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Dirichlet Averages of Generalized Multi–index Mittag–Leffler Functions

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Abstract

The object of this article is to investigate the Dirichlet averages of the generalized multi–index Mittag–Leffler functions introduced by Saxena and Nishimoto [23, 24]. Representations of such relations are obtained in terms of Riemann–Liouville integrals and hypergeometric functions of several variables. Some interesting special cases of the established results associated with generalized Mittag–Leffler functions due to Srivastava and Tomovski [32] and multi–index Mittag–Leffler functions due to Kiryakova [12] are deduced. Certain results given earlier by Kilbas *et al.* [8] also follow as special cases of our findings.

Key Words: Generalized Mittag–Leffler Function, Dirichlet Averages, Riemann–Liouville Fractional Integral, Multi–index Mittag–Leffler Function, Srivastava–Daoust Generalized Lauricella F_D Hypergeometric Function.
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1 Introduction

The Dirichler average of a function is a certain kind integral average with respect to Dirichlet measure. The concept of Dirichlet average was introduced by Carlson in 1977. It is studied, among others, by Carlson [1, 2, 4], zu Castell [5], Massopust and Forster [17], Neuman

[18], Neuman and Van Fleet [19] and others. A detailed and comprehensive account of various types of Dirichlet averages has been given by Carlson in his monograph [3]. In a recent paper Dirichlet averages of the generalized Mittag–Leffler function due to Prabhakar [21] are obtained by Kilbas and Kattuveettil [8]. Motivated essentially by the utility and usefulness of Dirichlet averages in the study of Dirichlet splines, B–splines and Stolarsky means etc., the authors investigate the Dirichlet averages of the following generalized multi–index Mittag–Leffler function defined and studied by Saxena and Nishimoto [23, 24] in the following manner:

$$E_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};z\big] = E_{\rho,k}\big[(\alpha_1,\delta_1),\cdots,(\alpha_m,\delta_m);z\big] = \sum_{n=0}^{\infty} \frac{(\rho)_{kn}\, z^n}{\prod\limits_{j=1}^{m} \Gamma(\alpha_j+\delta_j n)\cdot n!}\,, \tag{1}$$

where $z,\rho,k,\alpha_j,\delta_j \in \mathbb{C}$; $\Re\left(\sum_{j=1}^{m}\delta_j\right) > \max\{0,\Re(k)-1\}$; $\Re(\alpha_j)>0, j=\overline{1,m}$ and $(a)_{bn}$ denotes the generalized Pochhammer symbol or shifted factorial defined in terms of the Euler's Gamma function

$$(a)_b := \frac{\Gamma(a+b)}{\Gamma(a)} = \begin{cases} 1 & b=0,\ a\in\mathbb{C}\setminus\{0\} \\ a(a+1)\cdots(a+b-1) & b\in\mathbb{N},\ a\in\mathbb{C} \end{cases}.$$

The function $E_{\rho,k}[z]$ plays a very important role in the theory of Riemann–Liouville fractional calculus and integral transforms. In this connection, one can refer to the works [9, 10, 11, 22] and the papers by Kiryakova [12, 13, 14, 15]. Integral expressions of this function are recently given by Saxena *et al.* [27]. Let us mention that the multivariate generalization of (1) has been recently investigated by Saxena *et al.* in [26], where the authors provide generalization of the results given earlier by Kilbas *et al.* [10].

When $\rho=k=1$, relation (1) yields the multi–index Mittag–Leffler function defined and studied by Kiryakova [12] in a slightly different form:

$$E_{1,1}\big[(\alpha_j,\delta_j)_{1,m};z\big] = \sum_{n=0}^{\infty} \frac{z^n}{\prod\limits_{j=1}^{m} \Gamma(\alpha_j+\delta_j n)\cdot n!}\,, \tag{2}$$

where $z,\alpha_j,\delta_j\in\mathbb{C}$; $\Re\left(\sum_{j=1}^{m}\delta_j\right)>0, \Re(\alpha_j)>0, j=\overline{1,m}$.

Note 1. *The function* (2) *is introduced and studied in a series of articles by Kiryakova* [12, 13, 14]*. Due to importance of this function in fractional calculus, it is also studied, among others by Saxena* et al. [25, 27]*, Paneva–Koneska* [20] *and others.*

Further if we set $m=1$, (1) reduces to the following generalized Mittag–Leffler function defined by Srivastava and Tomovski [32]:

$$E_{\rho,k}\big[(\alpha,\delta);z\big] = E_{\alpha,\delta}^{\rho,k}[z] = \sum_{n=0}^{\infty} \frac{(\rho)_{kn}\, z^n}{\Gamma(\alpha+\delta n)\cdot n!}\,, \tag{3}$$

where $z, \rho, k, \alpha, \delta \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\delta) > \max\{0, \Re(k) - 1\}$.

For $k = 1$ function $E^{\rho,k}_{\delta,\alpha}[z]$ reduces to the generalized Mittag–Leffler function introduced by Prabhakar [21]

$$E_{\rho,1}\big[(\alpha,\delta); z\big] = E^{\rho,1}_{\alpha,\delta}[z] = \sum_{n=0}^{\infty} \frac{(\rho)_n\, z^n}{\Gamma(\alpha+\delta n)\cdot n!}\,, \tag{4}$$

where $z, \rho, \alpha, \delta \in \mathbb{C}$; $\min\{\Re(\alpha), \Re(\delta)\} > 0$. Now, when $\delta = 1$, the above function reduces to Kummer's confluent hypergeometric function $\frac{1}{\Gamma(\alpha)}\,{}_1F_1(\rho;\alpha;z)$ [6], see also Kilbas *et al.* [9, 10].

In (4) if we put $\rho = 1$, we get the Mittag–Leffler function defined by Wiman [33] (also consult [5, 18, 19]):

$$E_{1,1}\big[(\alpha,\delta); z\big] = E^{1,1}_{\alpha,\delta}[z] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha+\delta n)} \qquad \big(\alpha, \delta \in \mathbb{C},\, \min\{\Re(\delta), \Re(\alpha)\} > 0\big), \tag{5}$$

and finally with $\alpha = 1$ we earn the Mittag–Leffler function [7]:

$$E_\delta(z) = E^{1,1}_{1,\delta}[z] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\delta n)} \qquad \big(\delta \in \mathbb{C},\, \Re(\delta) > 0\big).$$

We will need some more notations in the further exposition. In the sequel, the symbol $\mathbb{E}_{n-1}$ will denote the Euclidean simplex, defined by

$$\mathbb{E}_{n-1} = \Big\{(u_1, \cdots, u_{n-1})\colon u_j \ge 0,\, j = \overline{1,n},\, u_1 + \cdots + u_{n-1} \le 1\Big\}\,. \tag{6}$$

Next, we need the concept of the Dirichlet average. Following [3, Definition 5.2-1] let Ω be a convex set in $\mathbb{C}$ and let $z = (z_1, \cdots, z_n) \in \Omega^n, n \ge 2$ and let f be a measurable function on Ω. Define

$$F(b; z) = \int_{\mathbb{E}_{n-1}} f(u \circ z)\, \mathrm{d}\mu_b(u)\,, \tag{7}$$

where $\mathrm{d}\mu_b$ is the Dirichlet measure

$$\mathrm{d}\mu_b(u) = \frac{1}{\mathrm{B}(b)}\, u_1^{b_1-1} \cdots u_{n-1}^{b_{n-1}-1} \big(1 - u_1 - \cdots - u_{n-1}\big)^{b_n-1} \mathrm{d}u_1 \cdots \mathrm{d}u_{n-1}\,, \tag{8}$$

with the multivariate Beta–function

$$\mathrm{B}(b) = \frac{\Gamma(b_1)\cdots\Gamma(b_n)}{\Gamma(b_1+\cdots+b_n)}, \qquad \Re(b_j) > 0,\, j = \overline{1,n}\,,$$

and

$$u \circ z = \sum_{j=1}^{n-1} u_j z_j + (1 - u_1 - \cdots - u_{n-1}) z_n\,.$$

For $n = 1, F(b; z) = f(z)$. For $n = 2$ we have

$$\mathrm{d}\mu_{\beta\beta'}(u) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\, u^{\beta-1}(1-u)^{\beta'-1}\mathrm{d}u\,. \tag{9}$$

Carlson [2] investigated the average (7) for $f(z) = z^k, k \in \mathbb{R}$, in the form

$$R_k(b; z) = \int_{\mathbb{E}_{n-1}} (u \circ z)^k \, \mathrm{d}\mu_b(u), \qquad \big(k \in \mathbb{R}\big)\,, \tag{10}$$

and for $n = 2$, he proved that [2, 3]

$$R_k(\beta, \beta'; x, y) = \frac{1}{\mathrm{B}(\beta, \beta')} \int_0^1 \big[ux + (1-u)y\big]^k \, u^{\beta-1}(1-u)^{\beta'-1} \, \mathrm{d}u, \tag{11}$$

where $\beta, \beta' \in \mathbb{C}, \min\{\Re(\beta), \Re(\beta')\} > 0$, $x, y \in \mathbb{R}$ and $\mathrm{B}(\cdot, \cdot)$ stands for the standard Beta–function.

The Dirichlet averages of the generalized multi–index Mittag–Leffler function (1) is given by

$$M_{\rho,k}\big[(\alpha_j, \delta_j)_{1,m}; (\beta, \beta'; x, y)\big] = \int_{\mathbb{E}_1} E_{\rho,k}\big[(\alpha_j, \delta_j)_{1,m}; u \circ z\big] \mathrm{d}\mu_{\beta\beta'}(u)\,, \tag{12}$$

where $z = (x, y) \in \mathbb{C}^2, \rho, \alpha_j, \delta_j \in \mathbb{C}; \Re\left(\sum_{j=1}^m \delta_j\right) > \max\{0, \Re(k) - 1\}$, $\Re(\alpha_j)$, $j = \overline{1, m}$ and $\Re(k)$, $\Re(\beta)$, $\Re(\beta') > 0$.

We also need the Srivastava-Daoust generalization of the Lauricella hypergeometric function F_D in n variables defined by [29, p. 454]

$$\mathcal{S}^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}} \left(\begin{array}{c} [(a) : \theta', \cdots, \theta^{(n)}] : [(b') : \varphi']; \cdots; [(b^{(n)}) : \varphi^{(n)}] \\ \\ [(c) : \psi'; \cdots; \psi^{(n)}] : [(d') : \delta']; \cdots; [(d^{(n)}) : \delta^{(n)}] \end{array} \middle| \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right)$$

$$= \sum_{m_1,\cdots,m_n=0}^{\infty} \frac{\prod\limits_{j=1}^{A} (a_j)_{m_1\theta'_j+\cdots+m_n\theta_j^{(n)}} \prod\limits_{j=1}^{B'} (b'_j)_{m_1\varphi'_j} \cdots \prod\limits_{j=1}^{B^{(n)}} (b_j^{(n)})_{m_n\varphi_j^{(n)}}}{\prod\limits_{j=1}^{C} (c_j)_{m_1\psi'_j+\cdots+m_n\psi_j^{(n)}} \prod\limits_{j=1}^{D'} (d'_j)_{m_1\delta'_j} \cdots \prod\limits_{j=1}^{D^{(n)}} (d_j^{(n)})_{m_n\delta_j^{(n)}}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}, \tag{13}$$

where the parameters satisfy

$$\theta'_1, \cdots, \theta'_A, \cdots, \delta_1^{(n)}, \cdots, \delta_{D^{(n)}}^{(n)} > 0.$$

For convenience, we write (a) to denote the sequence of A parameters $a_1, \cdots, a_A$, with similar interpretations for $(b'), \cdots, (d^{(n)})$. Empty products should be interpreted as unity. Srivastava and Daoust [30, pp. 157–158] reported that the series in (13) converges absolutely

(i) for all $x_1, \cdots, x_n \in \mathbb{C}$ when

$$\Delta_\ell = 1 + \sum_{j=1}^{C} \psi_j^{(\ell)} + \sum_{j=1}^{D^{(\ell)}} \delta_j^{(\ell)} - \sum_{j=1}^{A} \theta_j^{(\ell)} - \sum_{j=1}^{B^{(\ell)}} \varphi_j^{(\ell)} > 0, \qquad \ell = \overline{1, n};$$

(ii) for $|x_\ell| < \eta_\ell$ when $\Delta_\ell = 0$, $\ell = \overline{1,n}$, where

$$\eta_\ell := \min_{\mu_1,\cdots,\mu_n>0} \left\{ \mu_\ell^{1+\sum\limits_{j=1}^{D^{(\ell)}} \delta_j^{(\ell)} - \sum\limits_{j=1}^{B^{(\ell)}} \varphi_j^{(\ell)}} \frac{\prod\limits_{j=1}^{C} \left(\sum\limits_{\ell=1}^{n} \mu_\ell \psi_j^{(\ell)} \right)^{\psi_j^{(\ell)}} \prod\limits_{j=1}^{D^{(\ell)}} \left(\delta_j^{(\ell)}\right)^{\delta_j^{(\ell)}}}{\prod\limits_{j=1}^{A} \left(\sum\limits_{\ell=1}^{n} \mu_\ell \theta_j^{(\ell)} \right)^{\theta_j^{(\ell)}} \prod\limits_{j=1}^{B^{(\ell)}} \left(\varphi_j^{(\ell)}\right)^{\varphi_j^{(\ell)}}} \right\}.$$

When all $\Delta_\ell < 0$, $\mathcal{S}^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}(x_1,\cdots,x_n)$ diverges except at the origin.

The further set of conditions for convergence of the series $\mathcal{S}^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}(x_1,\cdots,x_n)$ is given in Srivastava and Daoust [30]. We remark at this point that the Srivastava–Daoust $\mathcal{S}$ generalized Lauricella hypergeometric function for $n=2$ reduces to $\mathcal{S}$ generalized Kampé de Fériet hypergeometric function of two variables initially introduced in [28, 29]. A detailed account of the above function can be found in the paper [30] and in the book [31].

In sections 2, 3 we shall obtain certain integral representations for Dirichlet averages for $n=2$ and in section 4 in $n \geq 2$ dimensions by virtue of the related Dirichlet measure μ_b defined by (8), in terms of the Riemann–Liouville fractional integral of order $\alpha \in \mathbb{C}, \Re(\alpha) > 0$ which we define [11, 16, 22]:

$$\left\{I^\alpha_{a+} f\right\}(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1}\,\mathrm{d}t, \qquad \big(x > a,\, a \in \mathbb{R}\big). \tag{14}$$

2 Two–variate Dirichlet averages

Carlson's result (11) has been reconsidered and reformulated by Kilbas and Kattuveettil. We recall their findings. Let x, y be such that $y > x > 0, \beta, \beta' > 0$ and $k \in \mathbb{R}$. Let R_k and $I^{\beta'}_{0+}$ be given by (11) and (14) respectively. Then there holds the following formula [8, p. 475]:

$$R_k(\beta,\beta';x,y) = \frac{(y-x)^{1-\beta-\beta'}}{\mathrm{B}(\beta,\beta')} \left\{ I^{\beta'}_{0+}\big(t^{\beta-1}(1-t)^k\big)\right\}(y-x)\,. \tag{15}$$

Theorem 1. *Let $z,\rho,\alpha_j,\delta_j,\beta,\beta' \in \mathbb{C}$; $\Re\left(\sum_{j=1}^m \delta_j\right) > \max\{0, \Re(k)-1\}$, $\Re(k), \Re(\alpha_j) > 0, j = \overline{1,m}$ and $\Re(\beta), \Re(\beta') > 0$, finally let x, y be such that $x > y > 0$. Then the Dirichlet average of Mittag–Leffler function* (1) *is given by*

$$M_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};(\beta,\beta';x,y)\big] = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}} \left\{ I^{\beta'}_{0+}\big(t^{\beta-1}E_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};y+t\big]\right\}(x-y)\,. \tag{16}$$

Proof. According to (1) and (12), we have

$$M_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};(\beta,\beta';x,y)\big] = \frac{1}{\mathrm{B}(\beta,\beta')} \sum_{n=0}^\infty \frac{(\rho)_{kn}}{\prod\limits_{j=1}^m \Gamma(\alpha_j+\delta_j n)\cdot n!}$$
$$\times \int_0^1 [y+u(x-y)]^n u^{\beta-1}(1-u)^{\beta'-1}\,\mathrm{d}u =: H_1\,.$$

Put $u(x-y)=t$:

$$
\begin{aligned}
H_1 &= \frac{1}{\mathrm{B}(\beta,\beta')}\sum_{n=0}^{\infty}\frac{(\rho)_{kn}}{\prod\limits_{j=1}^{m}\Gamma(\alpha_j+\delta_j n)\cdot n!}\int_0^{x-y}(y+t)^n\left(\frac{t}{x-y}\right)^{\beta-1}\left(1-\frac{t}{x-y}\right)^{\beta'-1}\frac{\mathrm{d}u}{x-y}\\
&= \frac{(x-y)^{1-\beta-\beta'}}{\mathrm{B}(\beta,\beta')}\sum_{n=0}^{\infty}\frac{(\rho)_{kn}}{\prod\limits_{j=1}^{m}\Gamma(\alpha_j+\delta_j n)\cdot n!}\int_0^{x-y}(y+t)^n t^{\beta-1}(x-y-t)^{\beta'-1}\,\mathrm{d}t\\
&= \frac{(x-y)^{1-\beta-\beta'}}{\mathrm{B}(\beta,\beta')}\int_0^{x-y}t^{\beta-1}\cdot E_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};y+t\big]\cdot(x-y-t)^{\beta'-1}\,\mathrm{d}t\,,
\end{aligned}
$$

which yields the assertion of the Theorem 1. □

Now, we present some illustrative special cases.

When $\rho = k = 1$, Theorem 1 yields the results for the following multi–index Mittag–Leffler functions [12]:

Corollary 1. *Let the conditions of* Theorem 1 *are satisfied with* $\rho = k = 1$*, then the following result holds:*

$$
\begin{aligned}
&M_{1,1}\big[(\alpha_j,\delta_j)_{1,m};(\beta,\beta';x,y)\big]\\
&\qquad= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\left\{I_{0+}^{\beta'}\big(t^{\beta-1}E_{1,1}\big[(\alpha_j,\delta_j)_{1,m};y+t\big]\right\}(x-y)\,. \qquad (17)
\end{aligned}
$$

If we set $m = k = 1$, Theorem 1 transforms into a result by Kilbas *et al.* [8, p. 476].

Corollary 2. *Let the conditions of* Theorem 1 *are satisfied with* $m = k = 1, \alpha_1 = \alpha, \delta_1 = \delta$*, then the following result holds:*

$$
M_{\rho,1}^{\alpha,\delta}(\beta,\beta';x,y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\left\{I_{0+}^{\beta'}\big(t^{\beta-1}E_{\alpha,\delta}^{\rho,1}(y+t)\right\}(x-y)\,. \qquad (18)
$$

If $y = 0$, Theorem 1 gives

Corollary 3. *Let the conditions of* Theorem 1 *are satisfied with* $y = 0$*, then we have*

$$
M_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};(\beta,\beta';x,0)\big] = \frac{(\beta)_n}{(\beta+\beta')_n}\,E_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};x\big]\,. \qquad (19)
$$

When $x = 0$, Theorem 1 yields

Corollary 4. *Let the conditions of* Theorem 1 *are satisfied with* $x = 0$*, then we have*

$$
M_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};(\beta,\beta';0,y)\big] = \frac{(\beta')_n}{(\beta+\beta')_n}\,E_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};y\big]\,. \qquad (20)
$$

Now, we consider a modification of the Dirichlet average $M_{\rho,k}\big[(\delta_j,\alpha_j)_{1,m};(\beta,\beta';x,y)\big]$ described in (12), in the form

$$ {}_\gamma M^\rho_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};(\beta,\beta';x,y)\big] = \int_{\mathbb{E}_1} (u\circ z)^{\gamma-1}\, E_{\rho,k}\big[(\alpha,\delta_j)_{1,m};(u\circ z)^\rho\big]\mathrm{d}\mu_{\beta\beta'}(u)\,, \tag{21} $$

where $z=(x,y)$, and $\gamma\in\mathbb{C}$, $\Re(\gamma)>0$.

Theorem 2. *Let $z,\rho,\alpha_j,\delta_j\in\mathbb{C};\Re\left(\sum_{j=1}^m\delta_j\right)>\max\{0,\Re(k)-1\}$, $\Re(\alpha_j)$, $j=\overline{1,m}$, $\Re(k)$, $\Re(\beta)$, $\Re(\beta')>0$ and further let $x>y\in\mathbb{R}$. Then for all γ such that $\Re(\gamma)>0$:*

$$ \begin{aligned} {}_\gamma M^\rho_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};(\beta,\beta';x,y)\big] &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}\,(x-y)^{1-\beta-\beta'} \\ &\quad\times\Big\{I^{\beta'}_{y+}\big(t^{\gamma-1}(t-y)^{\beta-1}\,E_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};t^\rho\big]\big)\Big\}(x). \end{aligned} \tag{22} $$

Proof. Recalling the definition (9) of Dirichlet measure associated with simplex $\mathbb{E}_1$:

$$ \mathrm{d}_{\beta\beta'}(u) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\,u^{\beta-1}(1-u)^{\beta'-1}\mathrm{d}u\,, $$

and the definition (21) of the modified Dirichlet average ${}_\gamma M^\rho_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};(\beta,\beta';x,y)\big]$, we clearly obtain

$$ \begin{aligned} &{}_\gamma M^\rho_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};(\beta,\beta';x,y)\big] \\ &\quad= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\int_0^1[ux+(1-u)y]^{\gamma-1}\sum_{n=0}^\infty\frac{(\rho)_{kn}[ux+(1-u)y]^{\rho n}}{\prod\limits_{j=1}^m\Gamma(\alpha_j+\delta_jn)\cdot n!}u^{\beta-1}(1-u)^{\beta'-1}\,\mathrm{d}u \\ &\quad= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{n=0}^\infty\frac{(\rho)_{kn}}{\prod\limits_{j=1}^m\Gamma(\alpha_j+\delta_jn)\cdot n!}\int_0^1[ux+(1-u)y]^{\gamma+\rho n-1}u^{\beta-1}(1-u)^{\beta'-1}\,\mathrm{d}u := H_2\,. \end{aligned} $$

Taking substitution $y+(x-y)u=t$, it gives

$$ \begin{aligned} H_2 &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\,(x-y)^{1-\beta-\beta'}\sum_{n=0}^\infty\frac{(\rho)_{kn}}{\prod\limits_{j=1}^m\Gamma(\alpha_j+\delta_jn)\cdot n!}\int_y^x t^{\gamma+\rho n-1}(t-y)^{\beta-1}(x-t)^{\beta'-1}\,\mathrm{d}t \\ &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\,(x-y)^{1-\beta-\beta'}\int_y^x t^{\gamma-1}(t-y)^{\beta-1}\left\{\sum_{n=0}^\infty\frac{(\rho)_{kn}(t^\rho)^n}{\prod\limits_{j=1}^m\Gamma(\alpha_j+\delta_jn)\cdot n!}\right\}(x-t)^{\beta'-1}\,\mathrm{d}t \\ &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\,(x-y)^{1-\beta-\beta'}\int_y^x t^{\gamma-1}(t-y)^{\beta-1}E_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};t^\rho\big]\,(x-t)^{\beta'-1}\,\mathrm{d}t\,, \end{aligned} $$

which is equivalent to the statement of the Theorem 2. □

For the Wiman's Mittag–Leffler function (5), that is when $\rho=k=1$ in Theorem 2, we obtain the result by Kiryakova [12].

Corollary 5. *Let* $z, \alpha_j, \delta_j \in \mathbb{C}; \Re\left(\sum_{j=1}^m \delta_j\right) > 0$, $\Re(\alpha_j)$, $j = \overline{1,m}$, $\Re(\beta)$, $\Re(\beta'), \Re(\gamma) > 0$ *and further let* $x > y \in \mathbb{R}$. *Then we have*

$$ {}_\gamma M_{1,1}\big[(\alpha_j,\delta_j)_{1,m};(\beta,\beta';x,y)\big] = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}\,(x-y)^{1-\beta-\beta'} \times \left\{ I_{y+}^{\beta'}\big(t^{\gamma-1}(t-y)^{\beta-1}\,E_{1,1}\big[(\alpha_j,\delta_j)_{1,m};t\big]\big)\right\}(x)\,. $$

When $m = k = 1$, Theorem 2 concerns the Prabhakar's Mittag–Leffler function (4), considered by Kilbas and Kattuveettil [8, p. 480].

Corollary 6. *Let* $z, \rho, \alpha, \delta \in \mathbb{C}; \Re(\delta) > 0$, $\Re(\alpha)$, $\Re(\beta)$, $\Re(\beta'), \Re(\gamma) > 0$ *and let* $x > y \in \mathbb{R}$. *Then we have*

$$ {}_\gamma M_{\rho,1}^{\rho}\big[(\alpha,\delta);(\beta,\beta';x,y)\big] = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta')}\,(x-y)^{1-\beta-\beta'}\ \left\{ I_{y+}^{\beta'}\big(t^{\gamma-1}(t-y)^{\beta-1}\,E_{\alpha,\delta}^{\rho,1}\big[t^{\rho}\big]\big)\right\}(x)\,. $$

Corollary 7. *Let* $z, \rho, \alpha_j, \delta_j \in \mathbb{C}; \Re\left(\sum_{j=1}^m \delta_j\right) > \max\{0, \Re(k)-1\}$, $\Re(\alpha_j)$, $j = \overline{1,m}$, $\Re(k)$, $\Re(\beta)$, $\Re(\beta'), \Re(\gamma) > 0$ *and further let* $y \in \mathbb{R}$. *Then*

$$ {}_\gamma M_{\rho,k}^{\rho}\big[(\alpha_j,\delta_j)_{1,m};(\beta,\beta';0,y)\big] = \frac{\Gamma(\beta+\beta')\Gamma(\beta'+\gamma-1)}{\Gamma(\beta')}\,y^{\gamma-1} \times E_{\rho,k;\beta'+\gamma-1,\rho}\big[(\alpha_j,\delta_j)_{1,m},\,(\beta+\beta'+\gamma-1,\rho);y^{\rho}\big]. $$

If we put $\beta + \beta' = 1$ in Corollary 2.3, we conclude

Corollary 8. *Let* $z, \rho, \alpha_j, \delta_j \in \mathbb{C}; \Re\left(\sum_{j=1}^m \delta_j\right) > \max\{0, \Re(k)-1\}$, $\Re(\alpha_j)$, $j = \overline{1,m}$, $\Re(k)$, $\Re(\beta)$, $\Re(\gamma) - \Re(\beta) > 0$ *and further let* $y \in \mathbb{R}$.*Then*

$$ {}_\gamma M_{\rho,k}^{\rho}\big[(\alpha_j,\delta_j)_{1,m};(\beta,1-\beta;0,y)\big] = \frac{\Gamma(\gamma-\beta)}{\Gamma(1-\beta)}\,y^{\gamma-1}E_{\rho,k;\gamma-\beta,\rho}\big[(\alpha_j,\delta_j)_{1,m},\,(\gamma,\rho);y^{\rho}\big]. $$

Corollary 9. *Let* $z, \rho, \alpha_j, \delta_j \in \mathbb{C}; \Re\left(\sum_{j=1}^m \delta_j\right) > \max\{0, \Re(k)-1\}$, $\Re(\alpha_j)$, $j = \overline{1,m}$, $\Re(k)$, $\Re(\beta)$, $\Re(\beta') > 0$ *and further let* $x \in \mathbb{R}$. *Then*

$$ {}_\gamma M_{\rho,k}^{\rho}\big[(\alpha_j,\delta_j)_{1,m};(\beta,\beta';x,0)\big] = \frac{\Gamma(\beta+\beta')\Gamma(\beta+\gamma-1)}{\Gamma(\beta)}\,x^{\gamma-1} \times E_{\rho,k;\beta+\gamma-1,\rho}\big[(\alpha_j,\delta_j)_{1,m},\,(\beta+\beta'+\gamma-1,\rho);x^{\rho}\big]. $$

Finally, taking $\beta + \beta' = 1$ in Corollary 2.5, we get

Corollary 10. *Let* $z, \rho, \alpha_j, \delta_j \in \mathbb{C}; \Re\left(\sum_{j=1}^m \delta_j\right) > \max\{0, \Re(k)-1\}$, $\Re(\alpha_j)$, $j = \overline{1,m}$, $\Re(k)$, $\Re(\beta)$, $\Re(\gamma) - \Re(\beta) > 0$ *and further let* $x > y \in \mathbb{R}$. *Then we have*

$$ {}_\gamma M_{\rho,k}^{\rho}\big[(\alpha_j,\delta_j)_{1,m};(\beta,1-\beta;x,0)\big] = \frac{\Gamma(\beta+\gamma-1)}{\Gamma(\beta)}\,x^{\gamma-1}E_{\rho,k;\beta+\gamma-1,\rho}\big[(\alpha_j,\delta_j)_{1,m},\,(\gamma,\rho);x^{\rho}\big]. $$

Another kind characterization of the modified Dirichlet average of the generalized multi–index Mittag–Leffler function presented as the following result.

Theorem 3. *Let* $z, \rho, \alpha_j, \delta_j \in \mathbb{C}; \Re\left(\sum_{j=1}^m \delta_j\right) > \max\{0, \Re(k)-1\}$, $\Re(\alpha_j)$, $j=\overline{1,m}$, $\Re(k)$, $\Re(\beta)$, $\Re(\beta') > 0$ *and assume that* $x > y \in \mathbb{R}$*. Then*

$$
{}_kM^{\rho}_{\rho,k}\big[(\alpha_j,\delta_j)_{1,m};(\beta,\beta';x,y)\big] =
\frac{y^{k-1}}{\prod\limits_{j=1}^m \Gamma(\alpha_j)}\, \mathcal{S}^{\,1:2;1}_{0:1+m;1}\left(\begin{array}{lll} [1-k:-\rho,1] & :[\rho:k] & ;[\beta:1] \\ - & :[1-k:-\rho];[(\alpha):\delta]_{1,m} & ;[\beta+\beta':1] \end{array}\left|\begin{array}{c} x^\rho \\ 1-\dfrac{x}{y}\end{array}\right.\right). \tag{23}
$$

Proof. In view of (11) and (21) (the second relation with $\gamma = k$), we have

$$
\begin{aligned}
&{}_kM^{\rho}_{\rho,k}\big[(\delta_j,\alpha_j)_{1,m};(\beta,\beta';x,y)\big] \\
&\quad = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{n=0}^\infty \frac{(\rho)_{kn}}{\prod\limits_{j=1}^m \Gamma(\alpha_j+\delta_j n)\cdot n!}\int_0^1 [ux+(1-u)y]^{k+\rho n-1}u^{\beta-1}(1-u)^{\beta'-1}\,\mathrm{d}u \\
&\quad = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{n=0}^\infty \frac{(\rho)_{kn}\, y^{\rho n+k-1}}{\prod\limits_{j=1}^m \Gamma(\alpha_j+\delta_j n)\cdot n!}\int_0^1 \left[1-\left(1-\frac{x}{y}\right)u\right]^{k+\rho n-1} u^{\beta-1}(1-u)^{\beta'-1}\,\mathrm{d}u \\
&\quad = y^{k-1}\sum_{n=0}^\infty \frac{(\rho)_{kn}\, y^{\rho n}}{\prod\limits_{j=1}^m \Gamma(\alpha_j+\delta_j n)\cdot n!}\; {}_2F_1\left(\begin{array}{c}\beta,\ -\rho n+1-k \\ \beta+\beta'\end{array}\left|1-\frac{x}{y}\right.\right) =: H_3\,.
\end{aligned}
$$

Now, since

$$
(1-k-\rho n)_r = \frac{\Gamma(1-k-\rho n+r)}{\Gamma(1-k-\rho n)} = \frac{(1-k)_{-\rho n+r}}{(1-k)_{-\rho n}}\,,
$$

we conclude

$$
\begin{aligned}
H_3 &= y^{k-1}\sum_{n=0}^\infty\sum_{r=0}^\infty \frac{(1-k)_{-\rho n+r}\,(\rho)_{kn}\,(\beta)_r}{(1-k)_{-\rho n}\cdot\prod\limits_{j=1}^m \Gamma(\alpha_j+\delta_j n)\cdot(\beta+\beta')_r}\cdot\frac{(y^\rho)^n\left(1-\frac{x}{y}\right)^r}{n!\,r!} \\
&= \frac{y^{k-1}}{\prod\limits_{j=1}^m \Gamma(\alpha_j)}\sum_{n=0}^\infty\sum_{r=0}^\infty \frac{(1-k)_{-\rho n+r}\,(\rho)_{kn}\,(\beta)_r}{(1-k)_{-\rho n}\cdot\prod\limits_{j=1}^m (\alpha_j)_{\delta_j n}\cdot(\beta+\beta')_r}\cdot\frac{(y^\rho)^n\left(1-\frac{x}{y}\right)^r}{n!\,r!}\,,
\end{aligned}
$$

which, in comparison with (13) clearly gives the assertion. □

3 Dirichlet average of multivariate function

Let us make the convention that here, and in what follows, (λ) denotes the n–tuple of some $\lambda_1, \cdots, \lambda_n$.

The Dirichlet average $M_{\rho,k}$ and its modification ${}_{\gamma}M^{\rho}_{\rho,k}$ are discussed here, where the complex variable vector is $(z) = (z_1, \cdots, z_n) \in \mathbb{C}^n$ and the prescribed parameters vector is (b). Our findings are based in part on the following result.

Lemma 1. [8, p. 483, Lemma 1] *Let n be a positive integer, $b_j, r_j \in \mathbb{C}$ such that $\Re(b_j)$, $\Re(r_j) + 1 > 0$, $j = \overline{1, n}$. When $\mathbb{E}_{n-1}$ denotes the Euclidean simplex (6) and $\mathrm{d}\mu_b(u)$ stands for the Dirichlet measure (8), then there holds the formula*

$$\int_{\mathbb{E}_{n-1}} u_1^{r_1} \cdots u_{n-1}^{r_n-1} (1 - u_1 - \cdots - u_{n-1})^{r_n} \, \mathrm{d}\mu_b(u) = \frac{(b_1)_{r_1} \cdots (b_n)_{r_n}}{(b_1 + \cdots + b_n)_{r_1+\cdots+r_n}} \,. \tag{24}$$

The Lauricella functions F_D defined for complex parameters $b \in \mathbb{C}^n$ in terms of the multiple series [31] is defined by

$$F_D(a, (b); c; (z)) = \sum_{m_1,\cdots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \cdot \frac{z_1^{m_1} \cdots z_n^{m_n}}{m_1! \cdots m_n!} \,, \tag{25}$$

Series (25) converges for all variables inside unit circle, that is for $\max_{1\le j\le n} |z_j| < 1$. Let us remind the Srivastava–Daoust generalization $\mathcal{S}$ (13) of the Lauricella F_D. Now, we will study the following Dirichlet average:

$${}_kM_{\rho,k}\left[(\alpha_j, \delta_j)_{1,m}; ((b); (1-z))\right] = \int_{\mathbb{E}_{n-1}} (1-u\circ z)^{k-1} \, E_{\rho,k}[(\alpha_j, \delta_j)_{1,m}; (1-u\circ z)^{\rho}] \, \mathrm{d}\mu_b(u)\,; \tag{26}$$

we will also need the directly verified formula

$$(1 - z_1 - \cdots - z_n)^{\eta} = \sum_{r_1,\cdots,r_n=0}^{\infty} (-\eta)_{r_1+\cdots+r_n} \frac{z_1^{r_1} \cdots z_n^{r_n}}{r_1! \cdots r_n!}, \qquad \Big(|z_1 + \cdots + z_n| < 1\Big). \tag{27}$$

Theorem 4. *Let $\rho, k, \alpha_j, \delta_j \in \mathbb{C}; \Re\left(\sum_{j=1}^m \delta_j\right) > \max\{0, \Re(k) - 1\}$, $\Re(\alpha_j)$, $j = \overline{1, m}$, $\Re(k)$, and let $(b) = (b_1, \cdots, b_n), (z) = (z_1, \cdots, z_n) \in \mathbb{C}^n$. Then the following result holds:*

$$\begin{aligned} &{}_kM_{\rho,k}\left[(\alpha_j, \delta_j)_{1,m}; ((b); (1-z))\right] \\ &= \frac{1}{\prod\limits_{j=1}^{m} \Gamma(\alpha_j)} \, \mathcal{S}^{0:\,2\,;(1)}_{2:m;(0)} \left(\begin{array}{c} - \qquad\qquad : [\rho : k], [k : \rho] \;; [(b) : 1] \\ [k : \rho; (-1)], \Big[\sum\limits_{j=1}^{n} b_j : 0; (1)\Big] : \quad [(\alpha) : \delta] \quad ; \quad (-) \end{array} \middle| \begin{array}{c} 1 \\ (-z)^{\mathrm{T}} \end{array} \right). \end{aligned} \tag{28}$$

Here $(\cdot)^{\mathrm{T}}$ denotes the matrix transpose operator.

Proof. Consider multivariate Dirichlet average

$${}_kM_{\rho,k}\left[(\alpha_j, \delta_j)_{1,m}; ((b); (1-z))\right] = \int_{\mathbb{E}_{n-1}} \sum_{n=0}^{\infty} \frac{(\rho)_{kn} (1 - u \circ z)^{k-1+\rho n}}{\prod\limits_{j=1}^{m} \Gamma(\alpha + \delta_j n) \cdot n!} \, \mathrm{d}\mu_b(u) =: H_4 \,.$$

Applying the Lemma 1 (24) and the polynomial expansion (27), letting $|u_1z_1+\cdots+u_nz_n|<1$, to H_4 we find that

$$\begin{aligned} H_4 &= \sum_{n=0}^\infty \frac{(\rho)_{kn}}{\prod\limits_{j=1}^m \Gamma(\alpha+\delta_j n)\cdot n!} \sum_{r_1,\cdots,r_n=0}^\infty (1-k-\rho n)_{r_1+\cdots+r_n}\, \frac{z_1^{r_1}\cdots z_n^{r_n}}{r_1!\cdots r_n!} \\ &\quad\times \int_{\mathbb{E}_{n-1}} u_1^{r_1}\cdots u_{n-1}^{r_n-1}(1-u_1-\cdots-u_{n-1})^{r_n}\,\mathrm{d}\mu_b(u) \\ &= \sum_{n=0}^\infty \frac{(\rho)_{kn}}{\prod\limits_{j=1}^m \Gamma(\alpha+\delta_j n)\cdot n!} \sum_{r_1,\cdots,r_n=0}^\infty \frac{(1-k-\rho n)_{r_1+\cdots+r_n}(b_1)_{r_1}\cdots(b_n)_{r_n}}{(b_1+\cdots+b_n)_{r_1+\cdots+r_n}}\, \frac{z_1^{r_1}\cdots z_n^{r_n}}{r_1!\cdots r_n!}\,. \end{aligned}$$

The n–fold inner sum (with respect to $r_1,\cdots,r_n$) forms a Lauricella F_D function, so

$$H_4 = \sum_{n=0}^\infty \frac{(\rho)_{kn}}{\prod\limits_{j=1}^m \Gamma(\alpha+\delta_j n)\cdot n!}\, F_D(1-k-\rho n;(b);b_1+\cdots+b_n;(z))\,.$$

Since

$$(1-k-\rho n)_{r_1+\cdots+r_n} = (-1)^{r_1+\cdots+r_n}\,\frac{(k)_{\rho n}}{(k)_{\rho n-r_1-\cdots-r_n}}\,,$$

and by applying another obvious transformations to H_4, we arrive at

$$H_4 = \frac{1}{\prod\limits_{j=1}^m \Gamma(\alpha_j)} \sum_{n,(r)=0}^\infty \frac{(\rho)_{kn}\,(k)_{\rho n}\,(b_1)_{r_1}\cdots(b_n)_{r_n}}{(k)_{\rho n-r_1-\cdots-r_n}(b_1+\cdots+b_n)_{r_1+\cdots+r_n}\cdot\prod\limits_{j=1}^m(\alpha_j)_{\delta_j n}}\, \frac{(-z_1)^{r_1}\cdots(-z_n)^{r_n}}{n!\,r_1!\cdots r_n!}\,,$$

which is equivalent to the asserted Srivastava–Daoust function expression (28). □

Corollary 11. *Under the conditions of* Theorem 4 *with $\rho=k=1$, the following result holds:*

$$\begin{aligned} &{}_1M_{1,1}^{(\alpha),\delta}[(b),(1-z)] \\ &\quad= \frac{1}{\prod\limits_{j=1}^m\Gamma(\alpha_j)}\, \mathcal{S}^{0:2;(1)}_{2:m;(0)}\left(\begin{array}{c} - \qquad : [1:1],[1:1]\ ;[(b):1] \\ [1:1;(-1)],\Big[\sum\limits_{j=1}^n b_j:0;(1)\Big]:\quad [(\alpha):\delta]\quad ;\quad (-) \end{array} \middle| \begin{array}{c} 1 \\ (-z)^{\mathrm{T}} \end{array}\right). \end{aligned}$$

Finally, for $m=k=1$ the assertion of the Theorem 4 reduces to the known result due to Kilbas and Kattuveettil [8, p. 482, Theorem 3], presented as the following result.

Corollary 12. *Under the conditions of* Theorem 4 *with $m=k=1$, we have:*

$$\begin{aligned} &{}_1M_{\rho,1}^{\alpha,\delta}[(b),(1-z)] \\ &\quad= \frac{1}{\Gamma(\alpha)}\, \mathcal{S}^{0:2;(1)}_{2:1;(0)}\left(\begin{array}{c} - \qquad : [\rho:1],[1:\rho]\ ;[(b):1] \\ [1:\rho;(-1)],\Big[\sum\limits_{j=1}^n b_j:0;(1)\Big]:\quad [\alpha:\delta]\quad ;\quad (-) \end{array} \middle| \begin{array}{c} 1 \\ (-z)^{\mathrm{T}} \end{array}\right). \end{aligned}$$

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