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The Representations of The Heisenberg Group over a Finite Field

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Abstract

We find all irreducible representations of the Heisenberg group over a finite field.

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1 Introduction.

Let F be a local p field where p is a prime number. Let W be a finite-dimensional vector space over F; and let $\langle , \rangle$ be a non-degenerate symplectic bilinear form on W. Let $G = Sp(W)$ be the group of isometries of W with respect to $\langle , \rangle$. The Weil representation of G which arises in considering unitary representations of the Heisenberg group attached to W, has an importatnt role in Θ-correspondences and automorphic forms. The nontrivial unitary representation theory of $\mathbf{H}$, the Heisenberg group, is given by the following Theorem.

Theorem 1. ($Stone - von\ Neumann$). *Let χ be a nontrivial character of F^+. Then up to an isomorphism there is only one equivalence class of irreducible unitary representations of* $\mathbf{H}$ *with central character χ.*

In this paper, we will use a specific complete polarization of the symplectic space W, to construct all smooth irreducible representations of the Heisenberg group over a finite field, F_{p^m} including p = 2. (See [5] and [6]for complete polarization of a symplectic space example.) Our approach, as in [5] can be applied to infinite cases as well, i.e. when the ground field is a local non-Archimedean p-field; However, to exhaust all representations, we use the finiteness of the ground field (in a finite case, the number of all irreducible representations are known.)

Although some of the Lemmas and corollaries are found in the literatures, in this paper, we present them with detailed and simple proofs for convenience and completeness.

2 Heisenberg Group

Let F be a finite field with $q = p^m$ elements; where p is a prime number and m is a positive integer. Let W be a finite-dimensional vector space over F; and let $\langle , \rangle$ be a non-degenerate symplectic bilinear form on W; i.e. $\langle , \rangle$ is a map from $W \times W \to F$ having the following properties:

(1)- $\langle , \rangle$ is linear in each variable.

(2)- $\langle , \rangle$ is non-degenerate; i.e. for any $w \in W, w \neq 0$ there is $w' \in W$ such that $\langle w, w' \rangle \neq 0$.

(3)- $\langle , \rangle$ is symplectic ; i.e. for all $w, w' \in W$ we have $\langle w, w' \rangle = - \langle w', w \rangle$.

The Heisenberg group, $H = H(W)$, attached to W is the group with underying set $W \times F$ and the following multiplication:

$$(w, a)(w', a') = (w + w', a + a' + \langle w, w' \rangle)$$

For all $w, w' \in W$ and $a, a' \in F$. See also [5] and [6].

Let $\dim W = 2n$, for some positive integer n. Thus $|H| = q^{2n+1}$. When $p = 2$ the group is abelian and its representations are all one dimentional (Characters). In this case; we find all representations in the next section.

From now on we assume p is an odd prime number.

Lemma 1. *Let $w \in W, w \neq 0$. Define $\varphi_w : W \to F$ by $\varphi_w(w') = \langle w, w' \rangle$. Then φ_w is onto.*

Proof. Let $b \in F$. If $b = 0$, then take $w' = 0$. So let $b \neq 0$. Since $\langle , \rangle$ is nondegenerate there is $w' \in W$ such that $\langle w, w' \rangle = 1$. Now we have $\varphi_w(bw') = \langle w, bw' \rangle = b \langle w, w' \rangle = b$. □

Lemma 2. *There are $q^{2n} + q - 1$ conjugacy classes in H.*

Proof. Let $(w, 0) \in H$, with $w \neq 0$ Then the conjugacy class that contains this element is

$$(w', a')(w, 0)(-w', -a') = (w, 2 \langle w', w \rangle)$$

From here and Lemma 1, we deduce that conjugacy classe of $(w, 0)$ is $\{(w, a) \in H \mid w \neq 0\}$. Thus there are $q^{2n} - 1$ classes of this form and each class contains q elements. Now let $w = 0$ and look for conjugacy classes for $(0, a), a \in F$. Then some computations as above shows that there are q different conjugacy classes of this type with one element in each class. □

Corollary 1. *There are two types of conjugacy classes in H with representative elements $(w, 0), w \neq 0$, and $(0, a), w \in W, a \in F$.*

Corollary 2. *There are $q^{2n} + q - 1$ irreducible representations of H. See [1], [2], [7]*

Lemma 3. *The center of H, $Z(H)$, is isomorphic to F^+.*

Proof. Let $(w,a)\in Z(H)$, then we must have

$$(w,a)(w',a')=(w',a')(w,a)$$

for all $(w',a')\in H$. Thus

$$(w'+w,a'+a+\langle w',w\rangle)=(w+w',a+a'+\langle w,w'\rangle)$$

From here we must have $\langle w,w'\rangle=\langle w',w\rangle$ or $\langle w,w'\rangle=0$. Since $\langle,\rangle$ is nondegenerate we get $w=0$. Thus

$$Z(H)=\{(0,a)\in H\mid a\in F\}.$$

Now define $f:Z(H)\to F^+$ by $f(0,a)=a$. It is easy to show that f is an isomorphism. See also [5] and [6]. □

Lemma 4. *The group of commutatores of $H,[H,H]$, is equal to $Z(H)$, the center of H.*

Proof. Let $(w,a),(w',a')\in H$. Then we have

$$(w,a)(w',a')(-w,-a)(-w',-a')=(0,\langle w,w'\rangle).$$

Now apply Lemma 1 to get $[H,H]=Z(H)$. See also [6] □

Corollary 3. *The quotient group $H/Z(H)$ is abelian group with $|H/Z(H)|=\frac{q^{2n+1}}{q}=q^{2n}$ elements.*

Lemma 5. *Any character (one-dimensional representation) ρ of H induces a character of $H/Z(H)$, and conversely any character of $H/Z(H)$ induces a character of H.*

Proof. Let $(w,a),(w',a')\in H$. Then we have:

$$\begin{aligned}
&\rho((w,a)(w',a')(-w,-a)(-w',-a'))\\
=&\ \rho(w,a)\rho(w',a')\rho(-w,-a)\rho(-w',-a')\\
=&\ \rho(w,a)\rho(w',a')(\rho(w,a))^{-1}(\rho(w',a'))^{-1}\\
=&\ 1\\
=&\ \rho(0,\langle w,w'\rangle)
\end{aligned}$$

Thus ρ is trivial on $[H,H]=Z(H)$. Let $\widetilde{(w,a)}$ be an element of $H/Z(H)$ whose representative is (w,a). Now one can easily check that $\widetilde{\rho}:H/Z(H)\to\mathbb{C}^\times,\widetilde{\rho}\left(\widetilde{(w,a)}\right)=\rho(w,a)$ is a well-defined character of $H/Z(H)$. Conversely any character $\widetilde{\rho}$, of $H/Z(H)$ induces a character of H by $\rho(w,a)=\widetilde{\rho}\left(\widetilde{(w,a)}\right)$. □

Lemma 6. *$H/Z(H)$ is isomorphic to additive group W.*

Proof. Define $\varphi:H\to W$ by $\varphi(w,a)=w$. Then one can check that φ is an onto homomorphis and its kernel is $Z(H)$. □

Corollary 4. *Any character of $H/Z(H)$ will be determined by its value on W. In fact a set of the representatives of $Z(H)$ in $H/Z(H)$ is $W \times \{0\} = \{(w,0)\,|w \in W\}$.*

Proof. Let $\widetilde{(w,a)} \in H/Z(H)$. Then $\widetilde{(w,a)} = \widetilde{(w,0)}$, because $(w,a) - (w,0) = (0,a) \in Z(H)$. □

All characters (One-dimensional representations) of H are given by the following theorem.

Theorem 2. *Let χ be a non-trivial character of F^+(See [4] for the existence of a non trivial character of $\mathbb{F}^+$.). For any $(w,0) \in W \times \{0\}$ Define $\widetilde{\psi_{(w,\chi)}} : H/Z(H) \to \mathbb{C}^\times$ by $\widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w',0)}\right) = \chi(\langle w,w'\rangle)$ for $\widetilde{(w',0)} \in H/Z(H)$. Then $\widetilde{\psi_{(w,\chi)}}$ is a character of $H/Z(H)$.*

Proof. First we will show that $\widetilde{\psi_{(w,\chi)}}$ is well defined. Let $\widetilde{(w_1,0)} = \widetilde{(w_2,0)} \in H/Z(H)$ then we must have $(w_1,0)-(w_2,0) \in Z(H)$. From here we have $(w_1 - w_2, 0+0+\langle w_1,w_2\rangle) \in Z(H)$ Thus $w_1 - w_2 = 0$. Hence $w_1 = w_2 = w'$; i.e.

$$\begin{aligned}\widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_1,0)}\right) &= \widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_2,0)}\right)\\ &= \chi(\langle w,w'\rangle).\end{aligned}$$

On the other hand we have:

$$\begin{aligned}\widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_1,0)}\widetilde{(w_2,0)}\right) &= \widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_1+w_2,0+0+\langle w_1,w_2\rangle)}\right)\\ &= \widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_1+w_2,0)}\right)\\ &= \chi(\langle w,w_1+w_2\rangle)\\ &= \chi(\langle w,w_1\rangle + \langle w,w_2\rangle)\\ &= \chi(\langle w,w_1\rangle)\,\chi(\langle w,w_2\rangle)\\ &= \widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_1,0)}\right)\widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_2,0)}\right).\end{aligned}$$

□

Lemma 7. *For any $a \in F^\times$ we have $\widetilde{\psi_{(aw,\chi)}} = \widetilde{\psi_{(w,\chi_a)}}$ where $\chi_a(x) = \chi(ax)$ for all $x \in F$ is, a character of F^+.*

Proof. For any $\widetilde{(w',0)} \in H/Z(H)$ we have:

$$\begin{aligned}\widetilde{\psi_{(aw,\chi)}}\left(\widetilde{(w',0)}\right) &= \chi(\langle aw,w'\rangle)\\ &= \chi(a\langle w,w'\rangle)\\ &= \chi_a(\langle w,w'\rangle)\\ &= \widetilde{\psi_{(w,\chi_a)}}\left(\widetilde{(w',0)}\right).\end{aligned}$$

□

Lemma 8. *Let χ be a character of F^+. If $\widetilde{\psi_{(w,\chi)}} = \widetilde{\psi_{(w',\chi)}}$, then $w = w'$.*

Proof. Let $\widetilde{\psi_{(w,\chi)}} = \widetilde{\psi_{(w',\chi)}}$. Then for any $\widetilde{(w'',0)} \in H/Z(H)$, we have:

$$\begin{aligned}\widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w'',0)}\right) &= \chi(\langle w, w''\rangle) \\ &= \widetilde{\psi_{(w',\chi)}}\left(\widetilde{(w'',0)}\right) \\ &= \chi(\langle w', w''\rangle).\end{aligned}$$

From here we get

$$\chi(\langle w - w', w''\rangle) = 1$$

Since $\langle , \rangle$ is non-degenerate we must have $w - w' = 0$,thus $w = w'$. □

Corollary 5. *There are q^{2n} characters of H.*

Proof. Since $|W \times \{0\}| = |\{(w,0)\,|w \in W\}| = q^{2n}$, the Lemmas 7 and 8 imply that there are q^{2n} characters for $H/Z(H)$. Now apply Lemma 5. □

There are $q-1$ more irreducible representations of H. These representations have dimensions bigger than one. We determine these representations as follows.

Lemma 9. *Let $B = \{\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_n, \beta_n\}$ be a basis of W having the following properties: (this basis exists because W is a non-degenerate symplectic space.)*

$$\begin{aligned}\langle \alpha_i, \alpha_j\rangle &= \langle \beta_i, \beta_j\rangle = 0, \text{ for all } i,j \\ \langle \alpha_i, \beta_j\rangle &= 0, \text{ for all } i,j, i \neq j \\ \langle \alpha_i, \beta_i\rangle &= 1, \text{ for all } i\end{aligned}$$

Let V be the subspace of W generated by $B_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Then:

1. *For all $v, v' \in V$, $\langle v, v'\rangle = 0$.*
2. *Let $w \in W$ but $w \notin V$. Then there exists $\alpha_k \in B_1$ such that $\langle w, \alpha_k\rangle \neq 0$.*

Proof. 1. This is a consequence of our choice of the basis B_1.

2. Let $w \in W$. Thus $w = \sum_{i=1}^{n} \lambda_i\alpha_i + \sum_{j=1}^{n} \mu_j\beta_j$, where $\lambda_i, \mu_j \in F$ and $\mu_k \neq 0$ for at least one $k, 1 \leq k \leq n$, because $w \notin V$. From here for this k we have:

$$\begin{aligned}\langle w, \alpha_k\rangle &= \left\langle \sum_{i=1}^{n} \lambda_i\alpha_i + \sum_{j=1}^{n} \mu_j\beta_j, \alpha_k \right\rangle \\ &= \sum_{i=1}^{n} \lambda_i \langle \alpha_i, \alpha_k\rangle + \sum_{j=1}^{n} \mu_j \langle \alpha_i, \alpha_k\rangle \\ &= 0 + \mu_k \\ &= \mu_k\end{aligned}$$

□

Corollary 6. *Let W' be the subspace of W generated by $B_2 = \{\beta_1, \beta_2, \cdots, \beta_n\}$. For any $a \in F$ and $w \in W'$, $w \neq 0$, there is $v \in V$ such that $\langle w, v\rangle = a$.*

Proof. By Lemma 1 there is $w' \in W$ such that $\langle w, w'\rangle = a$. Now write $w' = v_1 + v_2$ for some $v_1 \in V$ and $v_2 \in W'$(note that we have $W = V \bigoplus W'$). From here and properties of the basis B_2 we have:

$$\begin{aligned} a &= \langle w, w'\rangle \\ &= \langle w, v_1 + v_2\rangle \\ &= \langle w, v_1\rangle + \langle w, v_2\rangle \\ &= \langle w, v_1\rangle + 0 \\ &= \langle w, v_1\rangle \end{aligned}$$

Now set $v = v_1$. □

Lemma 10. *Let V be the subspace of W introduced in the Lemma 9. Set $K = \{(v, a) \in H \mid v \in V, a \in F\}$. Then K is a normal abelian subgroup of H.*

Proof. Let (v_1, a_1) and $(v_2, a_2) \in K$. Then by using first part of Lemma 9 we have:

$$\begin{aligned} (v_1, a_1)(-v_2, -a_2) &= (v_1 - v_2, a_1 - a_2 + \langle v_1, -v_2\rangle) \\ &= (v_1 - v_2, a_1 - a_2) \in K \end{aligned}$$

Since $K \neq \varnothing$ thus it is a subgroup of H. Also note that we have

$$\begin{aligned} (v_1, a_1)(v_2, a_2) &= (v_1 + v_2, a_1 + a_2) \\ &= (v_2 + v_1, a_2 + a_1) \\ &= (v_2, a_2)(v_1, a_1) \end{aligned}$$

So K is abelian. Now let $(w, a) \in H$. Then for any $(v, b) \in K$ we have

$$\begin{aligned} (w, a)(v, b)(-w, -a) &= (w + v - w, a + b - a + \langle w, v\rangle + \langle w + v, -w\rangle) \\ &= (v, b + \langle w, v\rangle) \in K. \end{aligned}$$

Thus K is normal. □

Definition 1. *Let K be as in Lemma 10. Define $\psi : K \to \mathbb{C}^{\times}$ by $\psi(v, a) = \chi(a)$, where χ is a nontrivial character of F^{+}.*

Lemma 11. *Let ψbe as in Definition 1 . Then ψ is a character of K. See also [5] and [6].*

Proof. Let (v_1, a_1) and $(v_2, a_2) \in K$. Then we have:

$$\begin{aligned}\psi\left(\left(v_1, a_1\right)\left(v_2, a_2\right)\right) &= \psi\left(v_1+v_2, a_1+a_2+\frac{1}{2}\left\langle v_1, v_2\right\rangle\right)\\ &= \psi\left(v_1+v_2, a_1+a_2+0\right)\\ &= \chi\left(a_1+a_2\right)\\ &= \chi\left(a_1\right)\chi\left(a_2\right)\\ &= \psi\left(v_1, a_1\right)\psi\left(v_2, a_2\right).\end{aligned}$$

□

Corollary 7. *There are at least $q-1$ nontrivial character of K.*

Proof. Because there are $q-1$ nontrivial character of F^+. □

Lemma 12. *Let ψ, K be as in Definition 1 . Let*

$$\mathcal{C}\left(H, K\right)=\left\{f: H \rightarrow \mathbb{C} \mid f\left(kh\right)=\psi\left(k\right) f\left(h\right),\ \textit{for all } k \in K, h \in H\right\}.$$

Then $\mathcal{C}(H, K)$ is a vector space over $\mathbb{C}$ of dimension q^n. See also [5]

Proof. It is easy to show $\mathcal{C}(H, K)$ is a vector space. We will show that $\dim \mathcal{C}(H, K) = q^n$. Let $S = \{s_1, s_2, \cdots, s_{q^n}\}$ be a set of representatives of cosets of K in H. Thus for any $h \in H$ there exist a unique $k \in K$ and $s \in S$ such that $h = ks$. Now for each $i, 1 \leq i \leq q^n$, define $f_i : H \rightarrow \mathbb{C}$ as follows

$$f_i\left(ks_j\right)=\chi\left(k\right)\delta_{ij}$$

where

$$\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

is Kronoker delta. Since for any $f \in \mathcal{C}(H, K)$, we then have

$$f=\sum_{i=1}^{q^n} f\left(s_i\right) f_i$$

and if we set $\sum_{i=1}^{q^n} \lambda_i f_i = 0$ we get $\lambda_i = 0$, for all $i, 1 \leq i \leq q^n$, the set of all these functions, $\{f_i\}_i, 1 \leq i \leq q^n$ is a basis for $\mathcal{C}(H, K)$. Thus $\dim \mathcal{C}(H, K) = q^n$. □

Definition 2. *Let Notations be as in Lemmas 12 and 8., and define $\rho : H \rightarrow GL(\mathcal{C}(H, K))$ by*

$$\left(\rho\left(h\right) f\right)\left(h'\right)=f\left(h'h\right), \qquad \textit{for all } f \in \mathcal{C}\left(H, K\right),\ \textit{and } h, h' \in H.$$

Theorem 3. *Let $(\rho, \mathcal{C}(H, K))$ be the function as defined in Definition 2. Then $(\rho, \mathcal{C}(H, K))$ is an irreducible representation of H of degree q^n.*

Proof. Let $h_1, h_2 \in H$. We then for all $f \in \mathcal{C}(H, K)$ and $h' \in H$ have:

$$\begin{aligned}(\rho(h_1h_2) f)(h') &= f((h'h_1) h_2) \\ &= (\rho(h_2) f)(h'h_1) \\ &= \rho(h_1)(\rho(h_2) f)(h')\end{aligned}$$

i.e. $\rho(h_1h_2) = \rho(h_1)\rho(h_2)$. Thus ρ is a representation of H. To Show $(\rho, \mathcal{C}(H,K))$ is irreducible; it is enough to show that $\psi^h \neq \psi$ for all $h \in H$ and $h \notin K$, where ψ^h is defined by

$$\psi^h(x) = \psi\left(h^{-1}xh\right), \quad \text{for all } x \in K.$$

See [7]. Since χ is nontrivial there is $a \in K$ such that $\psi(v, a) = \chi(a) \neq 1$, for all $v \in V$. (V is the same as in the Lemma 9.) Now Let W' be the same as in the Corollary 6. Let $h = (v, b) \in H \backslash K$; thus there are $v_1 \in V$ and $v_2 \in W', v_2 \neq 0$, such that $v = v_1 + v_2$.Then by the Lemma 1 there is some $v' \in V$ such that $\langle v_2, v' \rangle = a$. Now let $x = (v', 0) \in K$. We then have:

$$\begin{aligned}\psi^h(x) &= \psi\left(h^{-1}xh\right) \\ &= \psi((-v, -b)(v', 0)(v, b)) \\ &= \psi(v', \langle v, v' \rangle) \\ &= \chi(\langle v, v' \rangle).\end{aligned}$$

On the other hand we have

$$\begin{aligned}\langle v, v' \rangle &= \langle v_1 + v_2, v' \rangle \\ &= \langle v_1, v' \rangle + \langle v_2, v' \rangle \\ &= 0 + \langle v_2, v' \rangle \\ &= \langle v_2, v' \rangle \\ &= a.\end{aligned}$$

From here we get $\psi^h(x) = \chi(\langle v, v' \rangle) = \chi(a) \neq 1$, but $\psi(x) = \psi(v', 0) = \chi(0) = 1$, i.e. $\psi^h(x) \neq \psi(h^{-1}xh)$. See also [5] for another proof of irreducibility. □

Corollary 8. *Let$(\rho, \mathcal{C}(H,K))$ be the representation of H as defined in Definition 2. Then the central character of $(\rho, \mathcal{C}(H,K))$ is χ.*

Proof. Let $h = (0, a)$ be an element in $Z(H)$,the center of H. Then for all $f \in \mathcal{C}(H, K)$ and $(w, b) \in H$ we have:

$$\begin{aligned}(\rho(0, a) f)(w, b) &= f((w, b)(0, a)) \\ &= f((0, a)(w, b)) \\ &= \chi(a) f(w, b).\end{aligned}$$

i.e. $\rho(0, a) f = \chi(a) f$, for all $f \in \mathcal{C}(H, K)$. Thus $\rho_{|Z(H)} = \chi$. □

Corollary 9. *Different characters of F^+ induce different ρ's.*

Corollary 10. *Any irreducible representation of H is either a character as in Corollary 5 or a q^n- dimensional representation as in Theorem 3.*

Proof. The dimensions of irreducible representations of H, say $n_1, n_2, \cdots, n_k$, where k is the number of the irreducible representations of H must satisfy the following equation:

$$\begin{aligned} |H| &= q^{2n+1} \\ &= \sum_{i=1}^{k} n_i^2. \end{aligned}$$

From Corollary 5 we know there are q^{2n} one dimensional representations of H and from Theorem 3 we get $(q-1)$ irreducible representations of H of dimension q^n . Now note that:

$$\begin{aligned} \sum_{i=1}^{q-1} (q^n)^2 + \sum_{i=1}^{q^{2n}} 1^2 &= (q-1)\, q^{2n} + q^{2n} \\ &= q^{2n+1} \\ &= |H|\,. \end{aligned}$$

□

3 Representations of H when $p = 2$

Let $p = 2$ and let V, W' be as in Lemma 9 and Corollary 6. Thus $W = V \oplus W'$.

Lemma 13. *For any $w \in W$ we have $\langle w, w \rangle = 0$.*

Proof. Write $w = w_1 + w_2$, for $w_1 \in V$ and $w_2 \in W'$. We then have:

$$\begin{aligned} \langle w, w \rangle &= \langle w_1 + w_2, w_1 + w_2 \rangle \\ &= \langle w_1, w_1 \rangle + \langle w_1, w_2 \rangle + \langle w_2, w_1 \rangle + \langle w_2, w_2 \rangle \\ &= 0 + 2 \langle w_1, w_2 \rangle + 0 \\ &= 0. \end{aligned}$$

□

Definition 3. *Let χ be a nontrivial character of F^+. For each $w = w_1 + w_2 \in W$; define $\varphi_{(w,\chi)} : H \to \mathbb{C}^\times$ by*

$$\varphi_{(w,\chi)}(w', a) = \chi(a)\, \chi(\langle w_1, w_2' \rangle)\, \chi(\langle w_1', w_2' \rangle)\, \chi(\langle w_2, w_1' \rangle)\,.$$

For any $(w', a) \in H$, where $w' = w_1' + w_2' \in W$, $w_1' \in V$ and $w_2' \in W'$.

Theorem 4. *Let $\varphi_{(w,\chi)}$ be the function defined in Definition 3. Then $\varphi_{(w,\chi)}$ is a character of H whose restriction to F is χ.*

Proof. Let (w',a) and $(w'',b) \in H$ and write $w' = w_1' + w_2'$ and $w'' = w_1'' + w_2''$ for $w_1', w_1'' \in V$ and $w_2', w_2'' \in W'$. Then we have:

$$\begin{aligned}
(w',a)(w'',b) &= (w'+w'', a+b+\langle w',w''\rangle) \\
&= (w_1'+w_2'+w_1''+w_2'', a+b+\langle w_1'+w_2', w_1''+w_2''\rangle) \\
&= ((w_1'+w_1'')+(w_2'+w_2''), a+b+\langle w_1',w_1''\rangle+\langle w_1',w_2''\rangle+\langle w_2',w_1''\rangle+\langle w_2',w_2''\rangle) \\
&= ((w_1'+w_1'')+(w_2'+w_2''), a+b+\langle w_1',w_2''\rangle+\langle w_2',w_1''\rangle).
\end{aligned}$$

From here we have:

$$\begin{aligned}
& \varphi_{(w,\chi)}(w',a)(w'',b) \\
= & \varphi_{(w,\chi)}((w_1'+w_1'')+(w_2'+w_2''), a+b+\langle w_1',w_2''\rangle+\langle w_2',w_1''\rangle) \\
= & \chi(a+b+\langle w_1',w_2''\rangle+\langle w_2',w_1''\rangle)\chi(\langle w_1, w_2'+w_2''\rangle)\chi(\langle w_1'+w_1'', w_2'+w_2''\rangle)\chi(\langle w_2, w_1'+w_1''\rangle) \\
= & \chi(a+b)\chi(\langle w_1',w_2''\rangle+\langle w_2',w_1''\rangle)\chi(\langle w_1, w_2'+w_2''\rangle)\chi(\langle w_1'+w_1'', w_2'+w_2''\rangle)\chi(\langle w_2, w_1'+w_1''\rangle) \\
= & \chi(a+b)\chi(\langle w_1, w_2'+w_2''\rangle)\chi(\langle w_2, w_1'+w_1''\rangle)\chi(\langle w_1',w_2''\rangle+\langle w_2',w_1''\rangle)\chi(\langle w_1'+w_1'', w_2'+w_2''\rangle) \\
= & \chi(a+b)\chi(\langle w_1, w_2'+w_2''\rangle)\chi(\langle w_2, w_1'+w_1''\rangle)\chi(\langle w_1',w_2'\rangle+\langle w_1'',w_2''\rangle) \\
= & \chi(a)\chi(\langle w_1,w_2'\rangle)\chi(\langle w_1',w_2'\rangle)\chi(\langle w_2,w_1'\rangle)\chi(b)\chi(\langle w_1,w_2''\rangle)\chi(\langle w_1'',w_2''\rangle)\chi(\langle w_2,w_1''\rangle) \\
= & \varphi_{(w,\chi)}(w',a)\varphi_{(w,\chi)}(w'',b).
\end{aligned}$$

Thus $\varphi_{(w,\chi)}$ is a character of H. Inparticular we have: $\varphi_{(w,\chi)}(0,a) = \chi(a)$. □

Corollary 11. *For any $w \in W$ and any two distinct characters of F^+, χ_1 and χ_2 we have $\varphi_{(w,\chi_1)} \neq \varphi_{(w,\chi_2)}$.*

Lemma 14. *Let χ be a non-trivial character of F^+.Then for any two elements w and w' of W we have $\varphi_{(w,\chi)} \neq \varphi_{(w',\chi)}$.*

Proof. Let $w = w_1 + w_2 \in W$ and $w' = w_1' + w_2' \in W$ where $w_1, w_1' \in V$ and $w_2, w_2' \in W'$. Suppose $\varphi_{(w,\chi_1)} = \varphi_{(w,\chi_2)}$. For any $v \in V$ and $a \in F$ we then must have

$$\varphi_{(w,\chi_1)}(v,a) = \varphi_{(w,\chi_2)}(v,a)$$

But for the left hand side of this equation we have:

$$\begin{aligned}
\varphi_{(w,\chi_1)}(v,a) &= \chi(a)\chi(\langle w_1,0\rangle)\chi(\langle v,0\rangle)\chi(\langle w_2,v\rangle) \\
&= \chi(a)\chi(\langle w_2,v\rangle).
\end{aligned}$$

The same computations gives $\varphi_{(w,\chi_2)}(v,a) = \chi(a)\chi(\langle w_2',v\rangle)$. From here we must have $\chi(\langle w_2,v\rangle) = \chi(\langle w_2',v\rangle)$ for all $v \in V$. This and the Lemma 9 force to get $w_2 = w_2'$. The same argument by choosing $v \in W'$ gives $w_1 = w_1'$. □

Corollary 12. *The Theorem 4 gives* $(q-1)\,q^{2n}$ *characters of* H*.*

Proof. This is a consequence of the Corollary 11 and the Lemma 14. □

Definition 4. *Let* χ *be a nontrivial character of* F^{+}*; and set* $K=\{0\}\times F$*. By the Lemma 6* H/K *is isomorphic to additive group*W *. Thus any character of* H/K *is determined by its values on* $\{(w,0)\mid w\in W\}$ *.So for each* $w\in W$ *,* $\widetilde{\rho}_{(w,\chi)}:H/K\to\mathbb{C}^{\times}$ *defined by*

$$\widetilde{\rho}_{(w,\chi)}\left(\widetilde{(w',0)}\right)=\chi\left(\langle w,w'\rangle\right).$$

is a character of H/K *. Now define* $\rho_{(w,\chi)}:H\to\mathbb{C}^{\times}$ *by* $\rho_{(w,\chi)}\left(w',a\right)=\widetilde{\rho}_{(w,\chi)}\left(\widetilde{(w',0)}\right)$*.*

The rest of the characters of H are given in the following theorem.

Theorem 5. *Let* $\rho_{(w,\chi)}$*be the function as defined in the Definition 4. Then* $\rho_{(w,\chi)}$ *is a character of* H *whose restriction to* K *is trivial.*

Proof. The same proof as in the Theorem 2 works here. □

Corollary 13. *Theorem 5 determines* q^{2n} *characters of* H*.*

Proof. This result follows from Lemmas 7 and 8. □

Theorem 6. *Any representation of* H *when* $p=2$ *is either one of the characters defined in the Therorem 4 or one of the characters defined in the Theorem 5.*

Proof. Let ψ be a character of H. If the restriction of ψ to $\{0\}\times F$ is a nontrivial character, then it is one of the characters in the Theorem 4. If the restriction of ψ to $\{0\}\times F$ is trivial, then it is one of the characters in the Theorem 5. Moreover, the number of characters that are built in the Theorems 4 and 5 are:

$$(q-1)\,q^{2n}+q^{2n}=q^{2n+1}.$$

which is the same as the order of H . □

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