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A New Family of Number Sequences: Leonardo-Alwyn Numbers

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Abstract. In this study, we define a new type of number sequence called Leonardo-Alwyn sequence. We obtain the Binet formula, generating function and some relations for these numbers. Moreover, we give the matrix representation of the Leonardo-Alwyn numbers.

Key Words: Leonardo Number, Leonardo-Alwyn Number, John-Edouard Number, Ernst Number
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Introduction and Preliminaries

Number sequences arise in many different theoretical and applied areas, as well as in mathematical modeling of all the problems where there is a kind of invariance to shift in terms of space or of time. As in the computation of spline functions, time series analysis, signal and image processing, queueing theory, polynomial and power series computations and many other areas, typical problems modelled by number sequences are the numerical solution of certain differential and integral equations (see, for example, [7, 18, 23]).

The Fibonacci sequence has delighted mathematicians and scientists alike for centuries because of its beauty and tendency to appear in unexpected places. Fibonacci numbers are found in Pascals triangle, Pythagorean triples, computer algorithms, graph theory and many other areas of mathematics. They also occur in a variety of other fields such as physics, finance, architecture, computer sciences, color image processing, geostatistics, music, and art. There are many studies of the Fibonacci sequence in the literature because of its numerous applications as well as many generalizations, some of which can be found in [1–3, 8, 9, 11–13, 16, 24, 29].

Leonardo Fibonacci and Alwyn Horadam examined number sequences defined by recurrence relations, which were then studied over the years (see, for example, [4, 10, 19, 22, 25, 27, 28]). The Leonardo sequence, also known as Leonardo numbers, is a linear recurrent sequence of integers related to the Fibonacci sequence (see [30]). Its elements are defined by the following recurrence formula

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \quad n \geq 2$$

with $Le_0 = Le_1 = 1$. This sequence can also be defined in the following way:

$$Le_0 = Le_1 = 1, \qquad Le_2 = 3, \qquad Le_n = 2Le_{n-1} - Le_{n-3}, \quad n \geq 3.$$

Corresponding characteristic equation is given by

$$x^3 - 2x^2 + 1 = 0$$

and has three real roots: $x_1 = (1+\sqrt{5})/2$, $x_2 = (1-\sqrt{5})/2$ and $x_3 = 1$. Note that x_1 and x_2 are the roots of the characteristic equation of the Fibonacci sequence (see [5]).

Fibonacci and Lucas numbers are defined by the following recurrence relations

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+2} = F_{n+1} + F_n, \quad n \geq 0$$

and

$$L_0 = 2, \qquad L_1 = 1, \qquad L_{n+2} = L_{n+1} + L_n, \quad n \geq 0,$$

respectively. Besides, the n^{th} Fibonacci and Lucas numbers are formulized as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad L_n = \alpha^n + \beta^n, \qquad n \geq 1,$$

where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$ (see [20]).

Pell and Pell-Lucas numbers are defined by

$$P_0 = 0, \qquad P_1 = 1, \qquad P_{n+2} = 2P_{n+1} + P_n, \quad n \geq 0$$

and

$$Q_0 = Q_1 = 2, \qquad Q_{n+2} = 2Q_{n+1} + Q_n, \quad n \geq 0,$$

respectively. Equivalently, these numbers are formulized as

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad Q_n = \alpha^n + \beta^n, \qquad n \geq 1,$$

where $\alpha = 1 + \sqrt{2}$, $\beta = 1 - \sqrt{2}$ (see [21]).

Jacobsthal and Jacobsthal-Lucas numbers are defined recurrently as follows

$$J_0 = 0, \qquad J_1 = 1, \qquad J_{n+2} = J_{n+1} + 2J_n, \quad n \geq 0$$

and

$$K_0 = 2, \qquad K_1 = 1, \qquad K_{n+2} = K_{n+1} + 2K_n, \quad n \geq 0,$$

or, equivalently,

$$J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad K_n = \alpha^n + \beta^n,$$

where $\alpha = 2$, $\beta = -1$ (see [6]).

Horadam numbers are defined by the second-order linear homogeneous recurrence relation

$$H_{n+2} = aH_{n+1} + bH_n, \quad n \geq 0,$$

with $H_0 = r$, $H_1 = s$, where $a, b, r, s \in \mathbb{Z}$. Equivalently, they can be formulized as

$$H_n = \frac{(s - r\beta)\alpha^n - (s - r\alpha)\beta^n}{\alpha - \beta}, \quad n \geq 0,$$

where $\alpha = (a + \sqrt{a^2 + 4b})/2$, $\beta = (a - \sqrt{a^2 + 4b})/2$ (see [17]).

Horadam	$aH_{n-1} + bH_{n-2}$	a	b	r	s
Fibonacci	$F_{n-1} + F_{n-2}$	1	1	0	1
Pell-Lucas	$2Q_{n-1} + Q_{n-2}$	2	1	2	2
Jacobsthal	$J_{n-1} + 2J_{n-2}$	1	2	0	1

Table 1: Number sequences and their names

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34
Q_n	2	2	6	14	34	82	198	478	1154	2786
J_n	0	1	1	3	5	11	21	43	85	171

Table 2: The first ten terms of the Fibonacci, Pell-Lucas and Jacobsthal sequences

The Binet formula for a sequence of numbers whose characteristic equation is of the 3^{rd} order can be formed as follows

$$S_n = \frac{m_1\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{m_2\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{m_3\gamma^n}{(\gamma - \alpha)(\gamma - \beta)}, \quad n \geq 0,$$

where $m_1 = S_2 - (\beta + \gamma)S_1 + \beta\gamma S_0$, $m_2 = S_2 - (\alpha + \gamma)S_1 + \alpha\gamma S_0$ and $m_3 = S_2 - (\alpha + \beta)S_1 + \alpha\beta S_0$. Equivalently, S_n is defined by

$$S_0 = a, \qquad S_1 = b, \qquad S_2 = c, \qquad S_n = rS_{n-1} + sS_{n-2} + tS_{n-3}, \quad n \geq 3,$$

where a, b and c are arbitrary complex or real numbers and r, s and t are real numbers (see [26]).

In this work, Leonardo-Alwyn numbers will be defined and a variety of their algebraic properties will be presented. Some identities, such as Binets formula and generating function formula, as well as a matrix representation of the Leonardo-Alwyn number sequence, will be given.

1 The Leonardo-Alwyn Numbers

In this section, the Leonardo-Alwyn numbers, dedicated to Leonardo Fibonacci and Alwyn Horadam, will be introduced. These numbers will be defined similarly to the Leonardo number sequence. By adding the number 1 to the roots of number sequences with quadratic characteristic equations, a third-order characteristic equation will be obtained. Starting from this equation, the recurrence relationship between the numbers of the Leonardo-Alwyn sequence will be found. The elements of this sequence can also be derived from sequences of numbers whose two consecutive terms are equal; namely, the first two terms of this number sequence are determined to be the same.

The Leonardo-Alwyn sequence with the properties of well-known number sequences seems to be useful in different applications.

Definition 1 *Let integers $a \geq 1$ and b, c be such that $a+b-1 \neq 0$ and $a^2+4b \geq 1$. The Leonardo-Alwyn numbers LA_n are recursively defined by*

$$LA_n = aLA_{n-1} + bLA_{n-2} + c, \quad n \geq 2,$$

$$LA_0 = LA_1 = a = \begin{cases} H_0, & H_0 \neq 0, \\ H_1, & H_0 = 0, \end{cases}$$

where $H_0 = r$ and $H_1 = s$ are the first two Horadam numbers, $r, s \in \mathbb{Z}$ and $c = r + s$.

From equations

$$LA_n = aLA_{n-1} + bLA_{n-2} + c, \qquad LA_{n+1} = aLA_n + bLA_{n-1} + c, \quad n \geq 2,$$

one can obtain by subtraction an equivalent recurrence formula for LA_n.

Proposition 1 *The Leonardo-Alwyn sequence can be equivalently defined by*

$$LA_{n+1} = (a+1)LA_n + (b-a)LA_{n-1} - bLA_{n-2}, \quad n \geq 2,$$

where $LA_2 = a^2 + ab + c$ is an additional value. The associated characteristic polynomial is

$$p(X) = X^3 - (a+1)X^2 - (b-a)X + b = (X-1)(X^2 - aX - b),$$

which has roots $t_1 = (a + \sqrt{a^2+4b})/2$, $t_2 = (a - \sqrt{a^2+4b})/2$ and $t_3 = 1$.

The Leonardo sequence is obtained using the characteristic equation of the Fibonacci sequence. This sequence is named after Leonardo Bigollo Pisano, who obtained the Fibonacci sequence. The first names of Pell and Lucas are given since the John-Edouard sequence is also constructed using the characteristic equation of the Pell-Lucas sequence. Similarly, the characteristic equation of the Jacobsthal sequence is used in the Ernst sequence, and Jacobsthal's name is given to it. The Ernst sequence is also known as the Purkiss sequence (see, for example, [14], [15]).

The new family of number sequence is Leonardo-Alwyn, and its members are: Leonardo numbers LE_n (sequence A001595 in the ON-LINE Encyclopedia of integer sequences), John- Edouard numbers JE_n (new) and Ernst numbers ER_n (sequence A051049 in the ON-LINE Encyclopedia of integer sequences [semi-new: Ernst and Purkiss numbers have the same terms but different derivations]).

The sequences LE_n, JE_n and ER_n satisfy the following third-order linear recurrences:

$$LE_0 = LE_1 = 1, \quad LE_2 = 3, \quad LE_n = 2LE_{n-1} - LE_{n-3}, \quad n \geq 3;$$

$$JE_0 = JE_1 = 2, \quad JE_2 = 10, \quad JE_n = 3JE_{n-1} - JE_{n-2} - JE_{n-3}, \quad n \geq 3;$$

$$ER_0 = ER_1 = 1, \quad ER_2 = 4, \quad ER_n = 2ER_{n-1} + ER_{n-2} - 2ER_{n-3}, \quad n \geq 3.$$

Leonardo-Alwyn	$aLA_{n-1} + bLA_{n-2} + c$	a	b
Leonardo	$LE_{n-1} + LE_{n-2} + 1$	1	1
John-Edouard	$2JE_{n-1} + JE_{n-2} + 4$	2	1
Ernst	$ER_{n-1} + 2ER_{n-2} + 1$	1	2

Table 3: Number sequences and their names

n	0	1	2	3	4	5	6	7	8	9
LE_n	1	1	3	5	9	15	25	41	67	109
JE_n	2	2	10	26	66	162	394	954	2306	5570
ER_n	1	1	4	7	16	31	64	127	256	511

Table 4: The first ten terms of the Leonardo, John-Edouard and Ernst sequences

Definition 2 *The sequence LA_{-n} is defined as follows*

$$LA_{-n} = \frac{(b-a)}{b} LA_{-n+1} + \frac{(a+1)}{b} LA_{-n+2} - \frac{1}{b} LA_{-n+3}, \quad n \geq 1,$$

where $LA_0 = LA_1 = a$ and $LA_2 = a^2 + ab + c$.

The sequences LE_{-n}, JE_{-n} and ER_{-n} satisfy the following third order linear recurrences

$$LE_{-n} = 2LE_{-n+1} - LE_{-n+3},$$
$$JE_{-n} = -JE_{-n+1} + 3JE_{-n+2} - JE_{-n+3},$$
$$ER_{-n} = \frac{1}{2}ER_{-n+1} + ER_{-n+2} - \frac{1}{2}ER_{-n+3},$$

$n \geq 3$.

n	-1	-2	-3	-4	-5
LE_n	-1	1	-3	3	-7
JE_n	-6	10	-30	66	-166
ER_n	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{7}{8}$	$\frac{1}{16}$	$-\frac{31}{32}$

Table 5: The first five negative terms of the Leonardo, John-Edouard and Ernst sequence

Theorem 1 *The Binet's formula for LA_n, $n \geq 0$ numbers is*

$$LA_n = \alpha t_1^n + \beta t_2^n + \gamma t^n, \quad n \geq 0,$$

where

$$t_1 = \frac{a + \sqrt{a^2 + 4b}}{2}, \quad t_2 = \frac{a - \sqrt{a^2 + 4b}}{2}, \quad t_3 = 1,$$
$$\alpha = \frac{(a^2 + ab - a + c)(a^2 + 4b - (a-2)\sqrt{a^2 + 4b})}{2(a^2 + 4b)(a + b - 1)},$$
$$\beta = \frac{(a^2 + ab - a + c)(a^2 + 4b + (a-2)\sqrt{a^2 + 4b})}{2(a^2 + 4b)(a + b - 1)}, \quad \gamma = \frac{c}{1 - a - b}.$$

Moreover,

$$t_1 + t_2 + t_3 = a + 1, \quad t_1 t_2 t_3 = -b, \quad t_1 t_2 + t_1 t_3 + t_2 t_3 = a - b.$$

Proof. The proof is carried out using equation (1). □

The sequences LE_n, JE_n and ER_n satisfy the following Binet's formulas:

$$LE_n = \left(\frac{5 + \sqrt{5}}{5}\right) t_1^n + \left(\frac{5 - \sqrt{5}}{5}\right) t_2^n - 1,$$
$$JE_n = 2t_1^n + 2t_2^n - 2,$$
$$ER_n = t_1^n + \frac{1}{2}t_2^n - \frac{1}{2},$$

$n \geq 0$, where t_1 and t_2 represent the roots of sequences with quadratic characteristic equations associated with the sequences LE_n, JE_n and ER_n, respectively.

Theorem 2 *Let integers l_n be recursively defined by*

$$l_1 = l_2 = a, \qquad l_n = al_{n-1} + bl_{n-2}, \quad n \geq 2.$$

Then

$$LA_n = \left[1 + \frac{c}{a(a+b-1)}\right] l_{n+1} - \frac{c}{a+b-1}, \quad n \geq 0.$$

Proof. We show by induction on n. For $n > 0$ and $n = 1$, we have $LA_0 = LA_1 = l_1 = l_2 = a$, and it remains to check the equality

$$a = a\left[1 + \frac{c}{a(a+b-1)}\right] - \frac{c}{a+b-1},$$

which is obviously holds. Let the assertion be true for $(n-1)$ and n with $n \geq 1$. Then

$$\begin{aligned} LA_{n+1} &= a\left[1 + \frac{c}{a(a+b-1)} l_{n+1} - \frac{c}{a+b-1}\right] \\ &\quad + b\left[1 + \frac{c}{a(a+b-1)} l_n - \frac{c}{a+b-1}\right] + c \\ &= \left[1 + \frac{c}{a(a+b-1)}\right](al_{n+1} + bl_n) - \left[\frac{c}{a+b-1}\right](a+b) + c \\ &= \left[1 + \frac{c}{a(a+b-1)}\right] l_{n+2} - \frac{c}{a+b-1}. \end{aligned}$$

□

Corollary 1 *If $l_0 = l_1 = a$, then*

$$LA_n = \left[1 + \frac{c}{a(a+b-1)}\right] l_n - \frac{c}{a+b-1}, \quad n \geq 0.$$

Corollary 2 *For the sequences LE_n, JE_n and ER_n, the following equations hold*

$$\begin{aligned} LE_n &= 2F_{n+1} - 1, \\ JE_n &= 2Q_n - 2, \\ ER_n &= \frac{3J_{n+1} - 1}{2} \end{aligned}$$

for all $n \geq 0$.

Corollary 3 *The Leonardo numbers in the form $LE_n = 2F_{n+1} - 1$ are odd. The John-Edouard numbers in the form $JE_n = 2(Q_n - 1)$ are even. The elements of the Ernst sequence are even when n is even and odd otherwise. Moreover,*

$$ER_n = \begin{cases} 1 & \text{if } n = 0, \\ 2^n - 1 & \text{if } n \text{ is odd}, \\ 2^n & \text{if } n \text{ is even}. \end{cases}$$

Theorem 3 *The generating function for the Leonardo-Alwyn numbers is*

$$h(t) = \frac{LA_0 + [LA_1 - (a+1)LA_0]\,t + [LA_2 - (b-a)LA_0 - (a+1)LA_1]\,t^2}{1-(a+1)t-(b-a)t^2+bt^3}.$$

Proof. Let $h(t) = \sum_{n=0}^{\infty} LA_n t^n$ be the generating function for Leonardo-Alwyn. Then

$$(a+1)th(t) = (a+1)\sum_{n=0}^{\infty} LA_n t^{n+1},$$

$$(b-a)t^2h(t) = (b-a)\sum_{n=0}^{\infty} LA_n t^{n+2}$$

and

$$(-b)t^3h(t) = (-b)\sum_{n=0}^{\infty} LA_n t^{n+3}.$$

After the necessary calculations, the statement of the theorem follows. □

The generating functions for the sequences LE_n, JE_n and ER_n are given by

$$\sum_{n=0}^{\infty} LE_n t^n = \frac{1-t+t^2}{1-2t+t^3},$$

$$\sum_{n=0}^{\infty} JE_n t^n = \frac{2-4t+4t^2}{1-3t+t^2+t^3},$$

$$\sum_{n=0}^{\infty} ER_n t^n = \frac{1-t+t^2}{1-2t-t^2+2t^3}.$$

2 The Matrix Form of the Leonardo-Alwyn Numbers

In this section, we will give the matrix representation of the Leonardo-Alwyn numbers. Using it, we will obtain some properties of the Leonardo-Alwyn numbers.

The basic matrix for the Leonardo-Alwyn sequence is

$$Q = \begin{bmatrix} a+1 & 1 & 0 \\ b-a & 0 & 1 \\ -b & 0 & 0 \end{bmatrix}.$$

Due to the Cayley-Hamilton theorem, Leonardo-Alwyn's characteristic polynomial is given by

$$p(\lambda) = |\lambda I - Q| = \begin{vmatrix} \lambda - a - 1 & -1 & 0 \\ a-b & \lambda & -1 \\ b & 0 & \lambda \end{vmatrix} = \lambda^3-(a+1)\lambda^2-(b-a)\lambda+b = 0.$$

Theorem 4 *Let LA_n, $n \geq 0$ be the Leonardo-Alwyn numbers. Then the following equalities hold*

$$\begin{bmatrix} LA_{n+3} & LA_{n+2} & LA_{n+1} \\ LA_{n+2} & LA_{n+1} & LA_n \\ LA_{n+1} & LA_n & LA_{n-1} \end{bmatrix} = \begin{bmatrix} LA_3 & LA_2 & LA_1 \\ LA_2 & LA_1 & LA_0 \\ LA_1 & LA_0 & LA_{-1} \end{bmatrix} \begin{bmatrix} a+1 & 1 & 0 \\ b-a & 0 & 1 \\ -b & 0 & 0 \end{bmatrix}^n,$$

$$\begin{bmatrix} LA_{-n+3} & LA_{-n+2} & LA_{-n+1} \\ LA_{-n+2} & LA_{-n+1} & LA_{-n} \\ LA_{-n+1} & LA_{-n} & LA_{-n-1} \end{bmatrix} = \begin{bmatrix} LA_3 & LA_2 & LA_1 \\ LA_2 & LA_1 & LA_0 \\ LA_1 & LA_0 & LA_{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & -b \\ 1 & 0 & a+1 \\ 0 & 1 & b-a \end{bmatrix}^n.$$

Proof. We will use induction on n. The first equality hold for $n = 1$. Now suppose that it is true for $n > 1$. Then for $n + 1$, we can write

$$\begin{aligned} \begin{bmatrix} LA_3 & LA_2 & LA_1 \\ LA_2 & LA_1 & LA_0 \\ LA_1 & LA_0 & LA_{-1} \end{bmatrix} Q^{n+1} &= \begin{bmatrix} LA_3 & LA_2 & LA_1 \\ LA_2 & LA_1 & LA_0 \\ LA_1 & LA_0 & LA_{-1} \end{bmatrix} Q^n Q \\ &= \begin{bmatrix} LA_{n+3} & LA_{n+2} & LA_{n+1} \\ LA_{n+2} & LA_{n+1} & LA_n \\ LA_{n+1} & LA_n & LA_{n-1} \end{bmatrix} \begin{bmatrix} a+1 & 1 & 0 \\ b-a & 0 & 1 \\ -b & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} LA_{n+4} & LA_{n+3} & LA_{n+2} \\ LA_{n+3} & LA_{n+2} & LA_{n+1} \\ LA_{n+2} & LA_{n+1} & LA_n \end{bmatrix}. \end{aligned}$$

The second equality is verified similarly. □

Corollary 4 (Simsons identity) *Let LA_n, $n \geq 0$ be the Leonardo-Alwyn numbers. Then the following relations hold*

$$\begin{vmatrix} LA_{n+3} & LA_{n+2} & LA_{n+1} \\ LA_{n+2} & LA_{n+1} & LA_n \\ LA_{n+1} & LA_n & LA_{n-1} \end{vmatrix} = \begin{cases} 4(-1)^n & \text{if } LA_n \text{ is } LE_n, \\ 256(-1)^n & \text{if } LA_n \text{ is } JE_n, \\ 9(-2)^{n-1} & \text{if } LA_n \text{ is } ER_n, \end{cases}$$

$$\begin{vmatrix} LA_{-n+3} & LA_{-n+2} & LA_{-n+1} \\ LA_{-n+2} & LA_{-n+1} & LA_{-n} \\ LA_{-n+1} & LA_{-n} & LA_{-n-1} \end{vmatrix} = \begin{cases} 4(-1)^n & \text{if } LA_n \text{ is } LE_n, \\ 256(-1)^n & \text{if } LA_n \text{ is } JE_n, \\ 9(-2)^{n-1} & \text{if } LA_n \text{ is } ER_n. \end{cases}$$

Theorem 5 *Let LA_n, $n \geq 0$ be the Leonardo-Alwyn numbers. Then the following relations hold*

$$[LA_{n+2} \quad LA_{n+1} \quad LA_n] = [LA_2 \quad LA_1 \quad LA_0] \begin{bmatrix} a+1 & 1 & 0 \\ b-a & 0 & 1 \\ -b & 0 & 0 \end{bmatrix}^n,$$

$$[LA_{-n+2} \quad LA_{-n+1} \quad LA_{-n}] = [LA_2 \quad LA_1 \quad LA_0] \begin{bmatrix} 0 & 0 & -b \\ 1 & 0 & a+1 \\ 0 & 1 & b-a \end{bmatrix}^n.$$

Proof. The first equality holds for $n = 1$. Now suppose that it is true for $n > 1$. Then

$$\begin{aligned}
[LA_2 \quad LA_1 \quad LA_0] \begin{bmatrix} a+1 & 1 & 0 \\ b-a & 0 & 1 \\ -b & 0 & 0 \end{bmatrix}^{n+1} \\
&= [LA_{n+2} \quad LA_{n+1} \quad LA_n] \begin{bmatrix} a+1 & 1 & 0 \\ b-a & 0 & 1 \\ -b & 0 & 0 \end{bmatrix} \\
&= [LA_{n+3} \quad LA_{n+2} \quad LA_{n+1}] .
\end{aligned}$$

The second equality is verified similarly. □

3 Sums of Leonardo-Alwyn Numbers

In this section, we present some results concerning sums of terms of the Leonardo-Alwyn sequence.

Theorem 6 *Let LA_n, $n \geq 0$ be the Leonardo-Alwyn numbers. Then*

$$\sum_{i=0}^{n} LA_i = \begin{cases} 2F_{n+3} - n - 3 & \text{if } LA_n \text{ is } LE_n \\ 2P_{n+1} - 2n - 2 & \text{if } LA_n \text{ is } JE_n \\ \dfrac{3J_{n+3} - 2n - 5}{4} & \text{if } LA_n \text{ is } ER_n \end{cases}$$

$$\sum_{i=0}^{n} LA_{2i+1} = \begin{cases} 2F_{2n+2} - n - 1 & \text{if } LA_n \text{ is } LE_n \\ Q_{2n} - 2n - 8 & \text{if } LA_n \text{ is } JE_n \\ \dfrac{J_{2n+4} - 2n - 3}{2} & \text{if } LA_n \text{ is } ER_n \end{cases}$$

$$\sum_{i=0}^{n} LA_{2i} = \begin{cases} 2F_{2n+3} - n - 3 & \text{if } LA_n \text{ is } LE_n \\ Q_{2n+1} - 2n & \text{if } LA_n \text{ is } JE_n \\ J_{n+2} & \text{if } LA_n \text{ is } ER_n \end{cases}$$

Proof. We have

$$\begin{aligned}
LA_0 &= aLA_{-1} + bLA_{-2} + c, \\
LA_1 &= aLA_0 + bLA_{-1} + c, \\
LA_2 &= aLA_1 + bLA_0 + c, \\
&\dots \\
LA_n &= aLA_{n-1} + bLA_{n-2} + c.
\end{aligned}$$

Taking the sum of the equalities above, we obtain

$$\sum_{i=0}^{n} LA_i = \frac{(-a-b)LA_n + (a+b)LA_{-1} + b(LA_{-2} - LA_{n-1})}{1-a-b}.$$

From Corollary 2, it follows that finite sums for the sequences LE_n, JE_n and ER_n are given by

$$\sum_{i=0}^{n} LA_i = \begin{cases} 2F_{n+3} - n - 3 & \text{if } LA_n \text{ is } LE_n \\ 2P_{n+1} - 2n - 2 & \text{if } LA_n \text{ is } JE_n \\ \dfrac{3J_{n+3} - 2n - 5}{4} & \text{if } LA_n \text{ is } ER_n \end{cases}$$

Two other statements of the theorem are verified similarly. □

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