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Convergence acceleration of Fourier series revisited

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Abstract

A generalization of the Krylov-Eckhoff method is investigated for removing of the classical Gibbs phenomenon. Convergence acceleration scheme for Fourier expansions of piecewise smooth functions is derived. Numerical results are presented and discussed.

Key Words: Fourier series, Gibbs phenomenon, convergence acceleration.
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1 Introduction

Fourier series are used widely in many branches of mathematics. They are often used together with separation of variables to construct solutions to boundary value problems for differential equations as well as with spectral methods to find approximate solutions to these problems numerically. For practical purposes, solutions to these problems are usually obtained using a finite number of the terms in a Fourier series. This truncation procedure leads to nonuniformly valid approximations because when the approximated function has a point of discontinuity, the Gibbs phenomena is present. The "oscillations" caused by this phenomena typically propagate into regions away from the singularity and degrade the quality of the partial sum approximation.

In case of piecewise smooth functions the idea of overcoming the Gibbs phenomenon for convergence acceleration was suggested by A. Krylov as far back as in 1905 [1] but it was widely known only since 1960 after investigations of C. Lanczos [2]. The problem of practical applications was solved by K. Eckhoff in 1993 [3] by utilization of the Bernoulli polynomials for approximate reconstruction of jumps.

Let $f = f(x)$ is a function on $[-1, 1]$ with given points of singularities [1] $\{a_k\}$,

$$-1 = a_1 < \cdots < a_m = 1,\ 1 \leq m < \infty.$$

It is supposed that $a_{m+1} = a_1$ and if $a_1 = -1$ then $a_1 - 0 = 1 - 0,\ a_1 + 0 = -1 + 0$. Let $f \in C^{Q+1},\ Q \geq 0$ on each segment $[a_k, a_{k+1}]$. By $\{A_{sk}\}$ denote the following "jumps" of f and its derivatives in $\{a_s\}$

$$A_{sk}(f) = f^{(k)}(a_s - 0) - f^{(k)}(a_s + 0),\ k = 0, 1, \cdots, Q,\ s = 1, \cdots, m. \tag{1}$$

It is easy to see that the Fourier coefficients $\{f_n\}$ of f

$$f_n = \frac{1}{2}\int_{-1}^{1} f(t)e^{-i\pi nt}dt = \frac{1}{2}\sum_{s=1}^{m-1}\int_{a_s}^{a_{s+1}} f(t)e^{-i\pi nt}dt \tag{2}$$

have the following asymptotic representation $(n \neq 0)$

$$f_n = -\frac{1}{2}\sum_{s=1}^{m} e^{-i\pi n a_s}\sum_{k=0}^{Q}\frac{A_{sk}(f)}{(i\pi n)^{k+1}} + g_n,$$

where

$$g_n = \frac{1}{2(i\pi n)^{Q+1}}\int_{-1}^{1} f^{(Q+1)}(t)e^{-i\pi nt}dt = o(n^{-Q-1}),\ n \to \infty. \tag{3}$$

Consider now Bernoulli polynomials $\{B_k(x)\}$, $k \geq 0$ with the Fourier coefficients

$$B_{k,n} = \begin{cases} 0, & n = 0 \\ \dfrac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, & n \neq 0 \end{cases} \tag{4}$$

On the real line Bernoulli polynomials are considered as 2-periodic piecewise-smooth functions with "jump points" $\{a_k = 2k + 1\}$, $k = 0, \pm 1, \pm 2, ...$

According to (1-3) the function f can be represented in the form

$$f(x) = \sum_{\forall n} f_n e^{i\pi nx} = W_Q(x) + w(x) \tag{5}$$

where

$$W_Q(x) = \sum_{s=1}^{m}\sum_{k=0}^{Q} A_{sk}(f)B_k(x - a_s - 1)$$

and $w \in C^Q(R)$.

Hence the sequence

$$F_N(x) = W_Q(x) + \sum_{n=-N}^{N} w_n e^{i\pi nx} \tag{6}$$

[1] We call a point a_j as the Q-singularity point of a $g(x)$ if $\sum_{k=0}^{Q}|A_{jk}(g)|^2 \neq 0$. The points $x = \pm 1$ are identified

converges uniformly to $f(x)$ with the rate $o(N^{-Q-1})$, $N \to \infty$.

However for a realization of the corresponding algorithm we need to know not only the Fourier coefficients $\{f_n\}$, $|n| \le N$ but also the jump points $\{a_s\}$ and the corresponding jump values $\{A_{sk}(f)\}$.

The problem of determination of the approximate values $\{\tilde{A}_{sk}\}$ can be solved by utilization only of the Fourier coefficients (if all jump points $\{a_s\}$ are known) by solution of the following linear system, arising from (2), by an appropriate choice of indices $n = n_s$, $s = 1, 2, ..., m(Q+1)$, $\theta N \le |n_s| \le N$, $0 < \theta < 1$

$$f_n = -\frac{1}{2}\sum_{s=1}^{m} e^{-i\pi n a_s} \sum_{k=0}^{Q} \frac{\tilde{A}_{sk}}{(i\pi n)^{k+1}},\ n = n_1, n_2, ..., n_{m(Q+1)}$$

provided unique solvability[2].

We will call this approach as *Krylov-Eckhoff (KE)-method* and $W_Q(x)$ as a *correction function.* From above we have

Proposition 1. *KE-method is exact for a finite sum of piecewise-polynomials when Q is rather big and $N > (Q+1)/2$.*

Many investigations were devoted to similar problems (see, for example, [4]-[13] with references therein). In particular, D. Gottlieb and C.-W. Shu [4] proposed a way of overcoming the Gibbs phenomenon by a technique that involved a conversion to a series by the Gegenbauer polynomials. In [7] a method that used quasi-polynomials was suggested which was further generalized in [9] to the eigenfunction expansions using corresponding Green functions of the regular boundary value problems for ODE with smooth coefficients.

This paper proposes a general formulation of convergence acceleration scheme which actually contains the above mentioned Krylov-Eckhoff method as a particular case.

2 The scheme

The idea of our generalization is a realization of Krylov-Eckhoff scheme using a wide range of correction functions for acceleration in the case of $m \ge 1$ $Q-$singularity points (see Introduction). For a fixed integer Q consider a system of piecewise smooth *basis functions* $\{\Phi_r^j(x)\}$, $r = 1, 2, \cdots Q+1$, $j = 1, 2, \cdots, m$, $-1 \le x \le 1$ with corresponding Fourier coefficients $\{\phi_{rn}^j\}$, $n = 0, \pm 1, \pm 2, \cdots$. Suppose that we can quickly and efficiently find values of each $\Phi_r^j(x)$ on the segment $[-1, 1]$ (for example, when we have its explicit form) as well as find values of corresponding Fourier coefficients $\{\phi_{rn}^j\}$ for each n (for example, by an explicit form).

[2]In the case of smooth f the unique solvability follows from the fact that the matrix of the system is a Vandermonde matrix.

2.1 The case of smooth f

First consider the simplest case when $m = 1$. Then $\Phi_r^1(x) = \Phi_r(x)$, $\Phi_r^1 \in C^{Q+1}[-1, 1]$, and $f \in C^{Q+1}[-1, 1]$. We will show that (providing some simple conditions) the following function G

$$G(x) = \sum_{r=1}^{Q+1} c_r \, \Phi_r(x)$$

is a correction function if coefficients $\{c_r\}$ are determined appropriately.

Denote by $\phi_{rn}, n = 0, \pm 1, \cdots$ the Fourier coefficients of $\Phi_r(x)$. According to (3) we have the following asymptotic representation

$$\phi_{rn} = (-1)^{n+1} \sum_{k=0}^{Q} \frac{h_{rk}}{2(i\pi n)^{k+1}} + o(n^{-Q-1}), \; n \to \infty, \tag{7}$$

where $\{h_{rk}\}$ are the corresponding jumps of $\Phi_r(x)$ and its derivatives

$$h_{rk} = \Phi_r^{(k)}(1) - \Phi_r^{(k)}(-1).$$

Hence for Fourier coefficients $\{g_n\}$ of $G(x)$ we have

$$g_n = (-1)^{n+1} \sum_{k=0}^{Q} \frac{H_k}{2(i\pi n)^{k+1}} + o(n^{-Q-1}), \; n \to \infty, \tag{8}$$

where

$$H_k = \sum_{r=1}^{Q} c_r h_{rk}.$$

Representation (8) leads to the following system of linear equations for determination of the values $\widetilde{H}_k$ which approximate the values of H_k

$$(-1)^{n_s+1} \sum_{k=0}^{Q} \frac{\widetilde{H}_k}{2(i\pi n_s)^{k+1}} = f_{n_s}, \; s = 1, \cdots, Q+1. \tag{9}$$

The Vandermonde matrix of this system provides unique solvability.

It is clear, that here the acceleration problem is solved if the matrix $[h_{rk}]$, $r, k = 1, 2, \cdots, Q$ is invertible so approximate vales $\widetilde{c}_k$ of c_k can be found.

2.2 General case

In this case we suppose that each $\Phi_r^j(x)$ has only one Q-singularity point a_j.

Here we have the following asymptotic representation $(j = 1, 2, \cdots, m)$

$$\phi_{rn}^j = -e^{-i\pi a_j n} \sum_{k=0}^{Q} \frac{h_{rk}^j}{2(i\pi n)^{k+1}} + o(n^{-Q-1}), \; n \to \infty. \tag{10}$$

Hence for Fourier coefficients $\{g_n\}$ of G

$$G(x)=\sum_{j=1}^{m}\sum_{r=1}^{Q+1} c_r^j\,\Phi_r^j(x)$$

we have

$$g_n=-\sum_{j=1}^{m} e^{-i\pi a_j n}\sum_{k=0}^{Q}\frac{H_k^j}{2(i\pi n)^{k+1}}+o(n^{-Q-1}),\ n\to\infty, \tag{11}$$

where

$$H_k^j=\sum_{r=1}^{Q+1} c_r^j h_{rk}^j.$$

Representation (11) leads to the following system of linear equations for determination of the approximate values $\widetilde{H}_k^j$ of H_k^j

$$-\sum_{j=1}^{m} e^{-i\pi a_j n_s}\sum_{k=0}^{Q}\frac{\widetilde{H}_k^j}{2(i\pi n_s)^{k+1}}=f_{n_s},\ s=1,2,\cdots,(Q+1)m,\ n_s=O(N),\ N\to\infty. \tag{12}$$

It is easy to see that the matrix $\mathfrak{M}$ of this system is block-Vandermonde and does not depend on the type of correction functions.

Proposition 2. *The equation (12) is uniquely solvable if the $(Q+1)\,m\times(Q+1)\,m$-matrix $\mathfrak{M}$ and $(Q+1)\times(Q+1)$-matrices $[h_{rk}^j]$, $j=1,2,\cdots,m,$ are invertible and N is rather big.*

Proposition 3. *Described method is exact for any linear combination of functions $\{\Phi_j^j(x)\}$ if the $(Q+1)\times(Q+1)$-matrices $[h_{rk}^j]$, $j=1,2,\cdots,m,$ are invertible and $N>(Q+1)\,m/2$.*

Remark 1. *The solution of equation (12) is actually a repetition of corresponding part of KE-method with additional inverses of matrices $[h_{rk}^j]$.*

2.2.1 Some examples of basis functions

In addition to Bernoulli polynomials used by K. Eckhoff (when corresponding basis functions are polynomials) we consider some other simple cases.

- *Exponential basis functions*

$$\alpha_{rj}\in\mathbb{C},\ \Phi_r^j(x)=\begin{cases}-1<x<a_j, & \exp(i\,\pi\,\alpha_{rj}(x-a_j))\\ x\ge a_j, & \exp(i\,\pi\,\alpha_{rj}(x-a_j-2))\end{cases} \tag{13}$$

Fourier coefficients of basis functions are

$$\phi_{rn}^j=\begin{cases}\text{If } \alpha_{rj} \text{ is integer and } n=\alpha_{rj}, & 0\\ \text{Otherwise,} & \frac{\sin(\pi\,\alpha_{rj})}{i\pi(\alpha_{rj}-n)}\end{cases} \tag{14}$$

In this case $[h^j_{rk}] = 2\,[i^{k+1}\alpha^k_{\mathrm{rj}}\,\sin{(\alpha_{\mathrm{rj}})}]$ is multiplication of a Vandermonde matrix by diagonal matrices (from left side and right side). Here we see that suggested acceleration algorithm is valid, if for any fixed $j\,(=1,2,\cdots,m)$ the values α_{rj}, $r=1,2,\cdots,m$, are different and not integer.

- *Rational basis functions*

$$\alpha_{rj}\in\mathbb{C}\diagdown[-1,1],\ \Phi^j_r(x)=\begin{cases}-1<x<a_j, & \frac{1}{x-\alpha_{rj}}\\ x\geq a_j, & \frac{1}{x-\alpha_{rj}-2}\end{cases} \tag{15}$$

Here we have

$$\phi^j_{rn}=\frac{1}{2}\exp(-i\,\alpha_{rj}\,\pi\,n)(\Gamma(0,-i\,\pi\,n\,(\alpha_{rj}+1))-\Gamma(0,-i\,\pi\,n\,(\alpha_{rj}-1))) \tag{16}$$

where $\Gamma(a,z)=\int_z^{+\infty}t^{a-1}\,e^{-t}\,dt$ is the incomplete gamma function which is one of special functions included in the MATHEMATICA package. It has a branch cut discontinuity in the complex z plane running from $-\infty$ to 0 (see [13]). So it is necessary to calculate (16) separately on the cut line (for example, when n=0).

- *Bernoulli-like basis functions.*

It is convenient to use a system $\{\Phi^j_r(x)\}$ with triangular nonsingular $[h^j_{rk}]$ for each fixed j. Note, that in [12] the following functions $(k=0,1,\cdots)$ are considered

$$\Psi_l(x)=\begin{cases}l=2k+1, & (1-\cos(\pi(x+1)))^{k+1/2}\\ l=2k, & \sin(\pi x)(1-\cos(\pi(x+1)))^{k-1/2}\end{cases}$$

Fourier coefficients of this functions have explicit form and corresponding system $\{\Psi_r(x)\}, k=0,1,\cdots Q$ will be Bernoulli-like for any Q (see page 262 in [12]).

Remark 2. *It is easy to see that from any basis $\{\Phi^j_r(x)\}$ one can construct a Bernoulli-like system if Proposition 3 is fulfilled.*

Remark 3. *In consideration of known relations between the discrete and the continuous Fourier series one can expand above stated results to trigonometric interpolation on a uniform grid (similarly to [13]).*

3 Numerical results

Below "Classic" algorithm corresponds to the truncated sum

$$f_N(x)=\sum_{n=-N}^{N}f_n\,e^{i\pi nx}.$$

3.1 Smooth function

First consider a smooth function (with singularity only in $x = a_1 = \pm 1$)

$$f_1(x) = J_{\frac{7}{3}}\left(ix^2 - x + 1\right)\sinh\left(\frac{2}{3}\cos\left(\frac{1}{3} - x\right) - i(5x + 2i)\log(x+4)\right).$$

Figure 1 shows the graphs of real and imaginary parts of this function.

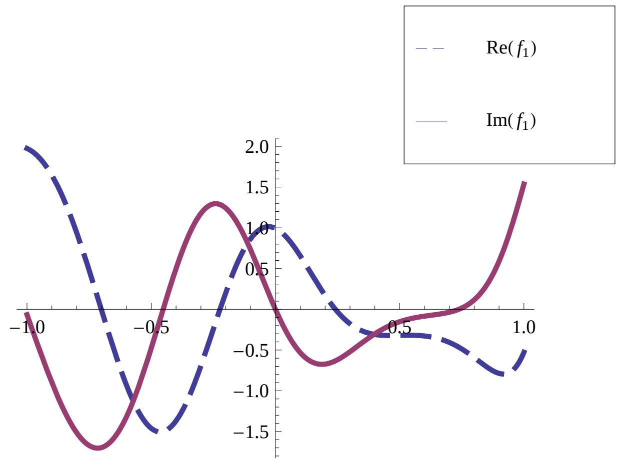

Figure 1: *The graphs of real and imaginary parts of $f_1(x)$.*

For acceleration algorithms with $Q = 3$ the following basis functions are used

Algorithms	**Basis functions**
KE-method	$\{x/2, x^2/4 - 1/12, x^3/12 - x/12, x^4/48 - x^2/24 + 7/720\}$
Bernoulli-like	$\{\mathrm{erf}(x), (e^x - e^{-1})\log(2-x), (1-x^2)^2/(x+5), (\cos(2) - \cos(2x))^3\}$
Exponential	$\{\exp(i\pi x/4), \exp(-i\pi x/2), \exp(4\,i\pi x/3), \exp(-5\,i\pi x/2)\}$
Rational	$\{1/(x+3),\ 1/(x-3),\ 1/(x+5),\ 1/(x-5)\}$

Table 1. *Basis functions for $f = f_1(x)$, $Q = 3$.*

In Table 2 working time of each algorithm corresponds to the sum of times for $N = 16$ and $N = 64$.

Error norm→	L_∞ -errors		L_2 -errors	
Algorithms ; time ↓ ‖N →	16	64	16	64
Classic ; 0.016 sec.	1.5 e-0	1.5 e0	8.8 e-2	4.4 e-2
KE-method ; 0.063 sec.	1.6 e-3	5.1 e-6	5.7 e-5	9.2 e-8
Bernoulli-like ; 0.17 sec.	1.7 e-3	5.2 e-6	6.0 e-5	9.4 e-8
Exponential ; 0.063 sec.	1.1 e-3	3.5 e-6	4.0 e-5	6.4 e-8
Rational ; 0.187 sec.	1.6 e-3	5.2 e-6	5.8 e-5	9.3 e-8

Table 2. *Errors and working times while approximating $f_1(x)$ when $N = 16, 64; Q = 3$.*

3.2 Function with one jump

Now consider a function with jumps in an additional point ($a_1 = \pm 1, a_2 = 1/3$)

$$f_2(x) = \begin{cases} x \le \frac{1}{3}, & \frac{2}{3-x} + x\sin((3+2i)x) \\ x > \frac{1}{3}, & e^{-x} - i\cos(4x) \end{cases}$$

Figure 2 shows the graphs of real and imaginary parts of f_2.

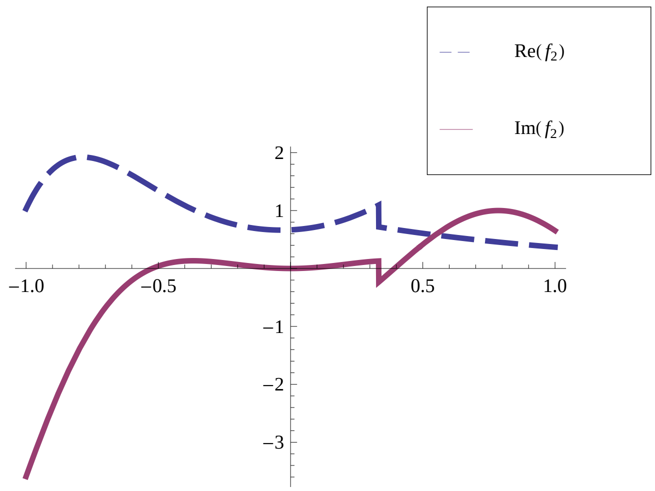

Figure 2: *The graphs of real and imaginary parts of* $f_2(x)$.

For acceleration algorithms with $Q = 2$ the following basis functions $\{\Phi_k^1(x)\}$, $\{\Phi_k^2(x)\}$, $k = 1, 2, 3$ are used.

Algorithms	**Basis functions**
KE-method	$\{x,\ x^2 - 1/3,\ x^3 - x\}$, $\{x,\ x^2 - 1/3,\ x^3 - x\}$
Exponential	$\{\exp(\frac{i\pi x}{2}), \exp(\frac{-3i\pi x}{2}), \exp(\frac{5i\pi x}{3})\}, \{\exp(\frac{-2i\pi x}{3}), \exp(\frac{4i\pi x}{5}), \exp(\frac{-7i\pi x}{4})\}$
Rational	$\{\frac{1}{(x-3)}, \frac{1}{(x+5)}, \frac{1}{(x-5)}\}, \{\frac{1}{(x+3)}, \frac{1}{(x-5/2)}, \frac{1}{(x+7/3)}\}$
Hybrid	$\{\exp(\frac{i\pi x}{2}),\ \log(3-x),\ \frac{1}{(x+3)}\}$, $\{x^2,\ x\ \exp(x),\ \frac{1}{(x-5)}\}$

Table 3. *Basis functions for* $f = f_2(x)$, $Q = 2$.

Error norm→	L_∞ -errors		L_2 -errors	
Algorithms ; time ↓ ‖N →	32	128	32	128
Classic ; 0.016 sec.	2.2 e 0	2.2 e 0	1.2 e-1	6.0 e-2
KE-method ; 1.61 sec.	2.1 e-4	2.9 e-6	1.0 e-5	7.1 e-8
Exponential ; 1.51 sec.	2.2 e-4	3.2 e-6	1.0 e-5	6.8 e-8
Rational ; 1.6 sec.	2.7 e-4	3.8 e-6	1.3 -5	8.8 e-8
Hybrid ; 2.22 sec.	1.5 e-4	2.6 e-6	7.3 e-6	5.0 e-8

Table 4. *Errors and working times of algorithms while approximating $f_2(x)$ for $N = 32, 128; Q = 2$.*

In Table 4 working time of each algorithm corresponds to the sum of times for $N = 32$ and $N = 128$.

4 Conclusion

The results of numerical experiments show that there is no any strict recommendation concerning the choice of basis functions in spite of repetition a part of KE-method in any cases (see Remark 1 above). Accuracy of the method depends on the approximated function rather than on the basis functions. L_2-error depends mainly on the distance between expanded function f and finite-dimensional space which has the system $\{\Phi_r^j(x)\}$, $j = 1, \cdots, m$; $r = 0, \cdots, Q$; as a basis.

It seems that the choice of parameters of some basis functions (for example, vales α_{rj} in *exponential basis functions* and *rational basis functions*) can give possibilities for construction of adaptive algorithms.

Working times of different algorithms mainly depend on the inner procedures of the packages that calculate the pointwise values and the Fourier coefficients ϕ_{rn}^j of the basis functions. We have used the MATHEMATICA package (see [14]) which calculates the values of (16) rather slow especially for large vales of N (see Table 4).

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