https://doi.org/10.1080/00029890.1979.11994723

[27] C. Peri, On the minimal convex shell of a convex body. Can. Math. Bull., **36** (1993), no. 4, pp. 466–472. https://doi.org/10.4153/cmb-1993-062-x

[28] C. Peri and S. Vassallo, Minimal properties for convex annuli of plane convex curves. Arch. Math. (Basel), **64** (1995), pp. 254–263. https://doi.org/10.1007/bf01188576

[29] C. Peri, J.M. Wills, and A. Zucco, On Blaschke's extension of Bonnesen's inequality. Geom. Dedicata, **48** (1993), pp. 349–357. https://doi.org/10.1007/bf01264078

[30] C. Peri and A. Zucco, On the minimal convex annulus of a planar convex body. Monatsh. Math., **114** (1992), pp. 125–133. https://doi.org/10.1007/bf01535579

[31] S. Rabinowitz, Some Bonnesen-style triangle inequalities. Missouri J. Math. Sci., **14** (2002), no. 2, pp. 75–87. https://doi.org/10.35834/2002/1402075

[32] J.R. Sangwine-Yager, Bonnesen-style inequalities for Minkowski relative geometry. Trans. Amer. Math. Soc., **307** (1988), pp. 373–382. https://doi.org/10.1090/s0002-9947-1988-0936821-5

[33] C.Zeng, On Bonnesen-style Aleksandrov-Fenchel inequalities in $\mathbb{R}^n$. Bull. Korean Math. Soc., **54** (2017), no. 3, pp. 799–816. https://doi.org/10.4134/bkms.b160317

[34] X.-M. Zhang, Bonnesen-style inequalities and pseudo-perimeters for polygons. J. Geom., **60** (1997), pp. 188–201. https://doi.org/10.1007/bf01252226

[35] J. Zhou, L. Ma, and W. Xu, On the isoperimetric deficit upper limit. Bull. Korean Math. Soc., **50** (2013), no. 1, pp. 175–184. https://doi.org/10.4134/bkms.2013.50.1.175

[36] J. Zhou, Y. Xia, and C. Zeng, Some new Bonnesen-style inequalities. J. Korean Math. Soc., **48** (2011), no. 2, pp. 421–430. https://doi.org/10.4134/jkms.2011.48.2.421

Salvatore Vassallo
Department of Mathematics for Economics, Finance, and Actuarial Sciences,
Catholic University of Milan
Largo Gemelli 1, 20123 Milan, Italy.
salvatore.vassallo@unicatt.it

Please, cite to this paper as published in
Armen. J. Math., V. **15**, N. 1(2023), pp. 1–15
https://doi.org/10.52737/18291163-2023.15.1-1-15