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Nonlocal Solvability of the Cauchy Problem for a System with Negative Functions of the Variable t

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Abstract. We obtain sufficient conditions for the existence and uniqueness of a local solution of the Cauchy problem for a quasilinear system with negative functions of the variable t and show that the solution has the same x-smoothness as the initial function. We also obtain sufficient conditions for the existence and uniqueness of a nonlocal solution of the Cauchy problem for a quasilinear system with negative functions of the variable t.

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Introduction

A problem with shift for mixed type equation with two degeneration lines was considered in [8].

We consider the system

$$\begin{cases} \partial_t u\,(t,x) + (a(t)u(t,x) + b(t)v(t,x))\partial_x u\,(t,x) = f_1(t,x), \\ \partial_t v(t,x) + (c(t)u(t,x) + g(t)v(t,x))\partial_x v(t,x) = f_2(t,x), \end{cases} \tag{1}$$

where $u(t,x)$, $v(t,x)$ are unknown functions, $f_1(t,x)$, $f_2(t,x)$, $a(t)$, $b(t)$, $c(t)$, $g(t)$ are given functions, $a(t), b(t), c(t), g(t) \in C([0,T])$ and

$$a(t) < 0,\ b(t) < 0,\ c(t) < 0,\ g(t) < 0 \text{ on } [0,T].$$

For system (1), we consider the following initial conditions:

$$u(0,x) = \varphi_1(x), \qquad v(0,x) = \varphi_2(x), \tag{2}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are given functions. Problem (1), (2) is considered on $\Omega_T = \{(t, x)\,|0 \le t \le T, x \in [0, +\infty), T > 0\}$.

In [5], by means of an additional argument method, there were found the conditions of nonlocal solvability of the Cauchy problem for the system

$$\begin{cases} \partial_t u\,(t, x) + (a(t)u(t, x) + b(t)v(t, x) + h_1(t))\partial_x u\,(t, x) = f_1(t, x), \\ \partial_t v(t, x) + (c(t)u(t, x) + g(t)v(t, x) + h_2(t))\partial_x v(t, x) = f_2(t, x), \end{cases} \tag{3}$$

subject to the initial conditions (2) on Ω_T, where $u(t, x)$ and $v(t, x)$ are unknown functions, $f_1(t, x)$, $f_2(t, x)$, $a(t)$, $b(t)$, $c(t)$, $g(t)$, $h_1(t)$, $h_2(t)$ are given functions, $a(t) > 0$, $b(t) < 0$, $c(t) > 0$, $g(t) < 0$, $h_1(t) \le 0$, $h_2(t) \le 0$ on $[0, T]$.

Systems (1), (3) appear in various problems in natural sciences. For instance, such systems are applied in models of shallow water [1].

In [5], the existence and uniqueness of a nonlocal solution of the Cauchy problem (3), (2) on Ω_T were proved under the following conditions

$$a(t) > 0,\ b(t) < 0,\ c(t) > 0,\ g(t) < 0,\ h_1(t) \le 0,\ h_2(t) \le 0 \text{ on } [0, T],$$

$$\varphi_1(x) \le 0,\ \varphi_2(x) \ge 0,\ \varphi_1'(x) \ge 0,\ \varphi_2'(x) \le 0 \text{ on } [0, +\infty),$$
$$f_1(t, x) \le 0,\ f_2(t, x) \ge 0,\ \partial_x f_1(t, x) \ge 0,\ \partial_x f_2(t, x) \le 0 \text{ on } \Omega_T.$$

In the present work, by means of the additional argument method, we determine the nonlocal solvability conditions for the Cauchy problem (1), (2) on Ω_T in the case when $a(t)$, $b(t)$, $c(t)$, $g(t)$ are continuous and negative functions on $[0, T]$. Also, we assume that

$$\varphi_1(x) \ge 0,\ \varphi_2(x) \ge 0,\ \varphi_1'(x) \le 0,\ \varphi_2'(x) \le 0 \text{ on } [0, +\infty),$$

$$f_1(t, x) \ge 0,\ f_2(t, x) \ge 0,\ \partial_x f_1(t, x) \le 0,\ \partial_x f_2(t, x) \le 0 \text{ on } \Omega_T.$$

We can avoid setting boundary conditions at $x = 0$ if

$$a(t) < 0,\ b(t) < 0,\ c(t) < 0,\ g(t) < 0 \text{ on } [0, T],$$

$$\varphi_1(x) \ge 0,\ \varphi_2(x) \ge 0 \text{ on } [0, +\infty), \qquad f_1(t, x) \ge 0,\ f_2(t, x) \ge 0 \text{ on } \Omega_T.$$

By means of the additional argument method, we obtain the following extended characteristic system (see [1]–[7] for details):

$$\frac{d\eta_1(s, t, x)}{ds} = a(s)w_1(s, t, x) + b(s)w_3(s, t, x), \tag{4}$$

$$\frac{d\eta_2(s, t, x)}{ds} = c(s)w_4(s, t, x) + g(s)w_2(s, t, x), \tag{5}$$

$$\frac{dw_1(s, t, x)}{ds} = f_1(s, \eta_1), \tag{6}$$

$$\frac{dw_2(s,t,x)}{ds} = f_2(s,\eta_2), \tag{7}$$

$$w_3(s,t,x) = w_2(s,s,\eta_1),\ w_4(s,t,x) = w_1(s,s,\eta_2), \tag{8}$$

$$\eta_1(t,t,x) = x,\ \eta_2(t,t,x) = x, \tag{9}$$

$$w_1(0,t,x) = \varphi_1(\eta_1(0,t,x)),\ w_2(0,t,x) = \varphi_2(\eta_2(0,t,x)). \tag{10}$$

Unknown functions η_i, $i=1,2$, and w_j, $j=\overline{1,4}$, depend not only on t and x, but also on additional argument s. Integrating equations (4)–(7) with respect to the argument s and taking into considerations conditions (8)–(10), we obtain an equivalent system of integral equations:

$$\eta_1(s,t,x) = x - \int_s^t (a(\nu)w_1 + b(\nu)w_3)d\nu, \tag{11}$$

$$\eta_2(s,t,x) = x - \int_s^t (c(\nu)w_4 + g(\nu)w_2)d\nu, \tag{12}$$

$$w_1(s,t,x) = \varphi_1(\eta_1(0,t,x)) + \int_0^s f_1(\nu,\eta_1)d\nu, \tag{13}$$

$$w_2(s,t,x) = \varphi_2(\eta_2(0,t,x)) + \int_0^s f_2(\nu,\eta_2)d\nu, \tag{14}$$

$$w_3(s,t,x) = w_2(s,s,\eta_1),\ w_4(s,t,x) = w_1(s,s,\eta_2). \tag{15}$$

Substituting (11) and (12) into (13)–(15)), we get

$$\begin{aligned} w_1(s,t,x) &= \varphi_1(x - \int_0^t (a(\nu)w_1 + b(\nu)w_3)d\nu) + \\ &\quad + \int_0^s f_1(\nu, x - \int_\nu^t (a(\tau)w_1 + b(\tau)w_3)d\tau)d\nu, \qquad (16) \\ w_2(s,t,x) &= \varphi_2(x - \int_0^t (c(\nu)w_4 + g(\nu)w_2)d\nu) + \\ &\quad + \int_0^s f_2(\nu, x - \int_\nu^t (c(\tau)w_4 + g(\tau)w_2)d\tau)d\nu, \qquad (17) \\ w_3(s,t,x) &= w_2(s,s,x - \int_s^t (a(\nu)w_1 + b(\nu)w_3)d\nu), \qquad (18) \\ w_4(s,t,x) &= w_1(s,s,x - \int_s^t (c(\nu)w_4 + g(\nu)w_2)d\nu). \qquad (19) \end{aligned}$$

Denote $\Gamma_T = \{(s,t,x) \mid 0 \le s \le t \le T, x \in [0,+\infty), T > 0\}$.

Lemma 1 *Assume that the system of integral equations* (16)–(19) *has a unique solution* $w_j \in C(\Gamma_T)$, $j = \overline{1,4}$, *and*

$$a(t) < 0,\ b(t) < 0,\ c(t) < 0,\ g(t) < 0 \ \text{on } [0,T],$$

$$\varphi_1(x) \geq 0,\ \varphi_2(x) \geq 0 \ \textit{on}\ [0,+\infty), \qquad f_1(t,x) \geq 0,\ f_2(t,x) \geq 0 \ \textit{on}\ \Omega_T.$$

Then $w_j(s,t,x), \eta_i(s,t,x) \in [0,+\infty)$ *on* Γ_T, $j = \overline{1,4}$, $i = 1,2$.

Proof. From (16) and conditions $\varphi_1(x) \geq 0$ on $[0,+\infty)$, $f_1(t,x) \geq 0$ on Ω_T, it follows that $w_1(s,t,x) \geq 0$ on Γ_T. From (17) and conditions $\varphi_2(x) \geq 0$ on $[0,+\infty)$, $f_2(t,x) \geq 0$ on Ω_T, we find that $w_2(s,t,x) \geq 0$ on Γ_T.

Since $w_1(s,t,x) \geq 0$ and $w_2(s,t,x) \geq 0$ on Γ_T, from (18) and (19), we conclude that $w_3(s,t,x) \geq 0$, $w_4(s,t,x) \geq 0$ on Γ_T. Since $w_1(s,t,x) \geq 0$, $w_3(s,t,x) \geq 0$ on Γ_T and $a(t) < 0$, $b(t) < 0$ on $[0,T]$, from (11), it follows that $\eta_1(s,t,x) \in [0,+\infty)$ on Γ_T. Finally, from $w_2(s,t,x) \geq 0$, $w_4(s,t,x) \geq 0$ on Γ_T, $c(t) < 0$, $g(t) < 0$ on $[0,T]$ and (12), we conclude that $\eta_2(s,t,x) \in [0,+\infty)$ on Γ_T. □

Lemma 2 *Let $w_1(s,t,x)$ and $w_2(s,t,x)$ satisfy the system of integral equations* (16)–(19)*). Assume that $w_1(s,t,x)$, $w_2(s,t,x)$ together with their first-order derivatives are continuously differentiable and bounded. Then the pair of functions*

$$u(t,x) = w_1(t,t,x), \qquad v(t,x) = w_2(t,t,x)$$

is a solution to the problem (1)*),* (2) *on Ω_{T_0}, where T_0 is a constant.*

Lemma 2 plays the key role in the additional argument method. It is proved in a standard way (cf., for example, [1]).

1 Existence of local solution

Let us introduce the following notations:

$$C_\varphi = \max\{\sup_{[0,+\infty)} \left|\varphi_i^{(l)}\right| \big| i = 1,2,\ l = \overline{0,2}\};$$

$$l = \max\{\sup_{[0,T]} |a(t)|,\ \sup_{[0,T]} |b(t)|,\ \sup_{[0,T]} |c(t)|,\ \sup_{[0,T]} |g(t)|\};$$

$$C_f = \max\{\sup_{\Omega_T} |f_1|, \sup_{\Omega_T} |f_2|, \sup_{\Omega_T} |\partial_x f_1|, \sup_{\Omega_T} |\partial_x f_2|\},$$

$$\|U\| = \sup_{\Gamma_T} |U(s,t,x)|,\ \|f\| = \sup_{\Omega_T} |f(t,x)|;$$

$\bar{C}^{\alpha_1,\alpha_2,\ldots\alpha_n}(\Omega_*)$ is the space of functions continuous and bounded, together with their derivatives up to order α_m w.r.t. m-th argument, $m = \overline{1,n}$, on unbounded subset $\Omega_* \subset R^n$, $n = 1,2...$;
$C([0,T])$ is the space of continuous functions on $[0,T]$.

In the next theorem, we provide conditions for the existence of local solution to the problem (1), (2).

Theorem 1 *Assume that*

$$\varphi_1, \varphi_2 \in \bar{C}^2([0, +\infty)), \quad a, b, c, g \in C([0, T]), \quad f_1, f_2 \in \bar{C}^{2,2}(\Omega_T),$$

$$T \le \min\Big(\frac{C_\varphi}{4C_f}, \frac{3}{40C_\varphi l}\Big),$$

$$a(t) < 0,\ b(t) < 0,\ c(t) < 0,\ g(t) < 0 \ \textit{on}\ [0, T],$$

$$\varphi_1(x) \ge 0,\ \varphi_2(x) \ge 0,\ \varphi_1'(x) \le 0,\ \varphi_2'(x) \le 0 \ \textit{on}\ [0, +\infty),$$

$$f_1(t, x) \ge 0,\ f_2(t, x) \ge 0,\ \partial_x f_1(t, x) \le 0,\ \partial_x f_2(t, x) \le 0 \ \textit{on}\ \Omega_T.$$

Then for each

$$T \le \min\Big(\frac{C_\varphi}{4C_f}, \frac{3}{40C_\varphi l}\Big),$$

the Cauchy problem (1), (2) *has a unique solution*

$$u(t, x), v(t, x) \in \bar{C}^{1,2}(\Omega_T)$$

which can be found from the system of integral equations (16)–(19).

The proof of Theorem 1 follows from the following lemma, the proof of which can be obtained in the same way it was done in [1]–[7].

Lemma 3 *Under conditions of Theorem 1, system* (16)–(19) *has a unique solution*

$$w_j \in C^{1,1,2}(\Gamma_T),\ j = \overline{1,4},\ T \le \min\Big(\frac{C_\varphi}{4C_f}, \frac{3}{40C_\varphi l}\Big).$$

2 Existence of nonlocal solution

In the next theorem, we provide conditions for the existence of nonlocal solution to the problem (1), (2).

Theorem 2 *Assume that*

$$\varphi_1, \varphi_2 \in \bar{C}^2([0, +\infty)), \quad a, b, c, g \in C([0, T]), \quad f_1, f_2 \in \bar{C}^{2,2}(\Omega_T),$$

$$a(t) < 0,\ b(t) < 0,\ c(t) < 0,\ g(t) < 0 \ \textit{on}\ [0, T],$$

$$\varphi_1(x) \ge 0,\ \varphi_2(x) \ge 0,\ \varphi_1'(x) \le 0,\ \varphi_2'(x) \le 0 \ \textit{on}\ [0, +\infty),$$

$$f_1(t, x) \ge 0,\ f_2(t, x) \ge 0,\ \partial_x f_1(t, x) \le 0,\ \partial_x f_2(t, x) \le 0 \ \textit{on}\ \Omega_T.$$

Then for any $T > 0$, the Cauchy problem (1), (2) *has a unique solution*

$$u(t, x), v(t, x) \in \bar{C}^{1,2}(\Omega_T)$$

which can be found from (16)– (19).

Proof. Differentiating (1) with respect to x and denoting

$$p(t,x) = \partial_x u(t,x), \qquad q(t,x) = \partial_x v(t,x),$$

we obtain the system of equations:

$$\begin{cases} \partial_t p + (a(t)u + b(t)v)\partial_x p = -a(t)p^2 - b(t)pq + \partial_x f_1, \\ \partial_t q + (c(t)u + g(t)v)\partial_x q = -g(t)q^2 - c(t)pq + \partial_x f_2, \\ p(0,x) = \varphi_1'(x), \quad q(0,x) = \varphi_2'(x). \end{cases} \tag{20}$$

We add following two equations to the system (11)– (15):

$$\begin{cases} \dfrac{d\gamma_1(s,t,x)}{ds} = -a(s)\gamma_1^2(s,t,x) - b(s)\gamma_1(s,t,x)\gamma_2(s,s,\eta_1) + \partial_x f_1(s,\eta_1), \\ \dfrac{d\gamma_2(s,t,x)}{ds} = -g(s)\gamma_2^2(s,t,x) - c(s)\gamma_1(s,s,\eta_2)\gamma_2(s,t,x) + \partial_x f_2(s,\eta_2), \end{cases} \tag{21}$$

with conditions

$$\gamma_1(0,t,x) = \varphi_1'(\eta_1), \qquad \gamma_2(0,t,x) = \varphi_2'(\eta_2). \tag{22}$$

System (21) can be written in the form

$$\begin{cases} \gamma_1(s,t,x) = \varphi_1'(\eta_1) + \displaystyle\int_0^s [-a(\nu)\gamma_1^2 - b(\nu)\gamma_1\gamma_2(\nu,\nu,\eta_1) + \partial_x f_1] d\nu, \\ \gamma_2(s,t,x) = \varphi_2'(\eta_2) + \displaystyle\int_0^s [-g(\nu)\gamma_2^2 - c(\nu)\gamma_2\gamma_1(\nu,\nu,\eta_2) + \partial_x f_2] d\nu. \end{cases} \tag{23}$$

As in [2]–[6], one can prove the existence of a continuously differentiable solution to the problem (23). Therefore,

$$\gamma_1(t,t,x) = p(t,x) = \frac{\partial u}{\partial x}, \qquad \gamma_2(t,t,x) = q(t,x) = \frac{\partial v}{\partial x}.$$

As in [5], one can prove that for all t and x on Ω_T

$$\|u\| \leqslant C_\varphi + TC_f, \qquad \|v\| \leqslant C_\varphi + TC_f. \tag{24}$$

Since $\varphi_1(x) \geq 0$, $\varphi_2(x) \geq 0$ on $[0,+\infty)$, $f_1(t,x) \geq 0$, $f_2(t,x) \geq 0$ on Ω_T, it follows from (13) and (14) that $w_1(s,t,x) \geq 0$, $w_2(s,t,x) \geq 0$ on Γ_T. Therefore, $u(t,x) = w_1(t,t,x) \geq 0$, $v(t,x) = w_2(t,t,x) \geq 0$ on Ω_T.

From (21), we have

$$\begin{cases} \gamma_1(s,t,x) = \varphi_1'(\eta_1)\exp\big(-\int\limits_0^s(a(\nu)\gamma_1+b(\nu)\gamma_2)d\nu\big)+ \\ \qquad + \int\limits_0^s \partial_x f_1 \exp\big(-\int\limits_\tau^s(a(\nu)\gamma_1+b(\nu)\gamma_2)d\nu\big)d\tau, \\ \gamma_2(s,t,x) = \varphi_2'(\eta_2)\exp\big(-\int\limits_0^s(c(\nu)\gamma_1+g(\nu)\gamma_2)d\nu\big)+ \\ \qquad + \int\limits_0^s \partial_x f_2 \exp\big(-\int\limits_\tau^s(c(\nu)\gamma_1+g(\nu)\gamma_2)d\nu\big)d\tau. \end{cases} \tag{25}$$

Since

$$a(t)<0,\ b(t)<0,\ c(t)<0,\ g(t)<0 \text{ on } [0,T],$$
$$\varphi_1'(x)\le 0,\ \varphi_2'(x)\le 0 \text{ on } [0,+\infty),$$
$$\partial_x f_1(t,x)\le 0,\ \partial_x f_2(t,x)\le 0 \text{ on } \Omega_T,$$

it follows from (25)) that $\gamma_1\le 0$, $\gamma_2\le 0$ on Γ_T. Therefore,

$$\|\gamma_i\| \leqslant C_\varphi + TC_f,\ i=1,2.$$

Since $\gamma_1(t,t,x)=\partial_x u$ and $\gamma_2(t,t,x)=\partial_x v$ for all t and x on Ω_T, the following estimates hold:

$$\|\partial_x u\| \leqslant C_\varphi + TC_f, \qquad \|\partial_x v\| \leqslant C_\varphi + TC_f. \tag{26}$$

Thus, $\partial_x u(t,x)\le 0,\ \partial_x v(t,x)\le 0$ on Ω_T.

As in [2]–[6], for all t and x, we obtain the following estimates

$$|\partial^2_{x^2}u| \le E_{11}ch\left(T\sqrt{C_{12}C_{21}}\right) + \frac{E_{21}C_{12}+C_{13}}{\sqrt{C_{12}C_{21}}}sh\left(T\sqrt{C_{12}C_{21}}\right) + C_{12}C_{23}T^2, \tag{27}$$

$$|\partial^2_{x^2}v| \le E_{21}ch\left(T\sqrt{C_{12}C_{21}}\right) + \frac{E_{11}C_{21}+C_{23}}{\sqrt{C_{12}C_{21}}}sh\left(T\sqrt{C_{12}C_{21}}\right) + C_{21}C_{13}T^2, \tag{28}$$

where E_{11}, E_{21}, C_{12}, C_{13}, C_{21}, C_{23} are constants.

Owing to the global estimates (24), (26)–(28), we can extend the solution to any given segment $[0,T]$. For the initial values take $u(T_0,x), v(T_0,x)\in \bar{C}^2([0,+\infty))$ such that

$$u(T_0,x)\ge 0,\ v(T_0,x)\ge 0,\ \partial_x u(T_0,x)\le 0,\ \partial_x v(T_0,x)\le 0 \text{ on } [0,+\infty).$$

Using Theorem 1, extend the solution to the segment $[T_0,\ T_1]$. Then take for the initial values $u(T_1,x), v(T_1,x)\in \bar{C}^2([0,+\infty))$ for which

$$u(T_1,x)\ge 0,\ v(T_1,x)\ge 0,\ \partial_x u(T_1,x)\le 0,\ \partial_x v(T_1,x)\le 0 \text{ on } [0,+\infty).$$

Using Theorem 1, extend the solution to the segment $[T_1,\ T_2]$.

Continuing in the similar way, we obtain that functions $u(T_k, x), v(T_k, x) \in \bar{C}^2([0, +\infty))$ such that

$$u(T_k, x) \geq 0,\ v(T_k, x) \geq 0,\ \partial_x u(T_k, x) \leq 0,\ \partial_x v(T_k, x) \leq 0 \text{ on } [0, +\infty),$$

satisfy the following estimates

$$|u(T_k, x)| \leqslant C_\varphi + TC_f, \qquad |v(T_k, x)| \leqslant C_\varphi + TC_f,$$

$$|\partial_x u(T_k, x)| \leqslant C_\varphi + TC_f, \qquad |\partial_x v(T_k, x)| \leqslant C_\varphi + TC_f.$$

The second-order derivatives satisfy estimates (27) and (28). As a result, one can extend the solution to any given segment $[0, T]$ in finitely many steps.

The uniqueness of the solution to the Cauchy problem (1), (2) is proved with the help of estimates similar to those used in the proof of the convergence of successive approximations. □

Let us bring an example.

Example. Consider the system

$$\begin{cases} \partial_t u\,(t, x) - ((t+2)u(t, x) + (t^3 + t + 4)v(t, x))\partial_x u\,(t, x) = \dfrac{2}{t + x + 1}, \\ \\ \partial_t v(t, x) - ((t+3)u(t, x) + (t^3 + t + 5)v(t, x))\partial_x v(t, x) = \dfrac{t + 7}{e^{8x} + 2}, \end{cases} \tag{29}$$

where $u(t, x)$ and $v(t, x)$ are unknown functions, with initial conditions

$$u(0, x) = \varphi_1(x) = \frac{1}{x + 1},\ v(0, x) = \varphi_2(x) = \frac{1}{e^{11x} + 2}. \tag{30}$$

on $\Omega_T = \{(t, x)\,|0 \leq t \leq T, x \in [0, +\infty), T > 0\}$.

We have

$$\varphi_1'(x) = -\frac{1}{(x+1)^2}, \qquad \varphi_2'(x) = -\frac{11e^{11x}}{(e^{11x} + 2)^2},$$

$$\partial_x f_1(t, x) = -\frac{2}{(t + x + 1)^2}, \qquad \partial_x f_2(t, x) = -\frac{8e^{8x}(t + 7)}{(e^{8x} + 2)^2}.$$

Since

$$\varphi_1, \varphi_2 \in \bar{C}^2([0, +\infty)), \quad a, b, c, g \in C([0, T]), \quad f_1, f_2 \in \bar{C}^{2,2}(\Omega_T),$$

$$a(t) = -t - 2 < 0, \qquad b(t) = -t^3 - t - 4 < 0,$$

$$c(t) = -t - 3 < 0, \qquad g(t) = -t^3 - t - 5 < 0 \text{ on } [0, T],$$

$$\varphi_1(x) = \frac{1}{x+1} > 0, \qquad \varphi_2(x) = \frac{1}{e^{11x}+2} > 0,$$

$$\varphi_1'(x) = -\frac{1}{(x+1)^2} < 0, \qquad \varphi_2'(x) = -\frac{11e^{11x}}{(e^{11x}+2)^2} < 0 \text{ on } [0,+\infty),$$

$$f_1(t,x) = \frac{2}{t+x+1} > 0, \qquad f_2(t,x) = \frac{t+7}{e^{8x}+2} > 0,$$

$$\partial_x f_1(t,x) = -\frac{2}{(t+x+1)^2} < 0, \qquad \partial_x f_2(t,x) = -\frac{8e^{8x}(t+7)}{(e^{8x}+2)^2} < 0 \text{ on } \Omega_T,$$

by Theorem 2, Cauchy problem (29), (30)) has a unique solution $u(t,x), v(t,x) \in \bar{C}^{1,2}(\Omega_T)$.

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