Introduction to Nonlinear Hyperbolic Partial Differential Equations

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Editorial Preface

These are the notes of the seven two-hours lectures that Karen Yagdjian delivered in the scope of the Summer School Nonlinear Hyperbolic Partial Differential Equations, which was organized by Michael Reissig at the Technical University Bergakademie Freiberg, Germany, during June-August, 2010. Karen Yagdjian received his Doctor of Physical and mathematical Sciences degree from Moscow State University in 1990. He worked in the Institute of Mathematics of Armenian Academy of Sciences until end of 2004. Since September of 2004 he is in Department of Mathematics of the University of Texas-Pan American, USA.

Participants of that school were young mathematicians, graduate and undergraduate students, and some invited and local scientists from Italy, Japan, Vietnam, USA, and Germany. The enthusiasm of Michael Reissig and the group of young mathematicians of Bergakademie Freiberg, combined with the financial support from Deutsche Forschungsgemeinschaft and the hospitality of Technical University Bergakademie Freiberg, gave life to that summer school.

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Preface

These are the notes of the lectures that I delivered in the scope of the Summer School "Nonlinear Hyperbolic Partial Differential Equations", which was organized by Michael Reissig at the Technical University Bergakademie Freiberg during June-August, 2010. The focus of the course was on the fundamental solutions for the linear operators with variable coefficients with an emphasize on some applications to the problem of the global in time existence of the solutions of the Cauchy problem for nonlinear equations.

The participants of this course were mostly young mathematicians, graduate and undergraduate students specializing in the partial differential equations. My thanks to Thomas Beyer, Christiane Böhme, Torsten Herrmann, Christian Jäh, Silke Konsulke, David Krieg, Simon Liebing, Simon Lomowski, Sascha Matthes, Jens Seidel, for their interests and constructive participation in lectures, seminars, and discussions. Special thanks to Michael Reissig, Daniele Del Santo, Fumihiko Hirosawa, Anahit Galstian, Christiane Böhme, Torsten Herrmann, and Christian Jäh, who gave numerous seminar presentations on various interesting related topics and made this Summer School unforgettable for all participants.

The preparation of these notes for publication recalls me how much I enjoyed the summer school at TU Bergakademie Freiberg and gives me the opportunity of thanking most heartily Michael Reissig for the invitation to Freiberg and for the warm hospitality, for his help and comments during my lectures and for the creating very friendly and fruitful atmosphere. He also helped me to minimize the number of misprints (although many more surely remain) in the manuscript and put plenty of efforts to link material of my lectures to that part of the audience, which has taken his courses on partial differential equations and on pseudo-differential operators offered in TU Bergakademie Freiberg.

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Lecture 1. June 25, 2010

1 Duhamels Principle and Fundamental Solutions of Hyperbolic Operators

1.1 The fundamental solutions at a glance

$$P(x,t,D_x,D_t)u(x,t)=f(x,t)$$

Example:

$$P=\partial_t^2-\Delta,\quad \Delta:=\sum_{i=1}^n \partial_{x_i}^2=\sum_{i=1}^n\left(\frac{\partial}{\partial x_i}\right)^2=\sum_{i=1}^n\frac{\partial^2}{\partial {x_i}^2}\,.$$

We look for $\mathcal{E}$ such that

$$P\mathcal{E}=I\ \text{(identity operator)}\,.$$

We will write formally $\mathcal{E}=\mathcal{E}(x,t,D_x,D_t)$. We call $\mathcal{E}$ *a fundamental solution of the operator* P.

In general, the operator can have several fundamental solutions.

If $\mathcal{E}$ is a fundamental solution of the operator P, then

$$u(x,t)=(\mathcal{E}f)(x,t)$$

since

$$P(x,t,D_x,D_t)u(x,t)=(P(x,t,D_x,D_t)\mathcal{E})f(x,t)=If(x,t)=f(x,t)\,.$$

Consider now the Cauchy problem

$$\begin{cases}\partial_t^2u(x,t)-\Delta u(x,t)=f(x,t), & (t,x)\in\mathbb{R}^{n+1}\,,\\ u(x,0)=\varphi_0(x),\quad u_t(x,0)=\varphi_1(x), & x\in\mathbb{R}^n\,.\end{cases}\tag{1.1}$$

To solve it we will look for the operators $\mathcal{E}_0(x,t,D_x,D_t)$ and $\mathcal{E}_1(x,t,D_x,D_t)$ such that

$$\begin{cases}(\partial_t^2-\Delta)\mathcal{E}_0(x,t,D_x,D_t)=0,\\ \mathcal{E}_0(x,0,D_x,D_t)=I\ \text{(identity operator)},\quad (\partial_t\mathcal{E}_0)(x,0,D_x,D_t)=0\end{cases}$$

and

$$\begin{cases}(\partial_t^2-\Delta)\mathcal{E}_1(x,t,D_x,D_t)=0,\\ \mathcal{E}_1(x,0,D_x,D_t)=0,\quad (\partial_t\mathcal{E}_1)(x,0,D_x,D_t)=I\ \text{(identity operator)}\end{cases}$$

We call $\mathcal{E}_0$ and $\mathcal{E}_1$ *the fundamental solutions of the Cauchy problem for the operator* P.

If $\mathcal{E}$ is a fundamental solution of the operator P while $\mathcal{E}_0$ and $\mathcal{E}_1$ are the fundamental solutions of the Cauchy problem for the operator P then the solution of the problem (1.1) is

$$u(x,t)=\mathcal{E}_0\left(\varphi_0-(\mathcal{E}f)|_{t=0}\right)+\mathcal{E}_1\left(\varphi_1-(\partial_t(\mathcal{E}f))\,|_{t=0}\right)+\mathcal{E}f(x,t)$$

In particular, if $\mathcal{E}$ is a fundamental solution of the operator P such that

$$\mathcal{E}(x,0,D_x,D_t)=0, \quad (\partial_t \mathcal{E})\,(x,0,D_x,D_t)=0\,,$$

then

$$u=\mathcal{E}_0\,(\varphi_0)+\mathcal{E}_1\,(\varphi_1)+\mathcal{E}f\,.$$

Moreover, it is easily seen that

$$u=\varphi_0+t\varphi_1+\mathcal{E}(f-P(\varphi_0+t\varphi_1)) \tag{1.2}$$

and, consequently, if we have such $\mathcal{E}$, then we do not need $\mathcal{E}_0$ and $\mathcal{E}_1$ to solve the Cauchy problem.

2. Operators and convolution

More precisely, let us look for the distribution $\mathcal{E}_1(x,t)\in \mathcal{D}'(\mathbb{R}^{n+1})$ (Pay attention, we use same notation!) such that $\mathcal{E}_1(x,t)\in C_t^\infty((-\infty,\infty);\mathcal{E}'(\mathbb{R}^n_x))$, that is

$$<\mathcal{E}_1(x,t),\varphi(x)>\in C_t^\infty(\mathbb{R}), \quad \text{for every} \quad \varphi\in C_0^\infty(\mathbb{R}^n_x)$$

and such that

$$\begin{cases} (\partial_t^2-\Delta)\mathcal{E}_1(x,t)=0, & (t,x)\in\mathbb{R}^{n+1},\\ \mathcal{E}_1(x,0)=0, \quad (\partial_t\mathcal{E}_1)\,(x,0)=\delta(x) & x\in\mathbb{R}^n, \end{cases}$$

and define the operator $\mathcal{E}_1(x,t,D_x,D_t)$ by

$$\mathcal{E}_1(x,t,D_x,D_t)\psi(x):=(\mathcal{E}_1(\cdot,t)*\psi(\cdot))(x,t).$$

We will write it formally (distribution notation)

$$\mathcal{E}_1(x,t,D_x,D_t)\psi(x)=\int_{\mathbb{R}^n}\mathcal{E}_1(x-y,t)\psi(y)\,dy, \qquad \forall\psi(x)\in C_0^\infty(\mathbb{R}^n)$$

Example: The fundamental solutions of the string operator $P=\partial_t^2-\partial_x^2$.

It is known, that for the Cauchy problem

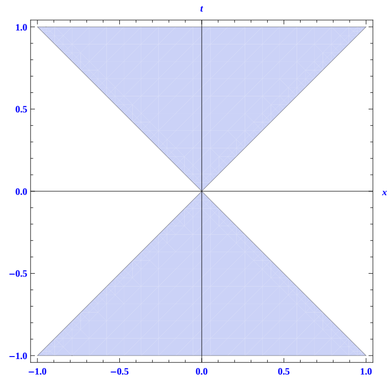
$$\begin{cases} \partial_t^2 u(x,t)-\partial_x^2 u(x,t)=0, & (t,x)\in\mathbb{R}^2,\\ u(x,0)=\varphi_0(x), \quad u_t(x,0)=\varphi_1(x), & x\in\mathbb{R}, \end{cases} \tag{1.3}$$

with smooth functions φ_0,φ_1 the solution is

$$u(x,t)=\frac{1}{2}\left(\varphi_0(x+t)+\varphi_0(x-t)\right)+\frac{1}{2}\int_{x-t}^{x+t}\varphi_1(y)\,dy\,.$$

We plug formally in the last representation formula $\varphi_0=0$ and $\varphi_1=\delta(x)$ and obtain the corresponding (formal) solution:

$$\mathcal{E}_1(x,t)=\frac{1}{2}\int_{x-t}^{x+t}\delta(y)\,dy.$$



Therefore, we define

$$\mathcal{E}_1(x,t) = \begin{cases} \dfrac{1}{2} & \text{if} \quad 0 \in [x-t, x+t], \\ 0 & \text{otherwise.} \end{cases}$$

The last function is locally integrable $\mathcal{E}_1 \in L^1_{loc}(\mathbb{R}^2)$, and defines a distribution by the integral:

$$< \mathcal{E}_1, \psi > = \int_{\mathbb{R}^2} \mathcal{E}_1(x,t)\psi(x,t)\, dx\, dt, \qquad \forall \psi(x,t) \in C_0^\infty(\mathbb{R}^2)$$

Proposition 1.1 *The operator defined by the formula*

$$\mathcal{E}_1(x,t,D_x,D_t)\varphi(x) = \int_{\mathbb{R}} \mathcal{E}_1(x-y,t)\varphi(y)\, dy, \qquad \forall \varphi(x) \in C_0^\infty(\mathbb{R}),$$

is a fundamental solution $\mathcal{E}_1(x,t,D_x,D_t)$ of the Cauchy problem for the operator P. Thus, in the operator notations that means

$$\begin{cases} (\partial_t^2 - \partial_x^2)\mathcal{E}_1 = 0, \\ \mathcal{E}_1|_{t=0} = 0, \quad (\partial_t \mathcal{E}_1)\,|_{t=0} = I(\textit{identity operator}), \end{cases}$$

while in the distributions notations

$$\begin{cases} (\partial_t^2 - \partial_x^2)\mathcal{E}_1(x,t) = 0, \quad \text{in} \quad D'(\mathbb{R}^2), \\ \mathcal{E}_1(x,0) = 0, \quad (\partial_t \mathcal{E}_1)\,(x,0) = \delta(x) \quad \text{in} \quad D'(\mathbb{R}). \end{cases}$$

Proof. Take $\psi(x,t) \in C_0^\infty(\mathbb{R}^2)$ and consider $< P\mathcal{E}_1, \psi >$ with the test-function ψ. Then

$$< P\mathcal{E}_1, \psi > = < (\partial_t^2 - \partial_x^2)\mathcal{E}_1, \psi > = < \mathcal{E}_1, (\partial_t^2 - \partial_x^2)\psi >$$

by the definition of the derivative of a distribution. Since $\mathcal{E}_1 \in L^1_{loc}(\mathbb{R}^2)$, we obtain

$$< \mathcal{E}_1, (\partial_t^2 - \partial_x^2)\psi > \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{E}_1(x,t)(\partial_t^2 - \partial_x^2)\psi(x,t)\, dx\, dt.$$

We make change of variables

$$\begin{cases} l = x+t \\ m = x-t \end{cases}, \qquad \begin{cases} x = \frac{1}{2}(l+m) \\ t = \frac{1}{2}(l-m) \end{cases}, \qquad \left|\frac{D(x,t)}{D(l,m)}\right| = \frac{1}{2}.$$

In the new variables the operator is

$$P = -4\frac{\partial^2}{\partial l \partial m} = -4\frac{\partial}{\partial l}\frac{\partial}{\partial m}.$$

Hence,

$$\begin{aligned}
& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{E}_1(x,t)(\partial_t^2 - \partial_x^2)\psi(x,t)\,dx\,dt \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{E}_1(x,t)\left(-4\frac{\partial}{\partial l}\frac{\partial}{\partial m}\right)\psi(x(l,m),t(l,m))\,\frac{1}{2}dl\,dm \\
= & \int_{\{l\le 0,m\ge 0\}\cup\{l\ge 0,m\le 0\}} \frac{1}{2}\left(-4\frac{\partial}{\partial l}\frac{\partial}{\partial m}\right)\psi(x(l,m),t(l,m))\,\frac{1}{2}dl\,dm \\
= & -\int_{\{l\le 0,m\ge 0\}\cup\{l\ge 0,m\le 0\}} \left(\frac{\partial}{\partial l}\frac{\partial}{\partial m}\right)\psi(x(l,m),t(l,m))\,dl\,dm
\end{aligned}$$

We denote $I := \{l \le 0, m \ge 0\}$ and $II := \{l \ge 0, m \le 0\}$ and consider two integrals, respectively. We have

$$\begin{aligned}
& \int_{\{l\le 0,m\ge 0\}} \left(\frac{\partial}{\partial l}\frac{\partial}{\partial m}\right)\psi(x(l,m),t(l,m))\,dl\,dm \\
= & \int_{-\infty}^{0} dl \int_{0}^{\infty} dm \left(\frac{\partial}{\partial l}\frac{\partial}{\partial m}\right)\psi(x(l,m),t(l,m)) \\
= & \int_{-\infty}^{0} \left(\frac{\partial}{\partial_l}\int_{0}^{\infty}\left(\frac{\partial}{\partial m}\right)\psi(x(l,m),t(l,m))\,dm\right) dl \\
= & \int_{-\infty}^{0} \frac{\partial}{\partial_l}\left(\psi(x(l,\infty),t(l,\infty)) - \psi(x(l,0),t(l,0))\right) dl \\
= & -\int_{-\infty}^{0} \frac{\partial}{\partial l}\left(\psi(x(l,0),t(l,0)\right) dl \\
= & \ \psi(x(-\infty,0),t(-\infty,0) - \psi(x(0,0),t(0,0)) \\
= & -\psi(0,0).
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \int_{\{l\ge 0,m\le 0\}} \left(\frac{\partial}{\partial l}\frac{\partial}{\partial m}\right)\psi(x(l,m),t(l,m))\,dl\,dm \\
= & \int_{0}^{\infty} dl \int_{-\infty}^{0} dm \left(\frac{\partial}{\partial l}\frac{\partial}{\partial m}\right)\psi(x(l,m),t(l,m)) \\
= & \int_{0}^{\infty} \left(\frac{\partial}{\partial l}\int_{-\infty}^{0}\left(\frac{\partial}{\partial m}\right)\psi(x(l,m),t(l,m))\,dm\right) dl
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \frac{\partial}{\partial l}\left(\psi(x(l,0),t(l,0)) - \psi(x(l,-\infty),t(l,-\infty))\right) dl \\
&= \int_{-\infty}^0 \frac{\partial}{\partial l}\left(\psi(x(l,0),t(l,0)\right) dl \\
&= \psi(x(0,0),t(0,0)) - \psi(x(-\infty,0),t(-\infty,0) \\
&= \psi(0,0)\,.
\end{aligned}
$$

Thus,

$$
\int_{\{l\le 0,m\ge 0\}} \left(\frac{\partial}{\partial l}\frac{\partial}{\partial m}\right)\psi(x(l,m),t(l,m))\,dl\,dm + \int_{\{l\ge 0,m\le 0\}} \left(\frac{\partial}{\partial l}\frac{\partial}{\partial m}\right)\psi(x(l,m),t(l,m))\,dl\,dm = 0
$$

implies

$$
< P\mathcal{E}_1, \psi >= 0 \qquad \forall \psi(x,t) \in C_0^\infty(\mathbb{R}^2)\,.
$$

Consequently $P\mathcal{E}_1 = 0$ in D'.

To check initial values we note that

$$
\mathcal{E}_1(x,0,D_x,D_t)\varphi(x) = \int_{\mathbb{R}} \mathcal{E}_1(x-y,0)\varphi(y)\,dy = 0, \qquad \forall \varphi(x) \in C_0^\infty(\mathbb{R})
$$

since the interval $[x-y, x-y]$ has measure zero. On the other hand

$$
\begin{aligned}
< (\partial_t\mathcal{E}_1)(x,0), \psi(x) > &= \lim_{t\to 0^+} \frac{1}{t}\left\{< \mathcal{E}_1(x,t), \psi(x) > - < \mathcal{E}_1(x,0), \psi(x) >\right\} \\
&= \lim_{t\to 0^+} \frac{1}{t}\left\{< \mathcal{E}_1(x,t), \psi(x) > -0\right\} \\
&= \lim_{t\to 0^+} \frac{1}{t} < \mathcal{E}_1(x,t), \psi(x) > \\
&= \lim_{t\to 0^+} \frac{1}{t} \int_{[x-t,x+t]\cap supp\psi} \frac{1}{2}\psi(y)dy = \psi(0) =< \delta, \psi > \ \ \forall \varphi(x) \in C_0^\infty(\mathbb{R}).
\end{aligned}
$$

Exercise: Prove, that $\mathcal{E}_1(x,t) \in C_t^\infty((-\infty,\infty); \mathcal{E}'(\mathbb{R}_x))$ (The operator P is partially hypoelliptic in the direction of time.) Proposition is proven. □

Exercise: Verify that the distribution $\mathcal{E}_1(x,t;x_0,t_0) := \mathcal{E}_1(x-x_0,t-t_0)$ solves the following Cauchy problem:

$$
\begin{cases}
(\partial_t^2 - \Delta_x)\mathcal{E}_1(x,t;x_0,t_0) = 0, \\
\mathcal{E}_1(x,t_0;x_0,t_0) = 0, \quad (\partial_t\mathcal{E}_1)(x,t_0;x_0,t_0) = \delta(x-x_0)\,.
\end{cases}
$$

Exercise: Verify that the distribution $\mathcal{E}_+(x,t;x_0,t_0)$ defined by

$$
\mathcal{E}_+(x,t;x_0,t_0) := \begin{cases}
\frac{1}{2} & \text{if} \quad 0 \in D_+(x_0,t_0), \\
0 & \text{otherwise,}
\end{cases}
$$

where

$$D_+(x_0,t_0) := \{(x,t)\in\mathbb{R}^{n+1}\,|\,|x-x_0|\le t-t_0\},$$

is a fundamental solution for the operator. Moreover, it solves the following Cauchy problem

$$\begin{cases} (\partial_t^2-\Delta_x)\mathcal{E}_+(x,t;x_0,t_0)=\delta(x-x_0)\delta(t-t_0), \\ \mathcal{E}_+(x,t_0;x_0,t_0)=0, \quad (\partial_t\mathcal{E}_+)\,(x,t_0;x_0,t_0)=0\,. \end{cases}$$

Moreover, $\operatorname{supp}\mathcal{E}_+(x,t;x_0,t_0)=D_+(x_0,t_0)$.

1.2 Representation formula for Three Space Dimensions

Theorem 1.2 *The classical solution $u=u(x,t)$ of the Cauchy problem*

$$\begin{cases} \partial_t^2 u(x,t)-\Delta u(x,t)=f(x,t), \quad x\in\mathbb{R}^3, \quad t\ge 0, \\ u(x,0)=0, \quad u_t(x,0)=0, \quad x\in\mathbb{R}^3\,, \end{cases} \tag{1.4}$$

is given by

$$u(x,t) \;=\; \frac{1}{4\pi}\int_{B_t(0)}\frac{f(x+y,t-|y|)}{|y|}\,dy\,. \tag{1.5}$$

Proof. First, for a given function $u=u(x,t)$ we define the spherical means of u about point x:

$$I_u(x,r,t) \;=\; \frac{1}{\omega_2}\int_{S^2}u(x+ry,t)\,dS_y\,,$$

where ω_2 denotes the area of the unit sphere $S^2\subset\mathbb{R}^3$. Then we define an operator Ω_r by

$$\Omega_r(u)(x,t) := rI_u(x,r,t)\,.$$

One can recover the functions according to

$$u(x,t) \;=\; \lim_{r\to 0}I_u(x,r,t)=\lim_{r\to 0}\frac{1}{r}\Omega_r(u)(x,t)\,, \tag{1.6}$$

$$u(x,0) \;=\; \lim_{r\to 0}\frac{1}{r}\Omega_r(u)(x,0)\,, \quad u_t(x,0)=\lim_{r\to 0}\frac{1}{r}\Omega_r(\partial_t u)(x,0)\,. \tag{1.7}$$

Exercise: Prove that $\Delta_x\Omega_r h=\frac{\partial^2}{\partial r^2}\Omega_r h$ for every function $h\in C^2(\mathbb{R}^3)$.

Therefore we arrive at the following mixed problem for the function $v(x,r,t):=\Omega_r(u)(x,r,t)$:

$$\begin{cases} v_{tt}(x,r,t)-v_{rr}(x,r,t)=F(x,r,t) \quad \text{for all} \quad t\ge 0\,,\; r\ge 0\,,\; x\in\mathbb{R}^3\,, \\ v(x,0,t)=0 \quad \text{for all} \quad t\ge 0\,, \quad x\in\mathbb{R}^3, \\ v(x,r,0)=0\,, \quad v_t(x,r,0)=0 \quad \text{for all} \quad r\ge 0\,, \quad x\in\mathbb{R}^3\,, \\ F(x,r,t):=\Omega_r(f)(x,t)\,, \quad F(x,0,t)=0\,, \quad \text{for all} \quad x\in\mathbb{R}^3\,. \end{cases}$$

It must be noted here that the spherical mean I_u defined for $r>0$ has an extension as an even function for $r<0$ and hence $\Omega_r(u)$ has a natural extension as an odd function. That allows replacing the mixed problem with the Cauchy problem. Namely, let functions $\widetilde{v}$ and $\widetilde{F}$ be the continuations of the functions v and the forcing term F, respectively, by

$$\widetilde{v}(x,r,t) = \left\{ \begin{array}{l} v(x,r,t),\ if\ r \geq 0 \\ -v(x,-r,t),\ if\ r \leq 0 \end{array} \right. , \quad \widetilde{F}(x,r,t) = \left\{ \begin{array}{l} F(x,r,t),\ if\ r \geq 0 \\ -F(x,-r,t),\ if\ r \leq 0 \end{array} \right. .$$

Then $\widetilde{v}$ solves the Cauchy problem

$$\left\{ \begin{array}{l} \widetilde{v}_{tt}(x,r,t) - \widetilde{v}_{rr}(x,r,t) = \widetilde{F}(x,r,t) \quad \mbox{for all} \quad t \geq 0\,, \quad r \in \mathbb{R}\,, \quad x \in \mathbb{R}^3\,, \\ \widetilde{v}(x,r,0) = 0\,, \quad \widetilde{v}_t(x,r,0) = 0 \quad \mbox{for all} \quad r \in \mathbb{R}\,, \quad x \in \mathbb{R}^3. \end{array} \right.$$

Hence, one has the representation

$$\widetilde{v}(x,r,t) \quad = \quad \int_0^t d\tau \left(\frac{1}{2} \int_{r-t+\tau}^{r+t-\tau} \widetilde{F}(x,r_1,\tau)\, dr_1 \right).$$

Since

$$u(x,t) = \lim_{r \to 0} \frac{\widetilde{v}(x,r,t)}{r},$$

we consider the case of $r < t$ in the above representation to obtain:

$$\begin{aligned}
u(x,t) &= \lim_{r \to 0} \frac{\widetilde{v}(x,r,t)}{r} \\
&= \lim_{r \to 0} \frac{1}{r} \int_0^t d\tau \left(\frac{1}{2} \int_{r-t+\tau}^{r+t-\tau} \widetilde{F}(x,r_1,\tau)\, dr_1 \right) \\
&= \lim_{r \to 0} \frac{1}{r} \int_0^t d\tau \left(\frac{1}{2} \int_{-t+\tau}^{t-\tau} \widetilde{F}(x,r+z,\tau)\, dz \right) \\
&= \lim_{r \to 0} \frac{1}{r} \int_0^t d\tau \frac{1}{2} \left\{ \int_{-t+\tau}^{0} \widetilde{F}(x,r+z,\tau)\, dz + \int_0^{t-\tau} \widetilde{F}(x,r+z,\tau)\, dz \right\} \\
&= \lim_{r \to 0} \frac{1}{r} \int_0^t d\tau \frac{1}{2} \left\{ \int_0^{t-\tau} \widetilde{F}(x,r-z,\tau)\, dz + \int_0^{t-\tau} \widetilde{F}(x,r+z,\tau)\, dz \right\} \\
&= \int_0^t d\tau \lim_{r \to 0} \frac{1}{2r} \left\{ \int_0^{t-\tau} \widetilde{F}(x,r-z,\tau)\, dz + \int_0^{t-\tau} \widetilde{F}(x,r+z,\tau)\, dz \right\} \\
&= \int_0^t d\tau \int_0^{t-\tau} \lim_{r \to 0} \frac{1}{2r} \left\{ \widetilde{F}(x,r-z,\tau) + \widetilde{F}(x,r+z,\tau) \right\} dz\,.
\end{aligned}$$

Then by definition of the function $\widetilde{F}$, we replace

$$\begin{aligned}
\lim_{r \to 0} \frac{1}{2r} \left\{ \widetilde{F}(x,r-z,\tau) + \widetilde{F}(x,r+z,\tau) \right\} &= \lim_{r \to 0} \frac{1}{2r} \left\{ -F(x,z-r,\tau) + F(x,z+r,\tau) \right\} \\
&= \lim_{r \to 0} \frac{1}{2r} \left\{ F(x,z+r,\tau) - F(x,z-r,\tau) \right\}
\end{aligned}$$

with

$$\left(\frac{\partial}{\partial r} F(x,r,\tau) \right)_{r=z}$$

in the last formula. The definitions of $F(x,r,t)$ and of the operator Ω_r yield:

$$\begin{aligned} u(x,t) &= \int_0^t d\tau \int_0^{t-\tau} \lim_{r\to 0} \frac{1}{2r} \left\{ \widetilde{F}(x,r-z,\tau) + \widetilde{F}(x,r+z,\tau) \right\} dz \\ &= \int_0^t d\tau \int_0^{t-\tau} \Big(\frac{\partial}{\partial r} F(x,r,\tau) \Big)_{r=z} dz \\ &= \int_0^t d\tau \int_0^{t-\tau} \Big(\frac{\partial}{\partial r} \Omega_r(f)(x,\tau) \Big)_{r=z} dz \\ &= \int_0^t d\tau \int_0^{t-\tau} \Big(\frac{\partial}{\partial r} \frac{r}{\omega_2} \int_{S^2} f(x+ry,\tau)\, dS_y \Big)_{r=z} dz\,, \end{aligned}$$

where $x \in \mathbb{R}^3$. Thus, the solution to the Cauchy problem is given by

$$u(x,t) = \int_0^t d\tau \int_0^{t-\tau} \Big(\frac{\partial}{\partial r} \frac{r}{4\pi} \int_{S^2} f(x+ry,\tau)\, dS_y \Big)_{r=z} dz\,. \tag{1.8}$$

(Note that if we have with some function K a similar formula, like

$$u(x,t) = \int_0^t d\tau \int_0^{t-\tau} K(\tau,z) \Big(\frac{\partial}{\partial r} \frac{r}{4\pi} \int_{S^2} f(x+ry,\tau)\, dS_y \Big)_{r=z} dz\,,$$

then we cannot apply Newton-Leibniz formula to get rid of the derivative and of the intermediate integral! But, on the other hand, it will be shown later that such function $K(\tau,z)$ opens a way for the generalizations.) Finally we show (**Exercise!**) that

$$u(x,t) = \frac{1}{4\pi} \int_{B_t(0)} \frac{f(x+y,t-|y|)}{|y|}\, dy\,,$$

where

$$B_t(0) := \{ y \in \mathbb{R}^3 \,|\, |y| \le t \}\,.$$

Theorem is proven. □

Lecture 2. July 2, 2010

1.3 Representation formula for n-Space Dimensions

Theorem 1.3 *For $\varphi_1 \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, $n = 2m+1$, $m \in \mathbb{N}$, the solution $u(x,t)$ of the Cauchy problem*

$$u_{tt} - \Delta u = 0, \quad u(x,0) = 0, \quad u_t(x,0) = \varphi_1(x)\,,$$

is given by

$$u(x,t) := \Big(\frac{1}{t}\frac{\partial}{\partial t} \Big)^{\frac{n-3}{2}} \frac{t^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{S^{n-1}} \varphi_1(x+ty)\, dS_y, \tag{1.9}$$

where $c_0^{(n)} = 1 \cdot 3 \cdot \ldots \cdot (n-2)$.
For $x \in \mathbb{R}^n$, $n = 2m$, $m \in \mathbb{N}$,

$$u(x,t) := \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}} \frac{2r^{n-1}}{\omega_{n-1} c_0^{(n)}} \int_{B_1^n(0)} \frac{1}{\sqrt{1-|y|^2}} \varphi_1(x+ty)\, dV_y\,, \tag{1.10}$$

where $c_0^{(n)} = 1 \cdot 3 \cdot \ldots \cdot (n-1)$.
For $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ *and for* $x \in \mathbb{R}^n$, $n = 2m+1$, $m \in \mathbb{N}$, *the solution* $u(x,t)$ *of the Cauchy problem*

$$u_{tt} - \Delta u = 0, \quad u(x,0) = \varphi_0(x), \quad u_t(x,0) = 0\,,$$

is given by

$$u(x,t) := \frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}} \frac{t^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{S^{n-1}} \varphi_0(x+ty)\, dS_y\,, \tag{1.11}$$

where $c_0^{(n)} = 1 \cdot 3 \cdot \ldots \cdot (n-2)$.
For $x \in \mathbb{R}^n$, $n = 2m$, $m \in \mathbb{N}$,

$$u(x,t) := \frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}} \frac{2t^{n-1}}{\omega_{n-1} c_0^{(n)}} \int_{B_1^n(0)} \frac{1}{\sqrt{1-|y|^2}} \varphi_0(x+ty)\, dV_y\,. \tag{1.12}$$

where $c_0^{(n)} = 1 \cdot 3 \cdot \ldots \cdot (n-1)$. *The constant* ω_{n-1} *is the area of the unit sphere* $S^{n-1} \subset \mathbb{R}^n$.

Sketch of the proof of the n-D analog of formula (1.5):

$$u(x,t) \;=\; \frac{2}{c_0^{(n)}} \int_0^t db \left(\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{m-1} r^{2m-1} \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(x+ry,b)\, dS_y \right)_{r=t-b}. \tag{1.13}$$

Consider the case of $x \in \mathbb{R}^n$, where $n = 2m+1$, $m \in \mathbb{N}$. For the given function $u = u(x,t)$ we define the spherical mean of u about point x:

$$I_u(x,r,t) \;=\; \frac{1}{\omega_{n-1}} \int_{S^{n-1}} u(x+ry,t)\, dS_y\,,$$

where ω_{n-1} denotes the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Then we define the operator Ω_r by

$$\Omega_r(u)(x,t) := \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{m-1} r^{2m-1} I_u(x,r,t)\,.$$

One can show that there are constants $c_j^{(n)}$, $j = 0, \ldots, m-1$, where $n = 2m+1$, with $c_0^{(n)} = 1 \cdot 3 \cdot 5 \cdots (n-2)$, such that

$$\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{m-1} r^{2m-1} \varphi(r) = r \sum_{j=0}^{m-1} c_j^{(n)} r^j \frac{\partial^j}{\partial r^j} \varphi(r)\,.$$

Exercise: Prove that $c_0^{(n)} = 1 \cdot 3 \cdot 5 \cdots (n-2)$.
One can recover the functions according to

$$u(x,t) = \lim_{r\to 0} I_u(x,r,t) = \lim_{r\to 0} \frac{1}{c_0^{(n)} r} \Omega_r(u)(x,t)\,, \tag{1.14}$$

$$u(x,0) = \lim_{r\to 0} \frac{1}{c_0^{(n)} r} \Omega_r(u)(x,0)\,, \quad u_t(x,0) = \lim_{r\to 0} \frac{1}{c_0^{(n)} r} \Omega_r(\partial_t u)(x,0)\,. \tag{1.15}$$

Exercise: Verify that $\Delta_x \Omega_r h = \frac{\partial^2}{\partial r^2}\Omega_r h$ for every function $h \in C^2(\mathbb{R}^n)$.

Therefore we arrive at the following mixed problem for the function $v(x,r,t) := \Omega_r(u)(x,r,t)$:

$$\begin{cases} v_{tt}(x,r,t) - v_{rr}(x,r,t) = F(x,r,t) \quad \text{for all} \quad t \ge 0\,,\ r \ge 0\,,\ x \in \mathbb{R}^n\,, \\ v(x,0,t) = 0 \quad \text{for all} \quad t \ge 0\,, \quad x \in \mathbb{R}^n, \\ v(x,r,0) = 0\,, \quad v_t(x,r,0) = 0 \quad \text{for all} \quad r \ge 0\,, \quad x \in \mathbb{R}^n\,, \\ F(x,r,t) := \Omega_r(f)(x,t)\,, \quad F(x,0,t) = 0\,, \quad \text{for all} \quad x \in \mathbb{R}^n\,. \end{cases}$$

Let the functions $\widetilde{v}$ and $\widetilde{F}$ be the continuations of the functions v and the forcing term F, respectively, by

$$\widetilde{v}(x,r,t) = \begin{cases} v(x,r,t), \ if\ r \ge 0 \\ -v(x,-r,t), \ if\ r \le 0 \end{cases}, \quad \widetilde{F}(x,r,t) = \begin{cases} F(x,r,t), \ if\ r \ge 0 \\ -F(x,-r,t), \ if\ r \le 0 \end{cases}.$$

Then $\widetilde{v}$ solves the Cauchy problem

$$\begin{cases} \widetilde{v}_{tt}(x,r,t) - \widetilde{v}_{rr}(x,r,t) = \widetilde{F}(x,r,t) \quad \text{for all} \quad t \ge 0\,, \quad r \in \mathbb{R}\,, \quad x \in \mathbb{R}^n\,, \\ \widetilde{v}(x,r,0) = 0\,, \quad \widetilde{v}_t(x,r,0) = 0 \quad \text{for all} \quad r \in \mathbb{R}\,, \quad x \in \mathbb{R}^n. \end{cases}$$

Hence, according to the result for the 1-d case, one has the representation

$$\widetilde{v}(x,r,t) = \int_0^t d\tau \left(\frac{1}{2} \int_{r-t+\tau}^{r+t-\tau} \widetilde{F}(x,r_1,\tau)\, dr_1 \right).$$

Since

$$u(x,t) = \lim_{r\to 0} \frac{\widetilde{v}(x,r,t)}{r},$$

we consider the case of $r < t$ in the above representation to obtain:

$$u(x,t) = \frac{1}{c_0^{(n)}} \int_0^t db \int_0^{t-b} dr_1 \lim_{r\to 0} \frac{1}{r} \left\{ \widetilde{F}(x,r+r_1,b) + \widetilde{F}(x,r-r_1,b) \right\}.$$

Then by definition of the function $\widetilde{F}$,

$$\lim_{r\to 0} \frac{1}{r} \Big\{ \widetilde{F}(x,r-r_1,b) + \widetilde{F}(x,r+r_1,b) \Big\} = 2 \Big(\frac{\partial}{\partial r} F(x,r,b) \Big)_{r=r_1}$$

in the last formula. The definitions of $F(x,r,t)$ and of the operator Ω_r yield:

$$u(x,t) = \frac{2}{c_0^{(n)}} \int_0^t db \int_0^{t-b} dr_1 \left(\frac{\partial}{\partial r} \Big(\frac{1}{r} \frac{\partial}{\partial r} \Big)^{m-1} r^{2m-1} I_f(x,r,b) \right)_{r=r_1},$$

where $x \in \mathbb{R}^n$, $n = 2m+1$, $m \in \mathbb{N}$. Thus

$$u(x,t) \;=\; \frac{2}{c_0^{(n)}} \int_0^t db \int_0^{t-b} dr_1 \left(\frac{\partial}{\partial r} \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{m-1} r^{2m-1} \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(x+ry,b)\, dS_y \right)_{r=r_1} .$$

It follows (1.13), that is,

$$\begin{aligned} u(x,t) &= \frac{2}{c_0^{(n)}} \int_0^t db \int_0^{t-b} dr \frac{\partial}{\partial r} \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{m-1} r^{2m-1} \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(x+ry,b)\, dS_y \\ &= \frac{2}{c_0^{(n)}} \int_0^t db \left(\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{m-1} r^{2m-1} \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(x+ry,b)\, dS_y) \right)_{r=t-b} . \end{aligned}$$

(Note that if we have with some function K the similar formula

$$u(x,t) \;=\; \frac{2}{c_0^{(n)}} \int_0^t db \int_0^{t-b} dr_1\, K(b,r_1) \left(\frac{\partial}{\partial r} \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{m-1} r^{2m-1} I_f(x,r,b) \right)_{r=r_1} ,$$

then we cannot apply Newton-Leibniz formula to get rid of the derivative and of the intermediate integral! On the other hand, such function opens a way for generalizations.)
Exercise: Employ the method of descent to complete the proof for the case of even n, $n = 2m$, $m \in \mathbb{N}$.
Theorem is proven. □

Fundamental solutions. We set $\varphi_0(x) = \delta(x)$ and $\varphi_1(x) = \delta(x)$ in (1.9) - (1.12), and we obtain: if n is odd, then

$$\mathcal{E}_1(x,t) := \frac{1}{\omega_{n-1} 1\cdot 3\cdot 5\cdot\ldots\cdot(n-2)} \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}} \frac{1}{t}\delta(|x|-t)\,,$$

$$\mathcal{E}_0(x,t) := \frac{1}{\omega_{n-1} 1\cdot 3\cdot 5\cdot\ldots\cdot(n-2)} \frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}} \frac{1}{t}\delta(|x|-t)\,,$$

while for n even we have

$$\begin{aligned} \mathcal{E}_1(x,t) &:= \frac{2}{\omega_{n-1} 1\cdot 3\cdot 5\cdot\ldots\cdot(n-1)} \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}} \frac{1}{\sqrt{t^2-|x|^2}} \chi_{B_t(x)}\,, \\ \mathcal{E}_0(x,t) &:= \frac{2}{\omega_{n-1} 1\cdot 3\cdot 5\cdot\ldots\cdot(n-1)} \frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}} \frac{1}{\sqrt{t^2-|x|^2}} \chi_{B_t(x)}\,. \end{aligned}$$

Here $\chi_{B_t(x)}$ denotes the characteristic function of the ball $B_t(x) := \{x \in \mathbb{R}^n;\ |x| \le t\}$. The constant ω_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. The distribution $\delta(|x|-t)$ is defined by

$$\langle \delta(|\cdot|-t), \psi(\cdot)\rangle = \int_{|x|=t} \psi(x)\, dx \quad \text{for} \quad \psi \in C_0^\infty(\mathbb{R}^n).$$

Exercise: Prove that the above formulas indeed give the fundamental solutions.

1.4 Duhamel's principle revised: Integral transform

The text of this subsection is taken from [27].
Duhamel's principle: The Cauchy problem with source term $f(x,t) \in C^\infty(\mathbb{R}^2)$:

$$u_{tt} - u_{xx} = f(x,t) \quad \text{in} \quad \mathbb{R}^2, \quad u(x,t_0) = 0,\ u_t(x,t_0) = 0 \quad \text{in} \quad \mathbb{R}. \tag{1.16}$$

The solution can be written as an integral

$$u(x,t) = \int_{t_0}^{t} v(x,t;\tau)\, d\tau$$

of the family of the solutions $v(x,t;\tau)$ of the problem without the source term, but with a parameter-dependent second initial datum

$$v_{tt} - v_{xx} = 0 \quad \text{in } \mathbb{R}^2, \quad v(x,\tau;\tau) = 0, \quad v_t(x,\tau;\tau) = f(x,\tau) \quad \text{in } \mathbb{R}\,.$$

(The parameter τ is both in the function f and in the initial hyperplane.) Hence,

$$u(x,t) = \int_{t_0}^{t} \left(\frac{1}{2} \int_{x-t+\tau}^{x+t-\tau} f(y,\tau)\, dy \right) d\tau\,.$$

We revise the Duhamel's principle. More precisely, we write

$$\begin{aligned} \int_{x-t+\tau}^{x+t-\tau} f(y,\tau)\, dy &= \int_{-(t-\tau)}^{t-\tau} f(x+z,\tau)\, dz \\ &= \int_{-(t-\tau)}^{0} f(x+z,\tau)\, dz + \int_{0}^{t-\tau} f(x+z,\tau)\, dz \\ &= \int_{0}^{t-\tau} f(x-z,\tau)\, dz + \int_{0}^{t-\tau} f(x+z,\tau)\, dz\,. \end{aligned}$$

Our *first observation* is: if we denote

$$w(x;t;\tau) := \frac{1}{2}(f(x+t,\tau) + f(x-t,\tau)),$$

then

$$u(x,t) = \int_{t_0}^{t} d\tau \int_{0}^{t-\tau} w(x,z;\tau)\, dz\,. \tag{1.17}$$

Here $w = w(x;t;\tau)$ is the solution of the Cauchy problem

$$w_{tt} - w_{xx} = 0 \quad \text{in } \mathbb{R}^2, \quad w(x,0;\tau) = f(x,\tau),\ w_t(x,0;\tau) = 0 \quad \text{in } \mathbb{R}. \tag{1.18}$$

This formula allows us to solve problems with the source term if we solve the problem for the **same** equation without source term but with the parameter-dependent **first** initial datum.

Exersise 1.4 *Prove that the formula*

$$u(x,t) = \int_{t_0}^{t} d\tau \int_{0}^{t-\tau} w(x,z;\tau)\,dz\,, \tag{1.19}$$

can be used also for $x \in \mathbb{R}^n$, $n \in \mathbb{N}$. More precisely, it holds also for the Cauchy problem

$$u_{tt} - \Delta u = f(x,t) \quad \text{in } \mathbb{R}^{n+1}, \quad u(x,t_0) = 0, \quad u_t(x,t_0) = 0 \quad \text{in } \mathbb{R}^n\,,$$

with the function $w = w(x;t;\tau)$ solving

$$\begin{cases} w_{tt} - \Delta w = 0 \quad \text{in} \quad \mathbb{R}^{n+1}, \\ w(x,0;\tau) = f(x,\tau),\ w_t(x,0;\tau) = 0 \quad \text{in} \quad \mathbb{R}^n. \end{cases} \tag{1.20}$$

Note that in the last problem the *initial time $t = 0$ is frozen*, while in the *Duhamel's principle it is varying with the parameter τ.*

The *second observation* is that in

$$u(x,t) = \int_{t_0}^{t} d\tau \int_{0}^{t-\tau} w(x,z;\tau)\,dz\,,$$

the *upper limit $t-\tau$ of the inner integral is* generated by the propagation phenomena with the speed which is equals to one. In fact, that is *a distance function between the points* reached from the origin at time t and τ.

Our *third observation* is that the *solution operator $G : f \longmapsto u$ can be regarded as a composition of two operators.*

- *The first operator*

  $$\mathcal{WE}:\ f \longmapsto w$$

  is a Fourier Integral Operator, which is a solution operator of the Cauchy problem with the first initial datum for the wave equation in the Minkowski spacetime.

- *The second operator*

  $$\mathcal{K}:\ w \longmapsto u$$

  is an integral operator given by $u(x,t) = \int_{t_0}^{t} d\tau \int_{0}^{t-\tau} w(x,z;\tau)dz.$

Thus, $G = \mathcal{K} \circ \mathcal{W}E$ and we have the diagram 1

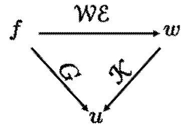


Figure 1: Diagram for the solution operator

The Aim is that *based on this diagram to generate a class of operators for which we will obtain explicit representation formulas for the solutions.* That means also that we will have representations for the fundamental solutions of the partial differential operators.

In fact, this diagram brings into a single hierarchy several different partial differential operators. Indeed, if we take into account the propagation cone by introducing the distance function $\phi(t)$, and if we provide the integral transform with the kernel $K(t;r,b)$ as follows:

$$\mathcal{K}[w](x,t) = \int_{t_0}^{t} db \int_{0}^{|\phi(t)-\phi(b)|} K(t;r,b)w(x,r;b)dr, \quad x \in \mathbb{R}^n, \ t > t_0, \tag{1.21}$$

then we actually can generate new representations for the solutions of different well-known equations. Below we illustrate the suggested scheme by several examples.

1.5 Examples of Integral transforms. Example 1: Klein-Gordon equation in the Minkowski spacetime.

1. Bessel functions: $J_\nu(z)$, $\nu \in \mathbb{R}$, is a solution of ODE

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2-\nu^2)y = 0,$$

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{z}{2}\right)^{2k+\nu},$$

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\nu+1)}\left(\frac{z}{2}\right)^{2k+\nu}.$$

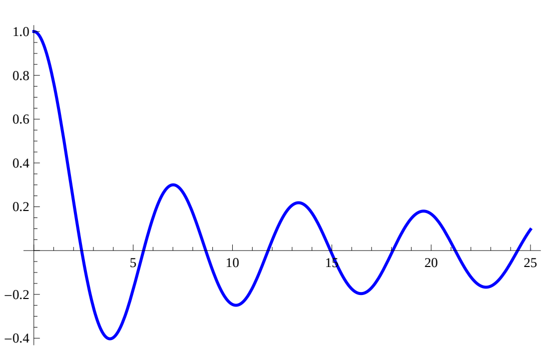


Figure 2: Graph of Function $J_0(x)$

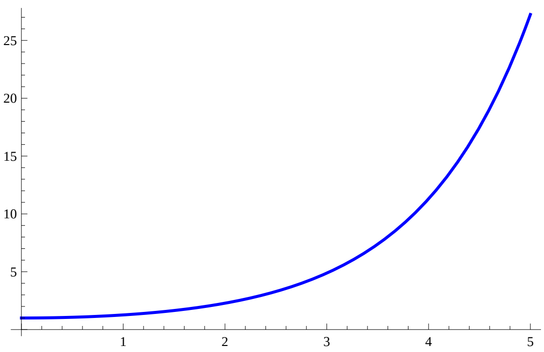


Figure 2: Graph of Function $I_0(x)$

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, \qquad \Re z > 0\,.$$

2. *The representation formulas.* If we choose the kernel $K(t;r,b)$ as

$$K(t;r,b) \;\; = \;\; J_0\left(\sqrt{(t-b)^2-r^2}\right)\,, \tag{1.22}$$

where $J_0(z)$ is the Bessel function of the first kind, and if we choose the distance function as $\phi(t)=t$, then we can prove (see Theorem 1.5 below) that the function

$$u(x,t)=\int_{t_0}^{t} db \int_0^{t-b} J_0\left(\sqrt{(t-b)^2-r^2}\right) w(x,r;b)dr, \;\; x\in\mathbb{R}, \;\; t>t_0,$$

solves the problem for the Klein-Gordon equation with a positive mass equals to 1 in the one-dimensional Minkowski spacetime,

$$u_{tt}-u_{xx}+u=f(x,t) \quad \mbox{in } \mathbb{R}^2, \quad u(x,t_0)=0, \quad u_t(x,t_0)=0 \quad \mbox{in } \mathbb{R}\,,$$

provided that $w(x,r;b)$ is a corresponding solution of the problem for the wave equation in the Minkowski spacetime. We emphasize that the function $w=w(x,t;b)$, with b regarded as a parameter, and the function $u=u(x,t)$ solve **different** equations. This is fundamental distinction from the Duhamel's principle.

If we now choose the kernel $K(t;r,b)$ as

$$K(t;r,b) \;\; = \;\; I_0\left(\sqrt{(t-b)^2-r^2}\right)\,, \tag{1.23}$$

where $I_0(z)$ is the modified Bessel function of the first kind, and the distance function as $\phi(t)=t$, then the function

$$u(x,t)=\int_{t_0}^{t} db \int_0^{t-b} I_0\left(\sqrt{(t-b)^2-r^2}\right) w(x,r;b)dr, \;\; x\in\mathbb{R}^n, \;\; t>t_0,$$

solves the problem for the Klein-Gordon equation with an imaginary mass in the one-dimensional Minkowski spacetime,

$$u_{tt}-u_{xx}-u=f(x,t) \quad \mbox{in } \mathbb{R}^2, \quad u(x,t_0)=0, \quad u_t(x,t_0)=0 \quad \mbox{in } \mathbb{R}\,,$$

provided that $w(x,r;b)$ is a corresponding solution of the problem (1.18) for the wave equation in the one-dimensional Minkowski spacetime.

According to the next theorem the representation formulas are valid also for the higher dimensional equations.

Theorem 1.5 (K.Y. [27] Rend. Istit. Mat. Univ. Trieste 42 Suppl. (2010))
The functions $u=u_{\Re}(x,t)$ and $u_{\Im}(x,t)$ defined by

$$u_{\Re}(x,t)=\int_{t_0}^{t} db \int_0^{t-b} J_0\left(m\sqrt{(t-b)^2-r^2}\right) w(x,r;b)dr, \;\; x\in\mathbb{R}^n, \tag{1.24}$$

$$u_{\Im}(x,t)=\int_{t_0}^{t} db \int_0^{t-b} I_0\left(m\sqrt{(t-b)^2-r^2}\right) w(x,r;b)dr, \;\; x\in\mathbb{R}^n, \tag{1.25}$$

$m=|M|$, are solutions of the problems

$$u_{tt}-\Delta u+M^2u=f(x,t) \mbox{ in } \mathbb{R}^{n+1}, \quad u(x,t_0)=0, \;\; u_t(x,t_0)=0 \;\mbox{ in } \mathbb{R}^n\,,$$

with $M^2>0$ and $M^2<0$, respectively. Here $w(x,t;b)$ is a solution of (1.20).

We emphasize that $u_{\Re}(x,t)$ and $w(x,r;b)$ solve **different** equations.

Proof of Theorem 1.5. We give a proof for the case of real mass $M = m = 1 > 0$, only, since the proof for the imaginary mass is similar. By straightforward calculations we obtain

$$\begin{aligned}\partial_t u_{\Re}(x,t) = & \int_{t_0}^{t} f(x,b)db + \int_{t_0}^{t} db \int_0^{t-b} w_z(x,r;b)dr \\ & + \int_{t_0}^{t} db \int_0^{t-b} J_0'\left(\sqrt{(t-b)^2-r^2}\right) \frac{t-b}{\sqrt{(t-b)^2-r^2}} w(x,r;b)dr.\end{aligned}$$

Then, one more differentiation and the equation for w imply

$$\begin{aligned}\partial_t^2 u_{\Re}(x,t) = & f(x,t) + \Delta \int_{t_0}^{t} db \int_0^{t-b} w(x,r;b)dr - \int_{t_0}^{t} \frac{t-b}{2} w(x,t-b;b)\, db \\ & + \int_{t_0}^{t} db \int_0^{t-b} \left\{ J_0'\left(\sqrt{(t-b)^2-r^2}\right) \partial_t \left(\frac{t-b}{\sqrt{(t-b)^2-r^2}}\right)\right. \\ & \left. + J_0''\left(\sqrt{(t-b)^2-r^2}\right)\left(\frac{t-b}{\sqrt{(t-b)^2-r^2}}\right)^2 \right\} w(x,r;b)dr\,.\end{aligned}$$

Consequently,

$$\begin{aligned} & \partial_t^2 u_{\Re}(x,t) - \Delta u_{\Re}(x,t) + u_{\Re}(x,t) \\ = & -\Delta \int_{t_0}^{t} db \int_0^{t-b} J_0\left(\sqrt{(t-b)^2-r^2}\right) w(x,r;b)dr + \int_{t_0}^{t} db \int_0^{t-b} J_0\left(\sqrt{(t-b)^2-r^2}\right) w(x,r;b)dr \\ & + f(x,t) + \Delta \int_{t_0}^{t} db \int_0^{t-b} w(x,r;b)dr - \int_{t_0}^{t} \frac{t-b}{2} w(x,t-b;b)\, db \\ & + \int_{t_0}^{t} db \int_0^{t-b} \left\{ J_0'\left(\sqrt{(t-b)^2-r^2}\right) \partial_t \left(\frac{t-b}{\sqrt{(t-b)^2-r^2}}\right)\right. \\ & \left. + J_0''\left(\sqrt{(t-b)^2-r^2}\right)\left(\frac{t-b}{\sqrt{(t-b)^2-r^2}}\right)^2 \right\} w(x,r;b)dr.\end{aligned}$$

Since $\Delta w = \partial_r^2 w(x,r;b)$ we obtain

$$\begin{aligned} & \partial_t^2 u_{\Re}(x,t) - \Delta u_{\Re}(x,t) + u_{\Re}(x,t) \\ = & f(x,t) + \int_{t_0}^{t} db \int_0^{t-b} \left(\partial_r J_0\left(m\sqrt{(t-b)^2-r^2}\right)\right) w_r(x,r;b)dr - \int_{t_0}^{t} \frac{t-b}{2} w(x,t-b;b)\, db \\ & + \int_{t_0}^{t} db \int_0^{t-b} J_0\left(m\sqrt{(t-b)^2-r^2}\right) w(x,r;b)dr \\ & + \int_{t_0}^{t} db \int_0^{t-b} \left\{ J_0'\left(\sqrt{(t-b)^2-r^2}\right) \partial_t \left(\frac{t-b}{\sqrt{(t-b)^2-r^2}}\right)\right. \\ & \left. + J_0''\left(\sqrt{(t-b)^2-r^2}\right)\left(\frac{t-b}{\sqrt{(t-b)^2-r^2}}\right)^2 \right\} w(x,r;b)dr.\end{aligned}$$

On the other hand

$$
\begin{aligned}
& \int_{t_0}^{t} db \int_0^{t-b} \left(\partial_r J_0\left(\sqrt{(t-b)^2-r^2}\right)\right) w_r(x,r;b)dr \\
= & \int_{t_0}^{t} \frac{t-b}{2} w(x,t-b;b)\, db \\
& - \int_{t_0}^{t} db \int_0^{t-b} \left\{ J''_0\left(\sqrt{(t-b)^2-r^2}\right) \left(\frac{r}{\sqrt{(t-b)^2-r^2}}\right)^2 \right. \\
& \left. + J'_0\left(\sqrt{(t-b)^2-r^2}\right) \partial_r \left(-\frac{r}{\sqrt{(t-b)^2-r^2}}\right) \right\} w(x,r;b)dr,
\end{aligned}
$$

then

$$
\begin{aligned}
& \partial_t^2 u_{\Re}(x,t) - \Delta u_{\Re}(x,t) + u_{\Re}(x,t) \\
= & f(x,t) + \int_{t_0}^{t} db \int_0^{t-b} \left(\partial_r J_0\left(m\sqrt{(t-b)^2-r^2}\right)\right) w_r(x,r;b)dr - \int_{t_0}^{t} \frac{t-b}{2} w(x,t-b;b)\, db \\
& + \int_{t_0}^{t} db \int_0^{t-b} J_0\left(m\sqrt{(t-b)^2-r^2}\right) w(x,r;b)dr \\
& + \int_{t_0}^{t} db \int_0^{t-b} \left\{ J'_0\left(\sqrt{(t-b)^2-r^2}\right) \partial_t \left(\frac{t-b}{\sqrt{(t-b)^2-r^2}}\right) \right. \\
& \left. + J''_0\left(\sqrt{(t-b)^2-r^2}\right) \left(\frac{t-b}{\sqrt{(t-b)^2-r^2}}\right)^2 \right\} w(x,r;b)dr \,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \partial_t^2 u_{\Re}(x,t) - \Delta u_{\Re}(x,t) + u_{\Re}(x,t) \\
= & f(x,t) \\
& + \int_{t_0}^{t} db \int_0^{t-b} \left\{ J'_0\left(\sqrt{(t-b)^2-r^2}\right) \frac{1}{\sqrt{(t-b)^2-r^2}} \right. \\
& \left. + J_0\left(\sqrt{(t-b)^2-r^2}\right) + J''_0\left(\sqrt{(t-b)^2-r^2}\right) \right\} w(x,r;b)dr \\
= & f(x,t).
\end{aligned}
$$

Theorem is proven. □

Let $s \in \mathbb{R}$ be a given number. For every function $\psi \in C_0^\infty(\mathbb{R}^n)$ we denote

$$
\|\psi(x)\|^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s |\hat{\psi}(\xi)|^2 d\xi \,,
$$

where $\hat{\psi}(\xi)$ is the Fourier transform of $\psi(x)$, that is,

$$
\psi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{x\cdot\xi} \hat{\psi}(\xi)\, d\xi, \quad \hat{\psi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-x\cdot\xi} \psi(x)\, dx \,.
$$

Exercise: Prove, that if s is a positive integer, $s \in \mathbb{N}$, then one has

$$\|\psi(x)\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sum_{\alpha_1+\alpha_2+\ldots+\alpha_n \le s} c_{\alpha_1\alpha_2\ldots\alpha_n} |\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\cdots\partial^{\alpha_n}_{x_n}\psi(x)|^2\,dx\,.$$

Find the constants $c_{\alpha_1\alpha_2\ldots\alpha_n}$.

Corollary 1.6 *For the solution (1.25) the following estimate holds*

$$\sum_{i=1}^n \|\partial_{x_i} u_{\Im}(x,t)\|^2_{H^s(\mathbb{R}^n)} \;\le\; \frac{1}{m}\int_{t_0}^t \sinh(m(t-b)) \sum_{i=1}^n \|\partial_{x_i} f(x,b)\|^2_{H^s(\mathbb{R}^n)}\,db\,.$$

Proof. For the solution $w = w(x,r;b)$ of the Cauchy problem for the wave equation

$$w_{tt} - \Delta w = 0 \quad w(x,0,b) = f(x,b), \quad w_t(x,0,b) = 0,$$

we have the conservation law

$$\begin{aligned} & \sum_{i=1}^n \|\partial_{x_i} w(x,t,b)\|^2_{H^s(\mathbb{R}^n)} + \|\partial_t w(x,t,b)\|^2_{H^s(\mathbb{R}^n)} \\ = & \sum_{i=1}^n \|\partial_{x_i} w(x,0,b)\|^2_{H^s(\mathbb{R}^n)} + \|\partial_t w(x,0,b)\|^2_{H^s(\mathbb{R}^n)}\,. \end{aligned}$$

That implies

$$\sum_{i=1}^n \|\partial_{x_i} w(x,t,b)\|^2_{H^s(\mathbb{R}^n)} + \|\partial_t w(x,t,b)\|^2_{H^s(\mathbb{R}^n)} \;=\; \sum_{i=1}^n \|\partial_{x_i} f(x,b)\|^2_{H^s(\mathbb{R}^n)}\,.$$

Hence, since $I_0\left(m\sqrt{(t-b)^2-r^2}\right)$ is positive, we obtain

$$\begin{aligned} \sum_{i=1}^n \|\partial_{x_i} u(x,t)\|^2_{H^s(\mathbb{R}^n)} &\le \int_{t_0}^t db \int_0^{t-b} I_0\left(m\sqrt{(t-b)^2-r^2}\right) \sum_{i=1}^n \|\partial_{x_i} f(x,b)\|^2_{H^s(\mathbb{R}^n)} dr \\ &\le \int_{t_0}^t \sum_{i=1}^n \|\partial_{x_i} f(x,b)\|^2_{H^s(\mathbb{R}^n)} db \int_0^{t-b} I_0\left(m\sqrt{(t-b)^2-r^2}\right) dr\,. \end{aligned}$$

We denote $\tau := t-b$. Then

$$\int_0^{t-b} I_0\left(m\sqrt{(t-b)^2-r^2}\right) dr \;=\; \int_0^{\tau} I_0\left(m\sqrt{\tau^2-r^2}\right) dr$$

and we make the change of variable $x = m\sqrt{\tau^2-r^2}$ in the integral

$$\int_0^{\tau} I_0\left(m\sqrt{\tau^2-r^2}\right) dr \;=\; \frac{1}{m}\int_0^{m\tau} \frac{x}{\sqrt{(m\tau)^2-x^2}} I_0(x)\,dx\,.$$

Exercise: Show that for $a > 0$ one has

$$\int_0^a \frac{x}{\sqrt{a^2 - x^2}} I_0(x)\, dx \quad = \quad \sinh(a)\,.$$

Finally, using the last identity with $a = m\tau$ we obtain

$$\sum_{i=1}^n \|\partial_{x_i} u(x,t)\|^2_{H^s(\mathbb{R}^n)} \quad \le \quad \frac{1}{m}\int_{t_0}^t \sinh(m(t-b)) \sum_{i=1}^n \|\partial_{x_i} f(x,b)\|^2_{H^s(\mathbb{R}^n)} db\,.$$

Corollary is proven. □

Problem. Show that the kernel $\sinh m(t-b)$ of this inequality is an optimal one, that is, if with some continuous function $K(t,b)$ the following estimate

$$\sum_{i=1}^n \|\partial_{x_i} u_{\Im}(x,t)\|^2_{H^s(\mathbb{R}^n)} \quad \le \quad \int_{t_0}^t K(t,b) \sum_{i=1}^n \|\partial_{x_i} f(x,b)\|^2_{H^s(\mathbb{R}^n)} db$$

holds for all t, t_0 and $f(x,b) \in C_0^\infty(\mathbb{R}^{n+1})$, then

$$K(t,b) \ge \frac{1}{m}\sinh(m(t-b)).$$

Problem. Derive for $u_{\Re}(x,t)$ a corresponding estimate.

3. *The fundamental solution for the Klein-Gordon equation in the Minkowski spacetime.* If we choose the kernel $K(t;r,b)$ as in (1.22) and choose the distance function as $\phi(t) = t$, then it can be easily verified (see Theorem 1.7 below) that the distribution

$$\mathcal{E}(x,t;x_0,t_0) = H(t-t_0)\int_0^{t-b} J_0\left(\sqrt{(t-b)^2 - r^2}\right) \mathcal{E}_0^{wave}(x-x_0,r)dr\,,$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$, is a forward fundamental solutions for the Klein-Gordon operator with a positive mass equals to 1 in the Minkowski spacetime,

$$\left(\partial_t^2 - \Delta + 1\right)\mathcal{E}(x,t;x_0,t_0) = \delta(x-x_0)\delta(t-t_0) \text{ in } \mathbb{R}^{n+1}, \quad \operatorname{supp}\mathcal{E} \subseteq D_+(x_0,t_0),$$

provided that $\mathcal{E}_0^{wave}(x,t)$ is the fundamental solution of the Cauchy problem corresponding to the first datum with the source at the origin, for the wave equation in the Minkowski spacetime.

We emphasize that the distributions $\mathcal{E}(x,t;x_0,t_0)$ and $\mathcal{E}^{wave}(x,t)$ solve **different** equations.

If we now choose the kernel $K(t;r,b)$ as in (1.23) and the distance function as $\phi(t) = t$, then the distribution

$$\mathcal{E}(x,t;x_0,t_0) = H(t-t_0)\int_0^{t-t_0} I_0\left(\sqrt{(t-b)^2 - r^2}\right) \mathcal{E}_0^{wave}(x-x_0,r)dr,$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$, is a forward fundamental solution for the Klein-Gordon operator with an imaginary mass in the Minkowski spacetime,

$$\left(\partial_t^2 - \Delta - 1\right)\mathcal{E}(x,t;x_0,t_0) = \delta(x-x_0)\delta(t-t_0) \quad \text{in } \mathbb{R}^{n+1}, \quad \operatorname{supp}\mathcal{E} \subseteq D_+(x_0,t_0).$$

The following theorem can be easily proved by direct substitution.

Theorem 1.7 (K.Y. [27] Rend. Istit. Mat. Univ. Trieste 42 Suppl. (2010))
The distributions $\mathcal{E}_{\Re}(x,t;x_0,t_0)$ *and* $\mathcal{E}_{\Im}(x,t;x_0,t_0)$ *defined by*

$$\mathcal{E}_{\Re}(x,t;x_0,t_0) = H(t-t_0)\int_0^{t-t_0} J_0\left(m\sqrt{(t-b)^2-r^2}\right)\mathcal{E}_0^{wave}(x-x_0,r)dr,$$
$$\mathcal{E}_{\Im}(x,t;x_0,t_0) = H(t-t_0)\int_0^{t-t_0} I_0\left(m\sqrt{(t-b)^2-r^2}\right)\mathcal{E}_0^{wave}(x-x_0,r)dr,$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$, *are forward fundamental solutions for the Klein-Gordon operator with a real or an imaginary mass*

$$\partial_t^2 - \Delta + M^2 \quad \text{in } \mathbb{R}^{n+1},$$

with $M^2 > 0$ *or* $M^2 < 0$*, respectively. Here* $m = |M| \geq 0$ *and* $\mathcal{E}_0^{wave}(x,t)$ *is the fundamental solution of the Cauchy problem corresponding to the first datum with the support at the origin, for the wave equation in the Minkowski spacetime.*

Lecture 3. July 16, 2010

1.6 Examples of Integral transforms. Example 2: Tricomi-type equations.

1. The Gauss's hypergeometric function.

$$z(1-z)\frac{d^2u}{dz^2} + [c-(a+b+1)z]\frac{du}{dz} - abu = 0,$$

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n = 1,2,3,\dots ,$$

$$F(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n \left(= {}_2F_1(a,b;c;z)\right) .$$

If $\Re c > \Re b > 0$, then there is the Euler formula

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-d)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^a dt\,.$$

2. The representation formulas. The first example linking to an operator with a variable coefficient is generated by the kernel

$$K(t;r,b) \;:=\; c_k\left((\phi(t)+\phi(b))^2-r^2\right)^{-\gamma} F\left(\gamma,\gamma;1;\frac{(\phi(t)-\phi(b))^2-r^2}{(\phi(t)+\phi(b))^2-r^2}\right), \qquad (1.26)$$

with $c_k = (k+1)^{-k/(k+1)}2^{k/(k+1)}$, the distance function $\phi = \phi(t)$ and the number γ defined as follows

$$\phi(t) = \frac{1}{k+1}t^{k+1} = \frac{2}{l+2}t^{\frac{l+2}{2}}, \qquad \gamma := \frac{k}{2k+2} = \frac{l}{2(l+2)}, \qquad 2k = l \in \mathbb{N}\cup\{0\}, \quad (1.27)$$

where $\mathbb{N}$ is the set of natural numbers, while $F(a,b;c;\zeta)$ is the Gauss's hypergeometric function.

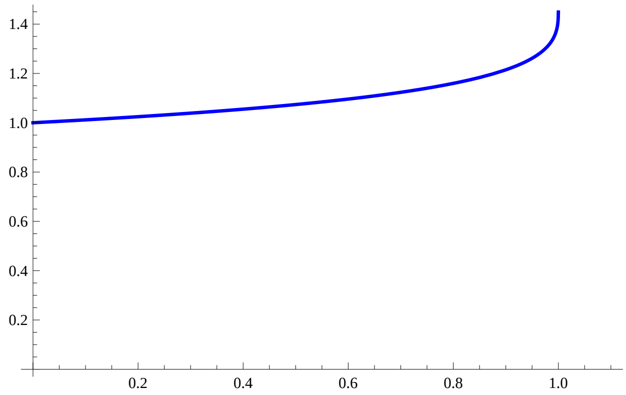


Figure 3: Graph of the function $F(1/3, 1/3; 1; x)$

Theorem 1.8 (K.Y. [17]J.D.E. 206(2004) Representation theorem) *For an integer non-negative l and for the smooth function $f = f(x,t)$, the function*

$$\begin{aligned} u(x,t) &= c_l \int_0^t db \int_0^{\phi(t)-\phi(b)} \left((\phi(t)+\phi(b))^2 - r^2\right)^{-\gamma} \\ &\quad \times F\left(\gamma, \gamma; 1; \frac{(\phi(t)-\phi(b))^2 - r^2}{(\phi(t)+\phi(b))^2 - r^2}\right) w(x,r;b)dr, \quad t > 0, \end{aligned}$$

solves the Tricomi-type equation

$$u_{tt} - t^l \Delta u = f(x,t) \quad \text{in} \quad \mathbb{R}^{n+1}_+ := \{(x,t)\,|\, x \in \mathbb{R}^n,\, t > 0\}, \tag{1.28}$$

and takes vanishing initial values

$$u(x,0) = 0, \quad u_t(x,0) = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{1.29}$$

Here the function $w(x,t;b)$ is a solution of the Cauchy problem

$$w_{tt} - \triangle w = 0, \quad w(x,0) = f(x,b), \quad w_t(x,0) = 0\,.$$

The next theorem gives representation formula for the problem without source term but with initial data.

Theorem 1.9 (K.Y.[23] Z. Angew. Math. Phys. 58 (2007)) *The solution to the Cauchy problem*

$$u_{tt} - t^l \triangle u = 0, \quad u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x) \tag{1.30}$$

can be represented as follows:

$$\begin{aligned} u(x,t) &= 2^{2-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 v_{\varphi_0}(x, \phi(t)s)(1-s^2)^{\gamma-1} ds \\ &\quad + t 2^{2\gamma} \frac{\Gamma(2-2\gamma)}{\Gamma^2(1-\gamma)} \int_0^1 v_{\varphi_1}(x, \phi(t)s)(1-s^2)^{-\gamma} ds, \quad x \in \mathbb{R}^n,\ t > 0\,. \end{aligned} \tag{1.31}$$

Here the function $v_\varphi(x,t)$ solves the Cauchy problem

$$v_{tt} - \triangle v = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = 0\,.$$

One can write (1.31) also as follows

$$\begin{aligned} u(x,t) &= \phi(t)^{1-2\gamma}2^{2-2\gamma}\frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)}\int_0^{\phi(t)} v_{\varphi_0}(x,\tau)(\phi^2(t)-\tau^2)^{\gamma-1}d\tau \qquad (1.32)\\ &\quad + t\phi(t)^{2\gamma-1}2^{2\gamma}\frac{\Gamma(2-2\gamma)}{\Gamma^2(1-\gamma)}\int_0^{\phi(t)} v_{\varphi_1}(x,\tau)(\phi^2(t)-\tau^2)^{-\gamma}d\tau, \quad x\in\mathbb{R}^n,\ t>0\,. \end{aligned}$$

We remind that here the function $v_\varphi(x,t)$ is given by formulas (1.11) and (1.12). More precisely, for $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for $x\in\mathbb{R}^n$, $n=2m+1$, $m\in\mathbb{N}$,

$$v_\varphi(x,t) := \frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\frac{t^{n-2}}{\omega_{n-1}c_0^{(n)}}\int_{S^{n-1}}\varphi(x+ty)\,dS_y$$

while for $x\in\mathbb{R}^n$, $n=2m$, $m\in\mathbb{N}$,

$$v_\varphi(x,t) := \frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}\frac{2t^{n-1}}{\omega_{n-1}c_0^{(n)}}\int_{B_1^n(0)}\frac{1}{\sqrt{1-|y|^2}}\varphi(x+ty)\,dV_y\,.$$

3. Huygens' Principle. Consider the formula that gives the solution for the Cauchy problem for the wave equation for odd n:

$$v(x^0,t^0) := \left\{\Big(\frac{\partial}{\partial t}\Big)^l\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\frac{t^{n-2}}{\omega_{n-1}c_0^{(n)}}\int_{S^{n-1}}\varphi(x^0+ty)\,dS_y\right\}_{t=t^0}, \quad l=0,1\,.$$

It shows that the value of the solution at point $x=x^0$ at a moment $t=t^0$ depends **only** on the values of the function φ and its derivatives on the sphere

$$S_{t^0}^{n-1}(x^0) := \{x\in\mathbb{R}^n\,|\,|x-x^0|=t^0\}\,.$$

This observation is called the Strong Huygens' Principle.

On the other hand, the corresponding formula for the Tricomi-type equation

$$\begin{aligned} u(x^0,t^0) &= c_l\int_0^1 v_{\varphi_0}(x^0,\phi(t^0)s)(1-s^2)^{\gamma-1}ds\\ &\quad + tc_l'\int_0^1 v_{\varphi_1}(x^0,\phi(t^0)s)(1-s^2)^{-\gamma}ds, \quad x\in\mathbb{R}^n, \end{aligned}$$

contains integrals, and, consequently, it collects information from the initial values via v_{φ_0} and v_{φ_1} for all times $\tau=\phi(t^0)s\in[0,\phi(t^0)]$ with positive weights $(1-s^2)^{\gamma-1}$ and $(1-s^2)^{-\gamma}$, respectively, where s is running over $[0,1]$. For instance, consider the first term with $v_{\varphi_0}(x^0,\phi(t^0)s)$. The last value depends on **only** values of φ_0 and its derivatives on the sphere

$$S_\tau^{n-1}(x^0) := \{x\in\mathbb{R}^n\,|\,|x-x^0|=\phi(t^0)s\}.$$

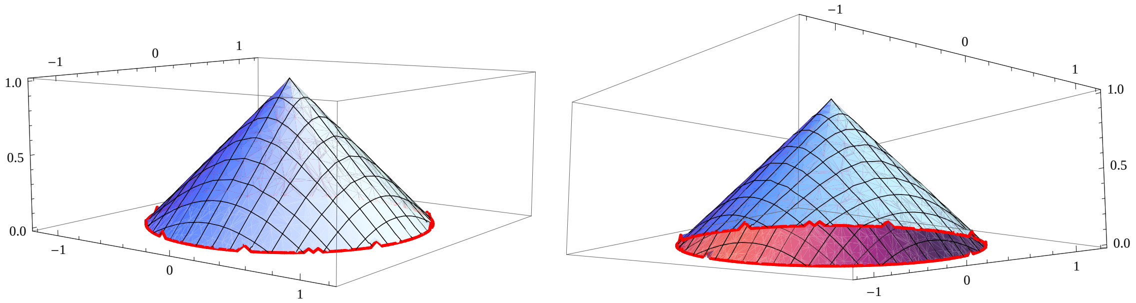


Figure 4: Cone of dependence for wave equation

Since s runs over $[0,1]$, these spheres fill the ball

$$B_{\phi(t^0)}(x^0) = \{x \in \mathbb{R}^n \,|\, |x - x^0| \le \phi(t^0)s\}.$$

Thus, the strong Huygens' Principle does not hold. At the same time, we have found a dependence domains for the point (x^0, t^0) with $t^0 \ge 0$:

$$\begin{aligned} D^n &:= \{(x,t) \in \mathbb{R}^{n+1} \,|\, |x - x^0| \le \phi(t^0) - \phi(t),\ \ 0 \le t \le t^0\} \subset \mathbb{R}^{n+1}, \\ D^{n-1} &:= \{x \in \mathbb{R}^n \,|\, |x - x^0| \le \phi(t^0)\} \subset \mathbb{R}^n. \end{aligned}$$

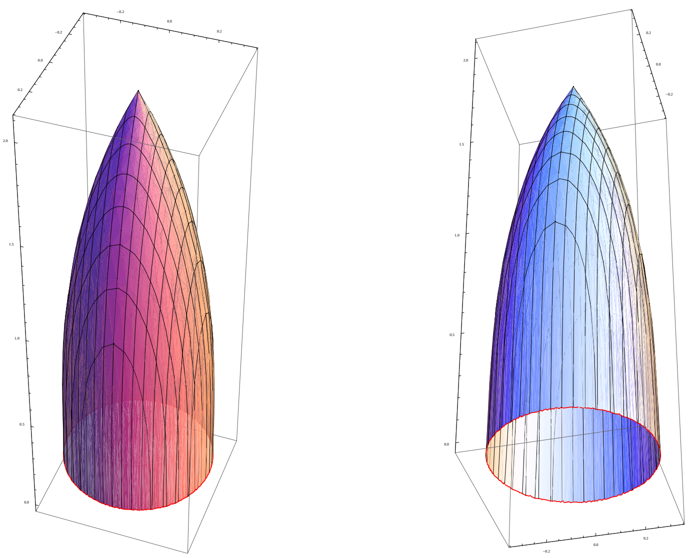


Figure 5: "Cone" of dependence for equation $u_{tt} - t^6 \Delta u = f$

Thus, *the value of the solution at the point* (x^0, t^0) *depends only on the values of the data* φ_0, φ_1 *on the domain* D^{n-1}*, and on the values of the source term* f *on the domain* D^n. Sometimes this observation is called a Weak Huygens' Principle.

4. Pointwise estimates for wave operator in the Minkowski spacetime. Denote

$$\|\psi(x)\|^2_{H^s(\mathbb{R}^n)} := \sum_{\alpha_1+\alpha_2+\ldots+\alpha_n \le s} \|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\cdots\partial^{\alpha_n}_{x_n}\psi(x)\|^2_{L^2(\mathbb{R}^n)}$$

and

$$\|\psi(x)\|_{\dot W^{s,p}(\mathbb{R}^n)} := \sum_{\alpha_1+\alpha_2+\ldots+\alpha_n=s} \|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}\psi(x)\|_{L^p(\mathbb{R}^n)}\,.$$

The definitions of the homogeneous Sobolev space $\dot W^{\frac{n+1}{2},1}(\mathbb{R}^n)$ and homogeneous Besov space $\dot B_{1,1}^{\frac{n-1}{2}}(\mathbb{R}^n)$ will be given later. The following is well known and we skip its proof.

Proposition 1.10 *For the Cauchy problem*

$$u_{tt} - \triangle u = 0, \quad u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x)$$

in n dimensions space the bound for u for odd n is given by

$$|u(x,t)| \;\leq\; C_n t^{-\frac{n-1}{2}} \left(\|\varphi_0\|_{\dot W^{\frac{n+1}{2},1}(\mathbb{R}^n)} + \|\varphi_1\|_{\dot W^{\frac{n-1}{2},1}(\mathbb{R}^n)} \right), \quad x \in \mathbb{R}^n,\ t>0\,,$$

while for even dimensions

$$|u(x,t)| \;\leq\; C_n t^{-\frac{n-1}{2}} \left(\|\varphi_0\|_{\dot B_{1,1}^{\frac{n+1}{2}}(\mathbb{R}^n)} + \|\varphi_1\|_{\dot B_{1,1}^{\frac{n-1}{2}}(\mathbb{R}^n)} \right), \quad x \in \mathbb{R}^n,\ t>0\,.$$

5. Pointwise estimates for the Tricomi-type operator.

We give just one example of the pointwise estimate. Let $\mathcal{M}(n)$ be a space of functions where we have the estimate similar to the given in Proposition 1.10 for the solution of the wave equation,

$$|u(x,t)| \;\leq\; C_k t^{-\frac{n-1}{2}} \left(\|\varphi_0\|_{\mathcal{M}(n)(\mathbb{R}^n)} + \|\varphi_1\|_{\mathcal{M}(n)(\mathbb{R}^n)} \right), \quad x \in \mathbb{R}^n,\ t>0\,.$$

Corollary 1.11 *For the solution $u = u(x,t)$ of (1.30) in $n = 2$ space dimension the bound for u is given by*

$$|u(x,t)| \;\leq\; C_k \left(t^{-\frac{1}{2}(k+1)} \|\varphi_0\|_{\mathcal{M}(2)(\mathbb{R}^2)} + t^{1-\frac{1}{2}(k+1)} \|\varphi_1\|_{\mathcal{M}(2)(\mathbb{R}^2)} \right), \quad x \in \mathbb{R}^2,\ t>0\,.$$

Proof of Corollary. First we set $\varphi_1 = 0$ and obtain

$$|u(x,t)| \;\leq\; \phi(t)^{1-2\gamma} C_l \int_0^{\phi(t)} |v_{\varphi_0}(x,\tau)|(\phi^2(t)-\tau^2)^{\gamma-1} d\tau\,.$$

According to the pointwise estimates for the solution of the wave equation:

$$|v_{\varphi_0}(x,\tau)| \;\leq\; c\tau^{-\frac{n-1}{2}} \|\varphi_0\|_{\mathcal{M}(2)(\mathbb{R}^n)}\,.$$

Hence,

$$\begin{aligned} |u(x,t)| \;&\leq\; C_l \phi(t)^{1-2\gamma} \int_0^{\phi(t)} \tau^{-\frac{n-1}{2}} \|\varphi_0\|_{\mathcal{M}(2)(\mathbb{R}^n)} (\phi^2(t)-\tau^2)^{\gamma-1} d\tau \\ &\leq\; C_l \phi(t)^{1-2\gamma} \|\varphi_0\|_{\mathcal{M}(2)(\mathbb{R}^n)} \int_0^{\phi(t)} \tau^{-\frac{n-1}{2}} (\phi^2(t)-\tau^2)^{\gamma-1} d\tau\,. \end{aligned}$$

Exercise: Prove that for $a > -1$, $b > -1$, and $c > 0$ one has

$$\int_0^c \tau^a (c-\tau^2)^b d\tau = C_{a,b} c^{1+a+2b} \,.$$

Thus, we plug $c = \phi(t)$ in the previous formula, and obtain

$$\int_0^{\phi(t)} \tau^{-\frac{n-1}{2}} (\phi^2(t)-\tau^2)^{\gamma-1} d\tau = c_k \phi(t)^{-\frac{n-1}{2}+2\gamma-1} \,.$$

Here we have used the assumption that the dimension is $n = 2$, that is $\dfrac{n-1}{2} < 1$. It follows

$$|u(x,t)| \;\; \le \;\; C_\gamma \|\varphi_0\|_{\mathcal{M}(2)(\mathbb{R}^n)} \phi(t)^{-\frac{n-1}{2}} = C'_\gamma \|\varphi_0\|_{\mathcal{M}(2)(\mathbb{R}^n)} t^{-\frac{1}{2}(k+1)} \,.$$

Now, if $\varphi_0 = 0$, then

$$|u(x,t)| \;\; \le \;\; C_l t \phi(t)^{2\gamma-1} \int_0^{\phi(t)} |v_{\varphi_1}(x,\tau)| (\phi^2(t)-\tau^2)^{-\gamma} d\tau, \quad x \in \mathbb{R}^n, \; t > 0 \,,$$

and we repeat the previous arguments. Corollary is proven. □

There are estimates similar to the given by Proposition 1.10 without singular (at $t = 0$) weight $t^{-\frac{n-1}{2}}$ but with the loss of derivatives. Those estimates allow to obtain corresponding inequalities for all dimensions n.

6. $H^s(\mathbb{R}^n)$ estimates for the Tricomi-type operator.

Corollary 1.12 *For the solution $u = u(x,t)$ of the Cauchy problem (1.28), (1.29) one has*

$$\sum_{i=1}^n \|\partial_{x_i} u(x,t)\|_{H^s(\mathbb{R}^n)} \;\; \le \;\; C \int_0^t \frac{t^{k+1} - b^{k+1}}{(t^{k+1}+b^{k+1})^{\frac{k}{k+1}}} \sum_{i=1}^n \|\partial_{x_i} f(x,b)\|_{H^s(\mathbb{R}^n)} \, db, \quad t > 0 \,.$$

Proof of Corollary 1.12. For the solution $w = w(x,r;b)$ of the Cauchy problem for the wave equation

$$w_{tt} - \Delta w = 0 \quad w(x,0,b) = f(x,b), \quad w_t(x,0,b) = 0,$$

we have the conservation law

$$\begin{aligned} & \sum_{i=1}^n \|\partial_{x_i} w(x,t,b)\|^2_{H^s(\mathbb{R}^n)} + \|\partial_t w(x,t,b)\|^2_{H^s(\mathbb{R}^n)} \\ = \;\; & \sum_{i=1}^n \|\partial_{x_i} w(x,0,b)\|^2_{H^s(\mathbb{R}^n)} + \|\partial_t w(x,0,b)\|^2_{H^s(\mathbb{R}^n)} \,. \end{aligned}$$

Hence,

$$\sum_{i=1}^n \|\partial_{x_i} w(x,t,b)\|^2_{H^s(\mathbb{R}^n)} \;\; \le \;\; \sum_{i=1}^n \|\partial_{x_i} w(x,0,b)\|^2_{H^s(\mathbb{R}^n)} = \sum_{i=1}^n \|\partial_{x_i} f(x,b)\|^2_{H^s(\mathbb{R}^n)} \,,$$

and, consequently,

$$\sum_{i=1}^{n} \|\partial_{x_i} w(x,t,b)\|_{H^s(\mathbb{R}^n)} \;\; \le \;\; \sum_{i=1}^{n} \|\partial_{x_i} f(x,b)\|_{H^s(\mathbb{R}^n)}\,.$$

It follows,

$$\begin{aligned}\sum_{i=1}^{n} \|\partial_{x_i} u(x,t)\|_{H^s(\mathbb{R}^n)} \;\; &\le \;\; C\int_0^t db \int_0^{\phi(t)-\phi(b)} \big((\phi(t)+\phi(b))^2 - r^2\big)^{-\gamma} \\ &\qquad \times F\left(\gamma,\gamma;1;\frac{(\phi(t)-\phi(b))^2-r^2}{(\phi(t)+\phi(b))^2-r^2}\right) \sum_{i=1}^{n} \|\partial_{x_i} w(x,r;b)\|_{H^s(\mathbb{R}^n)} dr\,.\end{aligned}$$

The argument of the hypergeometric function is bounded,

$$0 \le \frac{(\phi(t)-\phi(b))^2-r^2}{(\phi(t)+\phi(b))^2-r^2} \le 1 \quad \text{for all} \quad 0<b<t \quad \text{and} \quad 0<r\le \phi(t)-\phi(b)\,.$$

Since $2\gamma<1$, the function $F(\gamma,\gamma;1;\zeta)$ is bounded on the interval $\zeta\in[0,1]$, we have

$$\begin{aligned}&\sum_{i=1}^{n} \|\partial_{x_i} u(x,t)\|_{H^s(\mathbb{R}^n)} \\ &\le \;\; C\int_0^t db \int_0^{\phi(t)-\phi(b)} \big((\phi(t)+\phi(b))^2 - r^2\big)^{-\gamma} \sum_{i=1}^{n} \|\partial_{x_i} f(x,b)\|_{H^s(\mathbb{R}^n)} dr \\ &\le \;\; C\int_0^t db \sum_{i=1}^{n} \|\partial_{x_i} f(x,b)\|_{H^s(\mathbb{R}^n)} \int_0^{\phi(t)-\phi(b)} \big((\phi(t)+\phi(b))^2 - r^2\big)^{-\gamma} dr, \quad t>0.\end{aligned}$$

To estimate the integral

$$\int_0^{\phi(t)-\phi(b)} \big((\phi(t)+\phi(b))^2 - r^2\big)^{-\gamma} dr, \quad t>0,$$

we take into account that $z:=\phi(t)/\phi(b)=(t/b)^{k+1}>1$ and rewrite it as follows:

$$\int_0^{\phi(b)(z-1)} \big(\phi(b)^2(z+1)^2 - r^2\big)^{-\gamma} dr \;\; = \;\; \phi(b)^{1-2\gamma}\int_0^{z-1} \big((z+1)^2 - y^2\big)^{-\gamma} dy\,.$$

Since $1/2+\gamma<3/2$, the function $F\left(\frac{1}{2},\gamma;\frac{3}{2};x\right)$ is bounded on the interval $x\in[0,1]$, and we have

$$\begin{aligned}\int_0^{z-1} \big((z+1)^2 - y^2\big)^{-\gamma} dy \;\; &= \;\; (z-1)(z+1)^{-2\gamma} F\left(\frac{1}{2},\gamma;\frac{3}{2};\frac{(z-1)^2}{(z+1)^2}\right) \\ &\le \;\; C(z-1)(z+1)^{-2\gamma}\,,\end{aligned}$$

uniformly with respect to $z \in [1,\infty)$. It follows,

$$\begin{aligned}
\sum_{i=1}^{n} \|\partial_{x_i} u(x,t)\|_{H^s(\mathbb{R}^n)} &\le C\int_0^t db \sum_{i=1}^{n} \|\partial_{x_i} f(x,b)\|_{H^s(\mathbb{R}^n)} \phi(b)^{1-2\gamma}(z-1)(z+1)^{-2\gamma} \\
&\le C\int_0^t db \sum_{i=1}^{n} \|\partial_{x_i} f(x,b)\|_{H^s(\mathbb{R}^n)} \frac{\phi(t)-\phi(b)}{(\phi(t)+\phi(b))^{2\gamma}} \\
&\le C\int_0^t \frac{\phi(t)-\phi(b)}{(\phi(t)+\phi(b))^{2\gamma}} \sum_{i=1}^{n} \|\partial_{x_i} f(x,b)\|_{H^s(\mathbb{R}^n)}\, db \\
&\le C\int_0^t \frac{t^{k+1}-b^{k+1}}{(t^{k+1}+b^{k+1})^{\frac{k}{k+1}}} \sum_{i=1}^{n} \|\partial_{x_i} f(x,b)\|_{H^s(\mathbb{R}^n)}\, db\,.
\end{aligned}$$

Corollary is proven. □

Lecture 4. July 23, 2010

7. Fundamental solutions.

Theorem 1.13 (K.Y. [17] J.D.E. 206(2004) Fundamental solution)
The distribution $\mathcal{E}(x,t;x_0,t_0)$,

$$\begin{aligned}
\mathcal{E}(x,t;x_0,t_0) &= 2c_l H(t-t_0)\int_0^{\phi(t)-\phi(t_0)} \left((\phi(t)+\phi(t_0))^2 - r^2\right)^{-\gamma} \\
&\quad \times F\left(\gamma,\gamma;1;\frac{(\phi(t)-\phi(t_0))^2-r^2}{(\phi(t)+\phi(t_0))^2-r^2}\right)\mathcal{E}_0^{wave}(x-x_0,r)dr,
\end{aligned}$$

is the forward fundamental solution for the Tricomi-type equation (1.28), $x \in \mathbb{R}^n$, $t_0 \ge 0$, with the support in the forward light cone

$$D(x_0,t_0) := \{(x,t) \in \mathbb{R}^{n+1} \,|\, |x-x_0| \le \phi(t)-\phi(t_0)\}.$$

Thus,

$$\left(\partial_{tt} - t^l\Delta\right)\mathcal{E}(x,t;x_0,t_0) = \delta(x-x_0)\delta(t-t_0) \quad \text{in} \quad \mathbb{R}^{n+1}_+ := \{(x,t)\,|\, x\in\mathbb{R}^n,\, t>0\}.$$

Here $\mathcal{E}_0^{wave}(x,t)$ is the fundamental solution of the Cauchy problem corresponding to the first datum with the support at the origin, for the wave equation in the Minkowski spacetime,

$$\begin{cases} (\partial_t^2 - \Delta)\mathcal{E}_0^{wave}(x,t) = 0, \\ \mathcal{E}_0^{wave}(x,0,) = \delta(x)\,, \quad \left(\partial_t \mathcal{E}_0^{wave}\right)(x,0) = 0\,. \end{cases}$$

1.7 Examples of Integral transforms. Example 3: Klein-Gordon equations in the Robertson-Walker spacetime

1. The representation via Fourier transform (Galstian [6]) The equation

$$u_{tt} - e^{2t} \bigtriangleup u = 0, \tag{1.33}$$

is suggested and studied by Galstian as an example of hyperbolic equation with time dependent coefficient that can be solved via Fourier transform in terms of the *Bessel functions* and in terms of *confluent hypergeometric functions*. More precisely, in [6] the resolving operator for the Cauchy problem

$$\partial_t^2 u - e^{2t}\Delta u = 0, \qquad u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x)\,, \tag{1.34}$$

is written as a sum of the Fourier integral operators with the amplitudes given in terms of the Bessel functions and in terms of *confluent hypergeometric functions*. In particular, it is proved in [6] that for $t > 0$ the solution of the Cauchy problem (1.34) is given by

$$\begin{aligned} & u(x,t) \\ = & \sum_{j=0,1} -i\frac{2-j}{(2\pi)^n}\int_{\mathbb{R}^n} \Big\{ e^{i[x\cdot\xi+(e^t-1)|\xi|]} H_+\Big(\frac{1}{2};1;2ie^t|\xi|\Big) H_-\Big(\frac{3}{2}-j;3-2j;2i|\xi|\Big) \\ & \qquad - e^{i[x\cdot\xi-(e^t-1)|\xi|]} H_-\Big(\frac{1}{2};1;2ie^t|\xi|\Big) H_+\Big(\frac{3}{2}-j;3-2j;2i|\xi|\Big)\Big\} |\xi|^{2(1-j)}\mathcal{F}(\varphi_j)(\xi)d\xi\,. \end{aligned}$$

In the notations of [2] $H_-(\alpha;\gamma;z) = e^{i\alpha\pi}\Psi(\alpha;\gamma;z)$ and $H_+(\alpha;\gamma;z) = e^{i\alpha\pi}\Psi(\gamma-\alpha;\gamma;-z)$ with the confluent hypergeometric function $\Psi(a;c;z)$ defined in [2, Sec.6.5]. Here $\mathcal{F}(\varphi)(\xi)$ is a Fourier transform of $\varphi(x)$.

This equation under time inversion, $t \to -t$, becomes $u_{tt} - e^{-2t} \bigtriangleup u = 0$.

It turns out that (1.33) describes matter waves (particle) in the so-called Anti-de Sitter model of the universe that appears in the Mathematical Cosmology. Equation $u_{tt} - e^{-2t}\bigtriangleup u = 0$ describes particle in the so-called de Sitter model of the universe.

2. Real mass

$$u_{tt} - e^{-2t} \bigtriangleup u + M^2 u = f\,.$$

The integral transform and, in particular, its kernel and the Gauss's hypergeometric function, open a way to establish a bridge between the wave equation (massless equation) and the Klein-Gordon equation (massive equation) in the curved spacetime. Indeed, if we allow the parameter γ of the function $F(\gamma,\gamma;1;z)$ to be a complex number, $\gamma \in \mathbb{C}$, then this continuation into the complex plane produces the following representation formula.

Theorem 1.14 (A.G.-K.Y. [21] Comm. Math. Phys., 285 (2009).)
One can write

$$\begin{aligned} u(x,t) &= 2\int_0^t db \int_0^{e^{-b}-e^{-t}} dr\, w(x,r;b)(4e^{-b-t})^{iM} \left((e^{-t}+e^{-b})^2 - r^2\right)^{-\frac{1}{2}-iM} \\ & \qquad \times F\left(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(e^{-b}-e^{-t})^2 - r^2}{(e^{-b}+e^{-t})^2 - r^2}\right), \end{aligned} \tag{1.35}$$

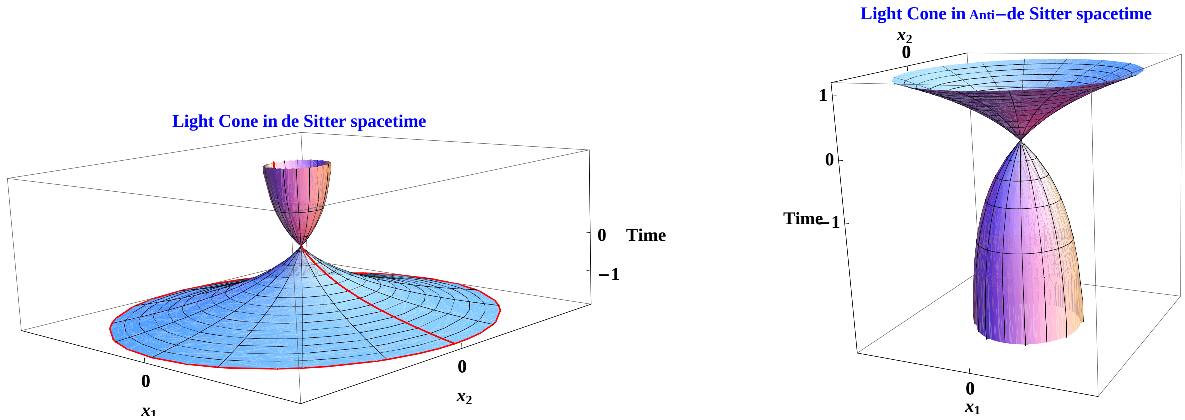


Figure 6: Graphs of the dependence and influence domains

where the function $w(x,t;b)$ is a solution to the Cauchy problem for the wave equation

$$w_{tt} - \triangle w = 0\,, \quad w(x,0;b) = f(x,b)\,, \quad w_t(x,0;b) = 0\,.$$

A similar formula holds for the fundamental solution $\mathcal{E}_+(x,t;x_0,t_0)$ for the Klein-Gordon operator in the de Sitter spacetime.

Theorem 1.15 (A.G.-K.Y. [21] Comm. Math. Phys., 285 (2009).)

$$\begin{aligned}\mathcal{E}_+(x,t;x_0,t_0) &= 2H(t-t_0)\int_0^{e^{-t_0}-e^{-t}} (4e^{-t_0-t})^{iM}\left((e^{-t_0}+e^{-t})^2-r^2\right)^{-\frac{1}{2}-iM} \\ &\quad \times F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-t_0}-e^{-t})^2-r^2}{(e^{-t_0}+e^{-t})^2-r^2}\Big)\mathcal{E}_0^{wave}(x-x_0,r)\,dr,\end{aligned} \tag{1.36}$$

where the distribution $\mathcal{E}_0^{wave}(x,t)$ is the fundamental solution of the Cauchy problem for the wave equation.

The non-negative *curved mass* $M \geq 0$ is defined as follows:

$$M^2 := m^2 - \frac{n^2}{4} \geq 0.$$

- The parameter m is a physical mass of particle.
- The fundamental solution $\mathcal{E}_-(x,t;x_0,t_0)$ with the support in the backward light cone admits a similar representation.
- The fundamental solutions $\mathcal{E}_+(x,t;x_0,t_0)$ and $\mathcal{E}_-(x,t;x_0,t_0)$ are constructed in [21] for the case of large masses $m \geq n/2$, that is $M \geq 0$.
- The integral makes sense in the topology of the space of distributions.
- The fundamental solutions for the Klein-Gordon operator in the anti-de Sitter spacetime can be obtained by time inversion, $t \to -t$, from the fundamental solutions for the Klein-Gordon operator in the de Sitter spacetime.

3. Imaginary mass
Moreover, the analytic continuation of this distribution in parameter M into $\mathbb{C}$ allows us to use it also in the case of small mass $0 \le m \le n/2$. The corresponding equation

$$u_{tt} - e^{-2t} \triangle u - M^2 u = f,$$

can be regarded as a Klein-Gordon equation with an imaginary mass. Equations with imaginary mass appear in several physical models such as ϕ^4 field model, tachion (super-light) fields, Landau-Ginzburg-Higgs equation and others.

More precisely, for small mass $0 \le m \le n/2$ we define the distribution $\mathcal{E}_+(x,t;x_0,t_0)$ by

$$\begin{aligned}\mathcal{E}_+(x,t;x_0,t_0) &= 2H(t-t_0)\int_0^{e^{-t_0}-e^{-t}} (4e^{-t_0-t})^{-M}\Big((e^{-t_0}+e^{-t})^2 - r^2\Big)^{-\frac{1}{2}+M} \\ &\quad \times F\Big(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-t_0}-e^{-t})^2-r^2}{(e^{-t_0}+e^{-t})^2-r^2}\Big)\mathcal{E}_0^{wave}(x-x_0,r)\,dr.\end{aligned}$$

1.8 Examples of Integral transforms. Example 3: Proof of case $n=1$.

The Riemann Function
In the characteristic coordinates l and m,

$$l = x + e^{-t}, \qquad m = x - e^{-t}, \tag{1.37}$$

the operator $\mathcal{S}$ of the equation reads as follows:

$$\mathcal{S} := \frac{\partial^2}{\partial t^2} - e^{-2t}\frac{\partial^2}{\partial x^2} + M^2 = -(l-m)^2\left\{\frac{\partial^2}{\partial l\,\partial m} - \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big) - \frac{1}{(l-m)^2}M^2\right\}.$$

In particular, in the new variables the equation

$$\left(\frac{\partial^2}{\partial t^2} - e^{-2t}\frac{\partial^2}{\partial x^2} + M^2\right)u = 0 \quad \text{implies} \quad \left\{\frac{\partial^2}{\partial l\,\partial m} - \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big)\right\}u - \frac{1}{(l-m)^2}M^2 u = 0\,.$$

We need the following lemma with $\gamma = \dfrac{1}{2} + iM$.

Lemma 1.16 (A.G.-K.Y. [21] Comm. Math. Phys., 285 (2009).)
The function

$$V(l,m;a,b) = (l-b)^{-\gamma}(a-m)^{-\gamma}F\Big(\gamma,\gamma;1;\frac{(l-a)(m-b)}{(l-b)(m-a)}\Big)$$

solves the equation

$$\left\{\frac{\partial^2}{\partial l\,\partial m} - \frac{1}{(l-m)}\gamma\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big)\right\}V(l,m;a,b) = 0\,.$$

Proof. In fact, the result follows by direct computations and basic properties of the hypergeometric function. We leave a proof to the reader. (A complete proof one can find in [21].) The lemma is proven. □

Lemma 1.17 (A.G.-K.Y. [21] Comm. Math. Phys., 285 (2009).)
For $\gamma \in \mathbb{C}$ such that $F(\gamma,\gamma;1;z)$ is well defined, the function

$$\begin{aligned}\widetilde{E}(l,m;a,b) &:= (a-b)^{\gamma-\frac{1}{2}}(l-m)^{\gamma-\frac{1}{2}}V(l,m;a,b)\\ &= (a-b)^{\gamma-\frac{1}{2}}(l-m)^{\gamma-\frac{1}{2}}(l-b)^{-\gamma}(a-m)^{-\gamma}F\Big(\gamma,\gamma;1;\frac{(l-a)(m-b)}{(l-b)(m-a)}\Big)\end{aligned}$$

solves the equation

$$\Big\{\frac{\partial^2}{\partial l\,\partial m}-\frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l}-\frac{\partial}{\partial m}\Big)\Big\}\widetilde{E}(l,m;a,b)+\frac{1}{(l-m)^2}\left(\frac{1}{2}-\gamma\right)^2\widetilde{E}(l,m;a,b)=0\,.$$

Proof. Indeed, straightforward calculations show

$$\begin{aligned}&(a-b)^{-\gamma+\frac{1}{2}}\Big\{\frac{\partial^2}{\partial l\,\partial m}-\frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l}-\frac{\partial}{\partial m}\Big)+\frac{1}{(l-m)^2}\left(\frac{1}{2}-\gamma\right)^2\Big\}\widetilde{E}\\ =\ &(l-m)^{\gamma-\frac{1}{2}}\left[V_{lm}-\frac{1}{(l-m)}\gamma\,(V_l-V_m)\right]=0\,.\end{aligned}$$

The lemma is proven. □

Consider now the operator

$$\mathcal{S}^*_{ch} := \frac{\partial^2}{\partial l\,\partial m}+\frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l}-\frac{\partial}{\partial m}\Big)-\frac{1}{(l-m)^2}(M^2+1)\,,$$

which is a formally adjoint to the operator

$$\mathcal{S}_{ch} := \frac{\partial^2}{\partial l\,\partial m}-\frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l}-\frac{\partial}{\partial m}\Big)-\frac{1}{(l-m)^2}M^2\,.$$

In the next lemma the Riemann function (see, e.g., [3, Ch.V, §5]) is presented.

Proposition 1.18 (A.G.-K.Y. [21] Comm. Math. Phys., 285 (2009).)
The function

$$\begin{aligned}&R(l,m;a,b)\\ =\ &(l-m)\widetilde{E}(l,m;a,b)\\ =\ &(a-b)^{iM}(l-m)^{1+iM}(l-b)^{-\frac{1}{2}-iM}(a-m)^{-\frac{1}{2}-iM}F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(l-a)(m-b)}{(l-b)(m-a)}\Big)\end{aligned}$$

defined for all l, m, a, $b\in\mathbb{R}$, such that $l>m$, is a unique solution of the equation $\mathcal{S}^*_{ch}R=0$, which satisfies the following conditions:

(i) $R_l = \dfrac{1}{2(l-m)}R$ *along the line $m=b$;*

(ii) $R_m = -\dfrac{1}{2(l-m)}R$ *along the line $l=a$;*

(iii) $R(a,b;a,b)=1$.

Proof. Indeed, if we denote $\gamma = \frac{1}{2} + iM$, then for the Riemann function we have

$$R(l,m;a,b) = (a-b)^{\gamma-\frac{1}{2}}(l-m)^{\gamma+\frac{1}{2}}V(l,m;a,b) = (l-m)\widetilde{E}(l,m;a,b)\,.$$

The operators $\mathcal{S}_{ch}$ and $\mathcal{S}_{ch}^*$ can be written as follows:

$$\begin{aligned}
\mathcal{S}_{ch} &= \frac{\partial^2}{\partial l\,\partial m} - \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big) + \frac{1}{(l-m)^2}\Big(\gamma - \frac{1}{2}\Big)^2\,,\\
\mathcal{S}_{ch}^* &= \frac{\partial^2}{\partial l\,\partial m} + \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big) - \frac{1}{(l-m)^2}\left(1 - \Big(\gamma - \frac{1}{2}\Big)^2\right)\,.
\end{aligned}$$

Direct calculations show that, if the function u solves the equation $\mathcal{S}_{ch}u = 0$, then the function $v = (l-m)u$ solves the equation $\mathcal{S}_{ch}^*v = 0$, and vice versa. Then Lemma 1.17 completes the proof. The lemma is proven. □

Proof of Theorem 1.15

Next we use the Riemann function $R(l,m;a,b)$ and the function $E(x,t;x_0,t_0)$ defined by

$$\begin{aligned}
E(x,t;x_0,t_0) &= (4e^{-t_0-t})^{iM}\Big((e^{-t}+e^{-t_0})^2 - (x-x_0)^2\Big)^{-\frac{1}{2}-iM} \qquad (1.38)\\
&\quad \times F\Big(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(e^{-t_0}-e^{-t})^2 - (x-x_0)^2}{(e^{-t_0}+e^{-t})^2 - (x-x_0)^2}\Big),
\end{aligned}$$

to complete the proof of (1.36), which gives the fundamental solution with support in the forward cone $D_+(x_0,t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and the fundamental solution with support in the backward cone $D_-(x_0,t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, defined by

$$D_\pm(x_0,t_0) := \Big\{(x,t) \in \mathbb{R}^{n+1}\,;\, |x-x_0| \le \pm(e^{-t_0} - e^{-t})\Big\}\,. \qquad (1.39)$$

with plus and minus, respectively.

We present a proof for $\mathcal{E}_+(x,t;0,t_0)$ since for $\mathcal{E}_-(x,t;0,t_0)$ it is similar. First, we note that the operator $\mathcal{S}$ is formally self-adjoint, $\mathcal{S} = \mathcal{S}^*$. We must show that

$$< \mathcal{E}_+, \mathcal{S}\varphi >= \varphi(0,t_0)\,, \qquad \text{for every} \quad \varphi \in C_0^\infty(\mathbb{R}^2)\,.$$

Since the distribution $\mathcal{E}_+(x,t;0,t_0)$ is locally integrable in $\mathbb{R}^2$, this is equivalent to show that

$$\int\!\!\int_{\mathbb{R}^2} \mathcal{E}_+(x,t;0,t_0)\mathcal{S}\varphi(x,t)\,dx\,dt = \varphi(0,t_0), \quad \text{for every} \quad \varphi \in C_0^\infty(\mathbb{R}^2). \qquad (1.40)$$

Let $(b,-b) := (e^{-t_0}, -e^{-t_0})$ be the image of the point $(0,t_0)$ in the characteristic coordinates l and m (see (1.37))

$$l = x + e^{-t}, \qquad m = x - e^{-t}\,.$$

The image of the interior of $D_+(0,t_0)$ in the characteristic coordinates l and m is the open triangle

$$T_+(b,-b) := \{(l,m) \in \mathbb{R}^2 \,|\, l > m,\ m > -e^{-t_0},\ l < e^{-t_0}\}\,.$$

For the functions φ and E in the new variables we use notations $\widetilde{\varphi}$ and $\widetilde{E}$, respectively, that is $\varphi(x,t) = \widetilde{\varphi}(l,m)$ and $E(x,t;0,t_0) = \widetilde{E}(l,m;b,-b)$. It is evident that $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^2)$ and

$$\operatorname{supp}\widetilde{\varphi} \subset \{(l,m) \in \mathbb{R}^2 \,|\, l > m\}.$$

In the mean time

$$\left|\frac{D(x,t)}{D(l,m)}\right| = (l-m)^{-1}$$

is the Jacobian of the transformation (1.37). Hence the integral in the left-hand side of (1.40) is equal to

$$\begin{aligned}
&\int\!\!\int_{\mathbb{R}^2} \mathcal{E}_+(x,t;0,t_0)\mathcal{S}\varphi(x,t)\,dx\,dt = \int_{t_0}^{\infty} dt \int_{e^{-t}-e^{-t_0}}^{e^{-t_0}-e^{-t}} E(x,t;0,t_0)\mathcal{S}\varphi(x,t)\,dx \\
= \;& -\int_{-\infty}^{b}\int_{-b}^{\infty} R(l,m;b,-b)\Big\{\frac{\partial^2}{\partial l\,\partial m} - \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big) - \frac{1}{(l-m)^2}M^2\Big\}\widetilde{\varphi}dl\,dm. \quad (1.41)
\end{aligned}$$

The rest of the proof is standard (see, e.g., [3, Ch.V, §5]), but we give it to make this section self-contained. We consider the first term of (1.41) and integrate it by parts

$$\begin{aligned}
&\int_{-b}^{\infty} dm \int_{-\infty}^{b} dl\;\, R(l,m;b,-b)\frac{\partial^2}{\partial l\,\partial m}\widetilde{\varphi} \\
= \;& -\varphi(b,-b) - \int_{-b}^{\infty} dm\,\left(\frac{\partial}{\partial m}R(b,m;b,-b)\right)\widetilde{\varphi}(b,m) \\
&+\int_{-\infty}^{b} dl\,\left(\frac{\partial}{\partial l}R(l,-b;b,-b)\right)\varphi(l,-b) + \int_{-b}^{\infty} dm \int_{-\infty}^{b} dl\,\left(\frac{\partial^2}{\partial l\,\partial m}R(l,m;b,-b)\right)\widetilde{\varphi}.
\end{aligned}$$

Then, for the second term in equation (1.41) we obtain

$$\begin{aligned}
&-\int_{-b}^{\infty} dm \int_{-\infty}^{b} dl\, R(l,m;b,-b)\frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big)\widetilde{\varphi} \\
= \;& -\int_{-\infty}^{b} R(l,-b;b,-b)\frac{1}{2(l+b)}\widetilde{\varphi}(l,-b)\,dl - \int_{-b}^{\infty} R(e^{-b},m;b,-b)\frac{1}{2(b-m)}\widetilde{\varphi}(b,m)\,dm \\
&-\int_{-b}^{\infty} dm \int_{-\infty}^{b} dl\,\frac{1}{(l-m)^2}R(l,m;b,-b)\widetilde{\varphi}(l,m)\,. \\
&+\int_{-b}^{\infty} dm \int_{-\infty}^{b} dl\,\Big[\frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big)R(l,m;b,-b)\Big]\widetilde{\varphi}(l,m)\,.
\end{aligned}$$

Using properties of the Riemann function we derive

$$\begin{aligned}
&-\int_{-b}^{\infty} dm \int_{-\infty}^{b} dl\, R(l,m;b,-b)\left\{\frac{\partial^2}{\partial l\,\partial m} - \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big) - \frac{1}{(l-m)^2}M^2\right\}\widetilde{\varphi} \\
= \;& \widetilde{\varphi}(b,-b) = \varphi(0,t_0)\,.
\end{aligned}$$

Theorem is proven. $\square$

2 Function Spaces

2.1 Notations

0.1 Denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let α be a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$. We define

$$\begin{aligned}
\alpha + \beta &= (\alpha_1 + \beta_1, ..., \alpha_n + \beta_n)\,, \\
|\alpha| &= \alpha_1 + \cdots + \alpha_n\,, \\
\beta &\le \alpha, \quad \text{if} \quad \beta_i \le \alpha_i, i = 1, ..., n; \\
\alpha! &= \alpha_1!...\alpha_n! \\
C_\beta^\alpha &= \binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} \\
x^\alpha &= (x_1)^{\alpha_1} \ldots (x^n)^{\alpha_n}, \quad \text{for} \quad x \in \mathbb{R}^n\,, \\
\partial^\alpha &= \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}\,; \\
D^\alpha &= \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n} = (-i)^{|\alpha|}\partial_x^\alpha\,.
\end{aligned}$$

0.2 Let f be a function from $C(\Omega)$, and $\Omega \subseteq \mathbb{R}^n$ be an open set. Support of the continuous function f is defined as the closure in Ω of the set of all $x \in \Omega$, such that $f(x) \neq 0$, i.e

$$\operatorname{supp} f = \overline{\{x \in \Omega; f(x) \neq 0\}} \cap \Omega\,.$$

Thus, $f \equiv 0$ on $\Omega \setminus \operatorname{supp} f$.

2.2 Sobolev spaces

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $u \in L^1_{loc}(\Omega)$. For a given multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$ we define the distributional derivative

$$\partial^\alpha u = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u = \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}$$

to be the linear functional on $C_0^\infty(\Omega)$ given by

$$< \partial^\alpha u, \varphi > := (-1)^{|\alpha|} \int_\Omega u(x) \partial^\alpha \varphi(x)\, dx\,, \qquad \forall \varphi \in C_0^\infty(\Omega)\,.$$

We say that $\partial^\alpha u \in L^p(\Omega)$, $1 \le p \le \infty$, if there exists $v \in L^p(\Omega)$ such that

$$< \partial^\alpha u, \varphi > = \int_\Omega v(x) \varphi(x)\, dx\,, \qquad \forall \varphi \in C_0^\infty(\Omega)\,.$$

For $k \in \mathbb{N}$, $1 \le p \le \infty$, we define the Sobolev space as a liner space

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \,|\, \partial^\alpha u \in L^p(\Omega) \text{ for all } \alpha \text{ such that } |\alpha| \le k\}$$

with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{|\alpha|\le k} \|\partial^\alpha u\|_{L^p(\Omega)}\,.$$

1. The spaces $W^{k,p}(\Omega)$ are Banach spaces.
2. For $p < \infty$ smooth functions $u \in C^\infty \cap W^{k,p}(\Omega)$ are dense in $W^{k,p}(\Omega)$.
3. We denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the $W^{k,p}(\Omega)$-norm.
4. The special case of the Sobolev spaces with $p = 2$ usually are denoted as $H^k(\Omega) := W^{k,2}(\Omega)$
5. If $\Omega = \mathbb{R}^n$, then we write simply $W^{k,p}$ and H^k.
6. The space $W^{k,p}$ can be also defined via Fourier transform even for all real $k \in \mathbb{R}$ by means of the norm

$$\|u\|_{W^{k,p}} := \|(1-\Delta)^{s/2}u\|_{L^p}\,.$$

where Δ is the Laplacian on $\mathbb{R}^n$.
7. One can proved the space $H^s(\Omega)$ with scalar product

$$(u,v)_{H^s(\Omega)} \stackrel{\text{def}}{=} \sum_{|\alpha|\le s} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}\,.$$

Then, $H^s(\Omega)$ is a separable Hilbert space.
8. Denote by $\mathcal{B}^k$ the Banach space of all functions $u \in C^k(\mathbb{R}^n)$ such that $\partial^\alpha u$, $|\alpha| \le k$, is bounded on $\mathbb{R}^n$, that is,

$$\begin{aligned}\mathcal{B}^k \;\; := \;\; & \{u \in C^k(\mathbb{R}^n)\,|\,\text{for every } \alpha,\ |\alpha| \le k, \text{there exists } C_\alpha \text{ such that} \\ & \quad |\partial^\alpha u(x)| \le C_\alpha \text{ for all } x \in \mathbb{R}^n\}\,.\end{aligned}$$

Theorem 2.1 (Sobolev embedding theorem.)
If $s > \frac{n}{2} + k$, then the space H^s is continuously embedded in $\mathcal{B}^k$, that is $H^s \subset \mathcal{B}^k$. More precisely, for every α, $|\alpha| \le k$, the function $\partial^\alpha u(x)$ is continuous and

$$\sup_{\mathbb{R}^n} |\partial^\alpha u(x)| \le C(n,k,s)\|u\|_{H^s}.$$

Proof. For any function $u \in C_0^\infty$ we have

$$\partial^\alpha u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \xi^\alpha e^{ix\cdot\xi}\hat{u}(\xi)d\xi\,.$$

Then

$$\begin{aligned}|\partial^\alpha u(x)| \;\; & \le \;\; (2\pi)^{-n/2} \int_{\mathbb{R}^n} |\xi^\alpha||\hat{u}(\xi)|d\xi \\ & = \;\; (2\pi)^{-n} \int_{\mathbb{R}^n} |\xi^\alpha|(1+|\xi|^2)^{s/2}|\hat{u}(\xi)|(1+|\xi|^2)^{-s/2}d\xi \\ & \le \;\; (2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s|\hat{u}(\xi)|^2\,d\xi\right)^{1/2} \left(\int_{\mathbb{R}^n} |\xi^{2\alpha}|(1+|\xi|^2)^{-s}\,d\xi\right)^{1/2} \\ & \le \;\; C(n,k)\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s|\hat{u}(\xi)|^2\,d\xi\right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{|\alpha|-s}\,d\xi\right)^{1/2} \\ & \le \;\; C(n,k)\|u\|_{H^s}\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{k-s}\,d\xi\right)^{1/2}\,.\end{aligned}$$

The integral $\int_{\mathbb{R}^n} |(1+|\xi|^2)^{k-s}\, d\xi$ is convergent since $s > \dfrac{n}{2} + k$, and we set

$$C(n,k) \int_{\mathbb{R}^n} (1+|\xi|^2)^{k-s}\, d\xi = C(n,k,s)\,.$$

Exercise. Complete the proof of the theorem.
Theorem is proven. $\square$

If $1 \le p < n$, the Sobolev conjugate of p is

$$p^* := \frac{np}{n-p}\,.$$

Note that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}\,.$$

Theorem 2.2 (Gagliardo-Nirenberg-Sobolev inequality)
Assume $1 \le p < n$. There exists a constant $C = C(n,p)$, depending only on n and p, such that

$$\|u\|_{L^{p^*}} \le C(n,p)\|\nabla u\|_{L^p} \qquad \textit{for all} \quad u \in C_0^1(\mathbb{R}^n).$$

Proof. It is enough to check it for $u \in C_0^\infty(\mathbb{R}^n)$. We have

$$|u(x)| \le \int_{-\infty}^{\infty} |\partial_{x_j} u(x_1,\ldots,x_{j-1},y_j,x_{j+1},\ldots,x_n)|dy_j \qquad \forall u \in C_0^\infty(\mathbb{R}^n).$$

It follows

$$|u(x)|^n \le \prod_{j=1}^n \int_{-\infty}^{\infty} |\partial_{x_j} u| dy_j \qquad \forall u \in C_0^\infty(\mathbb{R}^n)$$

and

$$|u(x)|^{\frac{n}{n-1}} \le \sqcap_{j=1}^n \left(\int_{-\infty}^{\infty} |\partial_{x_j} u(x_1,\ldots,x_{j-1},y_j,x_{j+1},\ldots,x_n)|dy_j \right)^{\frac{1}{n-1}}.$$

Now we set $p=1$ and integrate with respect to x_1 and apply the generalized Hölder inequality,

$$\int_{\mathbb{R}^n} |u_1 \cdots u_m|\, dx \le \|u_1\|_{L^{p_1}} \cdots \|u_m\|_{L^{p_m}} \quad \forall p_j \in [1,\infty],\ \frac{1}{p_1} + \ldots + \frac{1}{p_m} = 1,\ u_j \in C_0^\infty(\mathbb{R}^n).$$

For the case of a general n and $p=1$

$$\begin{aligned}
& \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\, dx_1 \\
\le\ & \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\partial_{x_1} u| dy_1 \right)^{\frac{1}{n-1}} \cdots \left(\int_{-\infty}^{\infty} |\partial_{x_n} u| dy_n \right)^{\frac{1}{n-1}} dx_1 \\
=\ & \left(\int_{-\infty}^{\infty} |\partial_{x_1} u| dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\partial_{x_2} u| dy_2 \right)^{\frac{1}{n-1}} \cdots \left(\int_{-\infty}^{\infty} |\partial_{x_n} u| dy_n \right)^{\frac{1}{n-1}} dx_1 \\
=\ & \left(\int_{-\infty}^{\infty} |\partial_{x_1} u| dy_1 \right)^{\frac{1}{n-1}} \left\{ \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\partial_{x_2} u|\, dx_1 dy_2 \right) \cdots \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\partial_{x_n} u|\, dx_1 dy_n \right) \right\}^{\frac{1}{n-1}},
\end{aligned}$$

where

$$p_2 = \ldots = p_n = n-1.$$

Then we integrate with respect to x_2:

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\, dx_1\, dx_2 \\
\leq\ & \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\partial_{x_1} u| dy_1\right)^{\frac{1}{n-1}} \\
& \times \left\{ \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_{x_2} u|\, dx_1 dy_2\right) \cdots \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_{x_n} u|\, dx_1 dy_n\right)\right\}^{\frac{1}{n-1}}\ dx_2 \\
=\ & \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_{x_2} u|\, dx_1 dy_2\right)^{\frac{1}{n-1}} \\
& \times \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\partial_{x_1} u| dy_1\right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_{x_3} u|\, dx_1 dy_3\right)^{\frac{1}{n-1}} \cdots \\
& \cdots \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_{x_n} u|\, dx_1 dy_n\right)^{\frac{1}{n-1}}\ dx_2\,.
\end{aligned}$$

Applying once more the generalized Hölder inequality, we find

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\, dx_1\, dx_2 \\
\leq\ & \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_{x_2} u|\, dx_1 dy_2\right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_{x_1} u| dy_1\, dx_2\right)^{\frac{1}{n-1}} \\
& \times \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_{x_3} u|\, dx_1\, dx_2 dy_3\right)^{\frac{1}{n-1}} \cdots \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_{x_n} u|\, dx_1\, dx_2 dy_n\right)^{\frac{1}{n-1}}.
\end{aligned}$$

We continue by integration with respect $x_3, \ldots, x_n$, eventually to find

$$\begin{aligned}
\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}}\, dx \ &\leq\ \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\partial_{x_i} u|\, dx_1 \ldots dy_i \ldots dx_n\right)^{\frac{1}{n-1}} \\
&=\ \|\nabla u\|_{L^1}^{\frac{n}{n-1}}\,.
\end{aligned}$$

The case of $p = 1$ is proven.

Consider now the case of $p \in (1, n)$. For every $\gamma > 1$, the function $v := |u|^\gamma$, is a C^1-function, and we can apply already proved case with $p = 1$:

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |u(x)|^{\frac{\gamma n}{n-1}}|\, dx\right)^{\frac{n-1}{n}} \ &\leq\ \int_{\mathbb{R}^n} |\nabla |u(x)|^\gamma|\, dx \\
&=\ \gamma \int_{\mathbb{R}^n} |u(x)|^{\gamma-1} |\nabla u(x)|\, dx \\
&\leq\ \gamma \left(\int_{\mathbb{R}^n} |u(x)|^{(\gamma-1)\frac{p}{p-1}}\, dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p\, dx\right)^{\frac{1}{p}}.
\end{aligned}$$

We set

$$\gamma = \frac{p(n-1)}{n-p} > 1 \quad \Longrightarrow \quad \frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1} = \frac{np}{n-p} = p^* .$$

Then

$$\left(\int_{\mathbb{R}^n} |u(x)|^{p^*} \, dx \right)^{\frac{n-1}{n}} \quad \leq \quad \gamma \left(\int_{\mathbb{R}^n} |u(x)|^{p^*} \, dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}}$$

implies the desired estimate

$$\left(\int_{\mathbb{R}^n} |u(x)|^{p^*} | \, dx \right)^{\frac{1}{p^*}} \quad \leq \quad \gamma \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}} .$$

Theorem is proven. □

Theorem 2.3 (Estimates for $W^{1,p}(\Omega)$, $1 \leq p < n$.)
Let Ω be a bounded, open subset of $\mathbb{R}^n$, and suppose that its boundary $\partial\Omega$ is C^1. Assume $1 \leq p < n$, then

$$W^{1,p}(\Omega) \subset L^{p^*}(\Omega).$$

Moreover, there is the estimate

$$\|u\|_{L^{p^*}(\Omega)} \leq C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}, \qquad \text{for all} \quad u \in W^{1,p}(\Omega) ,$$

where the constant $C(n,p,\Omega)$ depends only on n, p, and Ω.

Proof. We skip proof of the theorem. □

Theorem 2.4 (Poincaré's inequality. Estimates for $W_0^{1,p}(\Omega)$, $1 \leq p < n$.)
Let Ω be a bounded, open subset of $\mathbb{R}^n$. Assume $1 \leq p < n$. Then for every $q \in [1, p^*]$ there is a constant $C(n,p,q,\Omega)$ depending only on n, p, q, and Ω, such that the estimate

$$\|u\|_{L^q(\Omega)} \leq C(n,p,q,\Omega) \|\nabla u\|_{L^p(\Omega)}, \qquad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

Proof. We skip proof of the theorem. □

Remark 2.5 *On the space $W_0^{1,p}(\Omega)$ with a bounded, open Ω, the norm $\|\nabla u\|_{L^p(\Omega)}$ is equivalent to $\|u\|_{W^{1,p}(\Omega)}$.*

Exercise: Prove that the function $u = \log\log\left(1 + \frac{1}{|x|}\right)$, where $B_1(0) \subset \mathbb{R}^2$ is an open unit ball, belongs to $W^{1,2}(\Omega)$, but not to $L^\infty(\Omega)$.

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Theorem 2.6 (Rellich-Kondrachov Compactness Theorem.)
Let Ω *be a bounded, open subset of* $\mathbb{R}^n$. *Suppose that its boundary* $\partial\Omega$ *is* C^1. *Assume* $1 \le p < n$. *Then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

for every $q \in [1, p^*]$.

Proof. We skip proof of the theorem. □

Theorem 2.7 (General Sobolev embeddings.)
Let Ω *be a bounded, open subset of* $\mathbb{R}^n$. *Suppose that its boundary* $\partial\Omega$ *is* C^1. *Assume* $u \in W^{k,p}(\Omega)$.
(i) *If*

$$k < \frac{n}{p}\,,$$

then $u \in L^q(\Omega)$, *where*

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}\,.$$

Moreover, there is the estimate

$$\|u\|_{L^q(\Omega)} \le C(k,n,p,\Omega)\|u\|_{W^{k,p}(\Omega)},$$

with a constant $C(k,n,p,\Omega)$ *depending only on* k, n, p, q, *and* Ω.
(ii) *If*

$$k > \frac{n}{p}\,,$$

then

$$u \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{\Omega})\,,$$

where

$$\gamma = \begin{cases} \left[\dfrac{n}{p}\right] + 1 - \dfrac{n}{p}\,, & \text{if } \dfrac{n}{p} \text{ is not an integer,} \\ \text{any positive number} < 1, & \text{if } \dfrac{n}{p} \text{ is an integer.} \end{cases}$$

We have in addition the estimate

$$\|u\|_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{\Omega})} \le C\|u\|_{W^{k,p}(\Omega)}$$

with the constant C *depending only on* k, p, n, γ, *and* Ω.
(iii) *If*

$$k = \frac{n}{p}$$

then for k, $1 \le k \le n$,

$$u \in W^{k,p}(\Omega) \quad \Longrightarrow \quad u \in L^q(\Omega), \quad \textit{that is} \quad W^{k,p}(\Omega) \subset L^q(\Omega)\,, \qquad 1 \le p \le q < \infty$$

and

$$\|u\|_{L^q(\Omega)} \le C(k,n,p,\Omega)\|u\|_{W^{k,p}(\Omega)}\,.$$

Moreover, if $p = 1$ *so that* $k = n$*, embedding* $W^{k,p}(\Omega) \subset L^q(\Omega)$ *exists with* $q = \infty$ *as well; in fact,*

$$W^{n,1}(\Omega) \subset C^0_B(\Omega)\,.$$

Proof. We skip proof of the theorem. □

Remark 2.8 *In fact, the statement (i) is true for every* $q' \le q$*, that is*

$$\|u\|_{L^{q'}(\Omega)} \le C(k,n,p,\Omega)\|u\|_{W^{k,p}(\Omega)}.$$

Sobolev spaces with multiplication (algebra) property.

Theorem 2.9 (Algebra.)
Let Ω *be a bounded, open subset of* $\mathbb{R}^n$*. Then for* $s > \frac{n}{2}$ *the space* $H^s(\Omega)$ *is an algebra. Thus, for* $u, v \in H^s(\Omega)$ *we have* $uv \in H^s(\Omega)$ *and*

$$\|uv\|_{H^s(\Omega)} \le C\|u\|_{H^s(\Omega)}\|v\|_{H^s(\Omega)}.$$

Proof. Indeed, for $u, v \in H^k(\Omega)$ and any multi-index α with $|\alpha| \le s$, by the product rule we have

$$|\partial^\alpha(uv)| \le C_\alpha \sum_{\beta+\gamma=\alpha} |(\partial^\beta u)(\partial^\gamma v)|\,.$$

If $s - |\beta| > \frac{n}{2}$, then by (ii) of Sobolev embedding Theorem 2.7, we have

$$\|\partial^\beta u\|_{L^\infty(\Omega)} \le C\|\partial^\beta u\|_{C^{s-|\beta|-[\frac{n}{2}]-1,\tilde\gamma}(\overline{\Omega})} \le C\|\partial^\beta u\|_{H^{s-|\beta|}(\Omega)} \le C\|u\|_{H^s(\Omega)}.$$

Similarly, the following inequality holds if $s - |\gamma| > \frac{n}{2}$,

$$\|\partial^\gamma v\|_{L^\infty(\Omega)} \le C\|\partial^\gamma v\|_{C^{s-|\gamma|-[\frac{n}{2}]-1,\tilde\gamma}(\overline{\Omega})} \le C\|\partial^\gamma v\|_{H^{s-|\gamma|}(\Omega)} \le C\|v\|_{H^s(\Omega)}.$$

Hence,

$$\|(\partial^\beta u)(\partial^\gamma v)\|_{L^2(\Omega)} \le C\|\partial^\beta u\|_{L^\infty(\Omega)}\|\partial^\gamma v\|_{L^2(\Omega)} \le C\|u\|_{H^s(\Omega)}\|v\|_{H^s(\Omega)}\,.$$

In the remaining case we have $s - |\beta| \le \frac{n}{2}$ and $s - |\gamma| \le \frac{n}{2}$. On the other hand, for the nonnegative numbers a and b we have

$$a := \frac{1}{2} - \frac{s-|\beta|}{n} \ge 0 \quad \text{and} \quad b := \frac{1}{2} - \frac{s-|\gamma|}{n} \ge 0\,.$$

Meanwhile,

$$n + 2|\alpha| < 4s \quad \Longrightarrow \quad \frac{1}{2} - \frac{s-|\beta|}{n} + \frac{1}{2} - \frac{s-|\gamma|}{n} < \frac{1}{2} \quad \Longrightarrow \quad a + b < \frac{1}{2}\,.$$

Now we choose p and q such that $1 < p, q < \infty$ and

$$1 = \frac{1}{p} + \frac{1}{q}\,, \quad \frac{1}{p} \geq a \geq 0\,, \quad \frac{1}{q} \geq b \geq 0\,.$$

Then, in the case of $s - |\beta| < \frac{n}{2}$ we use (i) and in the case of $s - |\beta| = \frac{n}{2}$ we use (iii) of Theorem 2.7 to estimate $\|\partial^\beta u\|_{L^p(\Omega)}$. If we denote by p_1 a number defined by

$$\frac{1}{p_1} = \frac{1}{2} - \frac{s - |\beta|}{n}\,,$$

then $p \leq p_1$ and, according to Theorem 2.7 since Ω is bounded, we have

$$\|\partial^\beta u\|_{L^p(\Omega)} \leq C\|\partial^\beta u\|_{L^{p_1}(\Omega)} \leq C\|\partial^\beta u\|_{H^{s-|\beta|}(\Omega)} \leq C\|u\|_{H^s(\Omega)}\,.$$

Similarly, defining q_1 by $\dfrac{1}{q_1} = \dfrac{1}{2} - \dfrac{s - |\gamma|}{n}$ we obtain

$$\|\partial^\gamma v\|_{L^q(\Omega)} \leq C\|\partial^\gamma v\|_{L^{q_1}(\Omega)} \leq C\|\partial^\gamma v\|_{H^{s-|\gamma|}(\Omega)} \leq C\|v\|_{H^s(\Omega)}\,.$$

Thus,

$$\|(\partial^\beta u)(\partial^\gamma v)\|_{L^2(\Omega)} \leq C\|\partial^\beta u\|_{L^p(\Omega)}\|\partial^\gamma v\|_{L^q(\Omega)} \leq C\|u\|_{H^s(\Omega)}\|v\|_{H^s(\Omega)}\,.$$

Theorem is proven. □

Exercise. *Prove that for $s > \dfrac{n}{2}$ the space $H^s(\mathbb{R}^n)$ is an algebra. Thus, for $u, v \in H^s(\mathbb{R}^n)$ we have $uv \in H^s(\mathbb{R}^n)$ and*

$$\|uv\|_{H^s(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}.$$

Superposition of functions in Sobolev spaces.

If $P(z)$ is polynomial, then by induction on the degree, for any polynomial we find that

$$P \circ u \in H^s(\Omega) \quad \text{for} \quad u \in H^s(\Omega) \quad \text{and} \quad s > \frac{n}{2}\,,$$

$$\|P \circ u\|_{H^s(\Omega)} \leq C(1 + \|u\|^s_{H^s(\Omega)}) \sup_{l \leq s} \|P^{(l)} \circ u\|_{L^\infty(\Omega)}\,,$$

where $P^{(l)}(z) = \dfrac{d^l}{dz^l}P(z)$.

By density of polynomials in $C^k_{loc}(\mathbb{R})$, for any function $g \in C^k(\mathbb{R})$ and any $u \in H^s(\Omega)$, $k \geq s$, we conclude that the composition $g \circ u \in H^s(\Omega)$ and that

$$\|g \circ u\|_{H^s(\Omega)} \leq C\left(1 + \|u\|^s_{H^s(\Omega)}\right) \sup_{l \leq s} \|g^{(l)} \circ u\|_{L^\infty(\Omega)}\,,$$

where $g^{(l)}(z) = \dfrac{d^l}{dz^l}g(z)$.

Interpolation.

For every given α, $|\alpha| \le m$, and numbers q and r we define $p(\alpha)$ by

$$\frac{m}{p(\alpha)} = \frac{m-|\alpha|}{q} + \frac{|\alpha|}{r}\,. \tag{2.42}$$

Theorem 2.10 (Interpolation) *Let $1 \le r \le \infty$, $1 \le q \le \infty$, and let m be an integer ≥ 2. If $u \in L^q(\mathbb{R}^n)$ and $\partial^\gamma u \in L^r(\mathbb{R}^n)$ for all γ with $|\gamma| = m$, then $\partial^\alpha u \in L^{p(\alpha)}(\mathbb{R}^n)$ for every α, $|\alpha| \le m$. Moreover, for every j, $j = 0, 1, \ldots, m$,*

$$\sup_{|\alpha|=j} \|\partial^\alpha u\|_{L^{p(\alpha)}(\mathbb{R}^n)} \le 4^{j(m-j)} \left(\sup_{|\gamma|=m} \|\partial^\gamma u\|_{L^r(\mathbb{R}^n)} \right)^{\dfrac{j}{m}} \|u\|_{L^q(\mathbb{R}^n)}^{\dfrac{m-j}{m}}\,. \tag{2.43}$$

Corollary 2.11 *If $u, v \in L^\infty(\mathbb{R}^n)$ and $\partial^\alpha u$, $\partial^\alpha v \in L^r(\mathbb{R}^n)$ when $|\alpha| = m$, then*

$$\partial^\alpha (uv) \in L^r(\mathbb{R}^n), \quad \text{when} \quad |\alpha| = m\,,$$
$$\sum_{|\alpha|=m} \|\partial^\alpha (uv)\|_{L^r(\mathbb{R}^n)} \le C_m \Big(\|v\|_{L^\infty(\mathbb{R}^n)} \sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^r(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)} \sum_{|\alpha|=m} \|\partial^\alpha v\|_{L^r(\mathbb{R}^n)} \Big).$$

A more general version of the proceeding estimates is useful:

Corollary 2.12 *If v_1, v_2,..., $v_j \in L^\infty(\mathbb{R}^n)$ and $\partial^\alpha v_1$, $\partial^\alpha v_2$, ..., $\partial^\alpha v_j \in L^r(\mathbb{R}^n)$ when $|\alpha| = m$, then*

$$\partial^{\alpha_1} v_1 \partial^{\alpha_2} v_2 \cdots \partial^{\alpha_j} v_j \in L^r(\mathbb{R}^n)\,, \qquad \sum_1^j |\alpha_j| = m\,,$$
$$\|\partial^{\alpha_1} v_1 \partial^{\alpha_2} v_2 \cdots \partial^{\alpha_j} v_j\|_{L^r(\mathbb{R}^n)} \le 2^{jm^2/2} \max_{1 \le i \le j} \left\{ \Big(\prod_{k \ne i} \|v_k\|_{L^\infty(\mathbb{R}^n)} \Big) \sup_{|\alpha|=m} \|\partial^\alpha v_i\|_{L^r(\mathbb{R}^n)} \right\}.$$

Corollary 2.13 *Let $u \in L^\infty(\mathbb{R}^n, \mathbb{R}^N)$, let $F \in C^m(\mathbb{R}^N)$, and assume that $\partial^\alpha u \in L^r(\mathbb{R}^n, \mathbb{R}^N)$ when $|\alpha| = m$. Then*

$$\partial^\alpha F(u) \in L^r(\mathbb{R}^n), \quad \text{when} \quad |\alpha| = m\,,$$
$$\sup_{|\alpha|=m} \|\partial^\alpha F(u)\|_{L^r(\mathbb{R}^n)} \le C_m \Big(\sup_{1 \le |\gamma| \le m} |F^{(\gamma)}(u)| \|u\|_{L^\infty(\mathbb{R}^n)}^{|\gamma|-1} \Big) \sup_{|\alpha|=m} \|\partial^\alpha u\|_{L^r(\mathbb{R}^n)} \quad \text{if} \;\; m > 0, \tag{2.44}$$

while for $m = 0$ one has

$$\|F(u) - F(0)\|_{L^r(\mathbb{R}^n)} \le M \|u\|_{L^r(\mathbb{R}^n)}\,,$$

if M is a Lipschitz constant for F in the range of u.

Corollary 2.14 *The more general version of (2.44) where $F \in C^m(\mathbb{R}^{n+N})$ also depends on x, is*

$$\sup_{|\alpha|=m} \|\partial^\alpha (F(x,u(x)) - F(x,0))\|_{L^r(\mathbb{R}^n)}$$
$$\leq C'_m \sup_{\substack{1\leq|\gamma|\leq|\alpha| \\ |\alpha|+|\beta|=m}} \sup_{|v|\leq\|u\|_{L^\infty(\mathbb{R}^n)}} |\partial_x^\beta \partial_v^\gamma F(x,v)| \|u\|^{|\gamma|-1}_{L^\infty(\mathbb{R}^n)} \|\partial^\alpha u\|_{L^r(\mathbb{R}^n)}$$
$$+ C''_m \sup_{|\alpha|=m} \sup_{|v|\leq\|u\|_{L^\infty(\mathbb{R}^n)}} |\partial_x^\alpha \partial_v F(x,v)| \|\partial^\alpha u\|_{L^r(\mathbb{R}^n)}\,.$$

3 Energy Estimates for Linear Equation

Lemma 3.1 *Denote*

$$E_s(t) := \|\partial_t u\|_{H^{s-1}(\mathbb{R}^n)} + \|u\|_{H^s(\mathbb{R}^n)}\,.$$

Let $s > \dfrac{n+4}{2}$. Then for $f \in C([0,T]; H^{s-1}(\mathbb{R}^n)$, solutions $u \in C([0,T]; H^s(\mathbb{R}^n) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^n)$ of the equation

$$u_{tt} - \sum_{i,j=1,\dots,n} a_{ij}(x,t) u_{ij} = f(x,t), \qquad (x,t) \in \mathbb{R}^n \times [0,T], \tag{3.45}$$

satisfy the a priori estimate

$$E_s(t) \leq C \left(E_s(0) + \int_0^t \|f(\tau)\|_{H^{s-1}(\mathbb{R}^n)}\, d\tau \right), \qquad t \in [0,T]\,, \tag{3.46}$$

where the constant C depends on the coefficient functions a_{ij}, their L^∞-norms of the derivatives $\partial_x^\beta \partial_t^l a_{ij}(x,t)$, with $l \leq 1$, $|\beta| \leq s-1$, and numbers s, T.

Proof. In order to make an idea of the proof clear, we start with the simple case of an equation with constant coefficients $a_{ij}(x,t) = a_{ij}$. With $|\alpha| \leq s-1$ we apply ∂_x^α to the equation,

$$\partial_x^\alpha \left(u_{tt} - \sum_{i,j=1,\dots,n} a_{ij} u_{ij} \right) = \partial_x^\alpha f \quad \Longrightarrow \quad \partial_x^\alpha u_{tt} - \partial_x^\alpha \sum_{i,j=1,\dots,n} a_{ij} u_{ij} = \partial_x^\alpha f\,.$$

Then we multiply on $\partial_x^\alpha u_t$ and integrate with respect to x over $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n_x} \left(\partial_x^\alpha u_{tt} - \partial_x^\alpha \sum_{i,j=1,\dots,n} a_{ij} u_{ij} \right) \partial_x^\alpha u_t \, dx = \int_{\mathbb{R}^n_x} (\partial_x^\alpha f)\, \partial_x^\alpha u_t \, dx,$$

that is,

$$\int_{\mathbb{R}^n_x} (\partial_x^\alpha u_{tt})\, \partial_x^\alpha u_t \, dx - \int_{\mathbb{R}^n_x} \left(\sum_{i,j=1,\dots,n} a_{ij} \partial_x^\alpha u_{ij} \right) \partial_x^\alpha u_t \, dx = \int_{\mathbb{R}^n_x} (\partial_x^\alpha f)\, \partial_x^\alpha u_t \, dx\,.$$

Consider all terms of the last equation separately. For the first term we have

$$\int_{\mathbb{R}^n_x} (\partial_x^\alpha u_{tt})\, \partial_x^\alpha u_t \, dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n_x} (\partial_x^\alpha u_t)^2 \, dx \, .$$

The second term can be transformed as follows:

$$\begin{aligned} -\int_{\mathbb{R}^n_x} \sum_{i,j=1,\dots,n} a_{ij}\, (\partial_x^\alpha u_{ij})\, \partial_x^\alpha u_t \, dx &= -\int_{\mathbb{R}^n_x} \sum_{i,j=1,\dots,n} a_{ij}\, \big(\partial_x^\alpha \partial_{x_i}\partial_{x_j} u\big)\, \partial_x^\alpha u_t \, dx \\ = \int_{\mathbb{R}^n_x} \sum_{i,j=1,\dots,n} a_{ij}\, (\partial_x^\alpha \partial_{x_i} u)\, \partial_x^\alpha \partial_{x_j} u_t \, dx &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n_x} \sum_{i,j=1,\dots,n} a_{ij}\, (\partial_x^\alpha \partial_{x_i} u)\, \big(\partial_x^\alpha \partial_{x_j} u\big) \, dx \, . \end{aligned}$$

For the source term, using algebra property of the Sobolev space, inequality $|\alpha| \le s-1$, and the definition of the energy, we obtain

$$\left| \int_{\mathbb{R}^n_x} (\partial_x^\alpha f)\, \partial_x^\alpha u_t \, dx \right| \quad \le \quad \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^n_x)} \|\partial_x^\alpha u_t\|_{L^2(\mathbb{R}^n_x)} \, .$$

Thus, we have obtained

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n_x} (\partial_x^\alpha u_t)^2 \, dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n_x} \sum_{i,j=1,\dots,n} a_{ij}\, (\partial_x^\alpha \partial_{x_i} u)\, \big(\partial_x^\alpha \partial_{x_j} u\big) \, dx \le \|f\|_{H^{s-1}(\mathbb{R}^n_x)} \|\partial_x^\alpha u_t\|_{L^2(\mathbb{R}^n_x)} \, ,$$

or

$$\frac{1}{2}\frac{d}{dt}\left\{ \int_{\mathbb{R}^n_x} (\partial_x^\alpha u_t)^2 \, dx + \int_{\mathbb{R}^n_x} \sum_{i,j=1,\dots,n} a_{ij}\, (\partial_x^\alpha \partial_{x_i} u)\, \big(\partial_x^\alpha \partial_{x_j} u\big) \, dx \right\} \quad \le \quad \|f\|_{H^{s-1}(\mathbb{R}^n_x)} \|\partial_x^\alpha u_t\|_{L^2(\mathbb{R}^n_x)} \, .$$

We sum up all such inequalities for α with $|\alpha| \le s-1$, and arrive at the inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{ \int_{\mathbb{R}^n_x} \sum_{|\alpha|\le s-1} (\partial_x^\alpha u_t)^2 \, dx + \int_{\mathbb{R}^n_x} \sum_{|\alpha|\le s-1} a_{ij}\, (\partial_x^\alpha \partial_{x_i} u)\, \big(\partial_x^\alpha \partial_{x_j} u\big) \, dx \right\} \\ \le \quad &\|f\|_{H^{s-1}(\mathbb{R}^n_x)} \sum_{|\alpha|\le s-1} \|\partial_x^\alpha u_t\|_{L^2(\mathbb{R}^n_x)} \, . \end{aligned}$$

Exercise. Derive in a similar way a conservation law used above.
Moreover,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{ \int_{\mathbb{R}^n_x} \sum_{|\alpha|\le s-1} (\partial_x^\alpha u_t)^2 \, dx + \int_{\mathbb{R}^n_x} \sum_{|\alpha|\le s-1} a_{ij}\, (\partial_x^\alpha \partial_{x_i} u)\, \big(\partial_x^\alpha \partial_{x_j} u\big) \, dx \right\} \qquad (3.47) \\ \le \quad &\|f\|_{H^{s-1}(\mathbb{R}^n_x)} \left\{ \int_{\mathbb{R}^n_x} \sum_{|\alpha|\le s-1} (\partial_x^\alpha u_t)^2 \, dx + \int_{\mathbb{R}^n_x} \sum_{|\alpha|\le s-1} a_{ij}\, (\partial_x^\alpha \partial_{x_i} u)\, \big(\partial_x^\alpha \partial_{x_j} u\big) \, dx \right\}^{1/2} \, . \end{aligned}$$

Now we note that since the matrix a_{ij} is positive, the function $y = y(t)$ defined by

$$y(t) := \left\{ \int_{\mathbb{R}^n_x} \sum_{|\alpha|\le s-1} \left(\partial_x^\alpha u_t\right)^2 \, dx + \int_{\mathbb{R}^n_x} \sum_{\substack{|\alpha|\le s-1\\ i,j=1,\dots,n}} a_{ij} \left(\partial_x^\alpha \partial_{x_i} u\right) \left(\partial_x^\alpha \partial_{x_j} u\right) \, dx \right\}^{1/2} ,$$

is continuous and nonnegative. Then (3.47) implies

$$\frac{d}{dt} y^2(t) \le C y(t) \|f\|_{H^{s-1}(\mathbb{R}^n_x)}$$

for all $t \in [0,T]$. Then, the last inequality implies (check this statement)

$$y(t) \le C \int_0^t \|f(\tau)\|_{H^{s-1}(\mathbb{R}^n_x)} \, d\tau ,$$

that is

$$\begin{aligned} & \int_{\mathbb{R}^n_x} \sum_{|\alpha|\le s-1} \left(\partial_x^\alpha u_t\right)^2 \, dx + \int_{\mathbb{R}^n_x} \sum_{\substack{|\alpha|\le s-1\\ i,j=1,\dots,n}} a_{ij} \left(\partial_x^\alpha \partial_{x_i} u\right) \left(\partial_x^\alpha \partial_{x_j} u\right) \, dx \\ \le \; & C \left(\int_0^t \|f(\tau)\|_{H^{s-1}(\mathbb{R}^n_x)} \, d\tau \right)^2 . \end{aligned}$$

Now we have to take into account that in the left hand side of the last inequality the term $\|u(t)\|_{L^2(\mathbb{R}^n_x)}$ is absent, while it is present in the energy $E_s(t)$. We estimate that term as follows:

$$\begin{aligned} \|u(t)\|_{L^2(\mathbb{R}^n_x)} &= \|u(0) + \int_0^t u_t(\tau)\, d\tau\|_{L^2(\mathbb{R}^n_x)} \\ &\le \|u(0)\|_{L^2(\mathbb{R}^n_x)} + \int_0^t \|u_t(\tau)\|_{L^2(\mathbb{R}^n_x)} \, d\tau \\ &\le \|E_s(0)\|_{L^2(\mathbb{R}^n_x)} + \int_0^t E_s(\tau)\, d\tau . \end{aligned}$$

Since $|a_{ij}| \le C$ we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n_x} \sum_{|\alpha|\le s-1} \left(\partial_x^\alpha u_t\right)^2 \, dx + \int_{\mathbb{R}^n_x} \sum_{\substack{|\alpha|\le s-1\\ i,j=1,\dots,n}} a_{ij} \left(\partial_x^\alpha \partial_{x_i} u\right) \left(\partial_x^\alpha \partial_{x_j} u\right) \, dx \qquad (3.48) \\ \le \; & C E_s(0) + C \left(\int_0^t \|f(\tau)\|^2_{H^{s-1}(\mathbb{R}^n_x)} \, d\tau \right)^2 . \end{aligned}$$

On the other hand, since $\sum a_{ij}\xi_i\xi_j \ge c|\xi|^2$, the left hand side of the inequality (3.48) can be estimated from the below, and we obtain

$$E_s^2(t) \; \le \; C E_s^2(0) + C \left(\int_0^t \|f(\tau)\|^2_{H^{s-1}(\mathbb{R}^n_x)} \, d\tau \right)^2 .$$

Exercise. Complete the proof for variable coefficients $a_{ij} = a_{ij}(x,t)$.
Lemma is proven. □

Exercise. Derive the energy estimates from the representation formula for the solutions of the equation.

4 Uniqueness

In this section we consider real-valued functions and use the notations

$$x_0 := t, \;\; u' := (u_j) := (\partial_t u, \nabla u) =: (\partial_{x_0} u, \partial_{x_1} u, \dots, \partial_{x_n} u),$$
$$u'' := (u_{ij}) := (\partial_{x_i}\partial_{x_j} u)_{i,j=0,1,\dots,n}, \quad F_{jk}(x,u,u',u'') := \frac{\partial F(x,u,u',u'')}{\partial u_{jk}}\,.$$

Theorem 4.1 (Local Uniqueness.)
Let $u \in C^3$ be a solution of the differential equation

$$F(x,u,u',u'') = 0 \tag{4.49}$$

in a neighborhood of $0 \in \mathbb{R}^{n+1}$, with $F \in C^2$. Assume that it is strictly hyperbolic at the origin with respect to the plane $x_0 = 0$, that is the polynomial

$$\sum_{j,k=0}^{n} F_{jk}(0,u(0),u'(0),u''(0))\xi_j\xi_k$$

is strictly hyperbolic in the direction of the ξ_0 axis. Thus, the quadratic equation

$$\sum_{j,k=0}^{n} F_{jk}(0,u(0),u'(0),u''(0))\xi_j\xi_k = 0\,,$$

for ξ_0 has real distinct roots for all $\xi_1^2+\xi_2^2+\ldots+\xi_n^2 = 1$.

Then any other solution $v \in C^3$ of (4.49) with $\partial^\alpha u = \partial^\alpha v$ when $x_0 = 0$ and $|\alpha| \le 2$ is equal to u in some neighborhood of 0.

Proof. By Taylor's formula we can write

$$F(x,u,u',u'') - F(x,v,v',v'') = \sum_{|\alpha|\le 2} a_\alpha(x,u,u',u'',v,v',v'')\partial^\alpha(u-v)\,,$$

where $a_\alpha \in C^1$ as a function of x,u,u',u'',v,v',v''. Hence $u-v$ satisfies a linear differential equation with C^1 coefficients which by hypothesis is hyperbolic with respect to the plane $x_0 = 0$ at 0. Moreover, it is strictly hyperbolic with respect to the plane $x_0 = 0$ in some neighborhood of the origin because we consider real-valued functions, only. (Exercise: Prove the last statement.) Since the Cauchy data are 0 it follows from the energy estimates for linear equations proved before and from the Holmgren's transformation that $u - v = 0$ in some neighborhood of $0 \in \mathbb{R}^{n+1}$. □

Exercise. Prove that, if the equation is quasilinear, that is, linear in the highest order derivatives,

$$u_{tt} - \sum_{i,j=1,\dots,n} a_{ij}(u,\partial_t u,\nabla u)u_{ij} = f(u,\partial_t u,\nabla u), \qquad (x,t) \in \mathbb{R}^{n+1}\,,$$

the statement of the theorem is true if $u, v \in C^2$.

Lecture 6. August 6, 2010

5 Local Existence

Consider the Cauchy problem

$$u_{tt} - \sum_{i,j=1,\dots,n} a_{ij}(u,\nabla u)u_{ij} + u = f(u,\nabla u), \qquad (x,t) \in \mathbb{R}^{n+1}\,, \tag{5.50}$$

$$u(x,0) = \varphi_0(x), \qquad u_t(x,0) = \varphi_1(x), \qquad x \in \mathbb{R}^n, \tag{5.51}$$

where $u_{ij} := \partial_{x_i}\partial_{x_j} u$, and a_{ij} and f are smooth functions of their arguments. The matrix $a_{ij}(u,\eta)$ is assumed to be positive uniformly for all its arguments, that is,

$$0 < c_1 \le \sum_{i,j=1,\dots,n} a_{ij}(u,\eta)\xi_i\xi_j \quad \text{for all} \quad u \in \mathbb{R},\ \eta,\xi \in \mathbb{R}^n, \quad |\xi| = 1\,.$$

Example. $a_{ij}(u,\eta) = \delta_{ij}(1+|\eta|^2)$ or $a_{ij}(u,\eta) = \delta_{ij}\left(2 + u^4 + \sin(1+|\eta|^2)\right)$

5.1 Local Well-Posedness Theorem

Theorem 5.1 (Local Well-Posedness).
Given initial data

$$(\varphi_0, \varphi_1) \in H^s \times H^{s-1}$$

for some $s > \dfrac{n+4}{2}$*, there exists a number* $T > 0$ *such that the Cauchy problem (5.50),(5.51):*

$$u_{tt} - \sum_{i,j=1,\dots,n} a_{ij}(u,\nabla u)u_{ij} + u = f(u,\nabla u), \qquad (x,t) \in \mathbb{R}^{n+1}\,,$$

$$u(x,0) = \varphi_0(x), \qquad u_t(x,0) = \varphi_1(x), \qquad x \in \mathbb{R}^n,$$

is locally well-posed in the space $C([0,T];H^s) \times C([0,T];H^{s-1})$*, that is*

$$(u,u_t) \in C([0,T];H^s) \times C([0,T];H^{s-1})\,.$$

Proof. We use the notation for the "energy"

$$E_s(t) := \|\partial_t u\|_{H^{s-1}(\mathbb{R}^n)} + \|u\|_{H^s(\mathbb{R}^n)}\,.$$

Denote by X the space of functions $u \in L^2([0,T]\times\mathbb{R}^n)$ such that

$$u_t, u_{x_1}, \dots, u_{x_n} \in L^\infty([0,T];H^{s-1}) \bigcap C([0,T];L^2)$$

with the norm defined according to

$$\|u\|_X := \sup_{0\le t\le T} E_s(t)\,.$$

Then, denote for $M > 0$ the subset

$$X_M := \{u \in X \mid u(0) = \varphi_0,\ u_t(0) = \varphi_1\ \ \|u\|_X \le M\}\,.$$

We provide the set X_M with the metric $\rho(u,v)$ defined by

$$\rho(u,v) := \sup_{0\le t\le T} \left(\|u-v\|_{H^s} + \|\partial_t u - \partial_t v\|_{H^{s-1}}\right) .$$

Exercise. Prove that (X_M, ρ) is a complete metric space.
Define the map

$$L \,:\, X_M \ni v \longmapsto u \in X\,,$$

where u solves the Cauchy problem for the linear equation

$$u_{tt} - \sum_{i,j=1,\dots,n} a_{ij}(v,\nabla v)u_{ij} = f(v,\nabla v), \qquad (x,t)\in \mathbb{R}^n\times[0,T]\,,$$
$$u(x,0)=\varphi_0(x), \qquad u_t(x,0)=\varphi_1(x), \qquad x\in\mathbb{R}^n.$$

Step 1. First we prove that L maps X_M into itself provided that T is small enough. Indeed, according to the energy estimates, we have

$$E_s(t) \;\le\; C_0E_s(0) + C(\|v\|_X)\int_0^t \|f(v(x,\tau),\nabla v(x,\tau))\|_{H^{s-1}(\mathbb{R}^n)}d\tau\,.$$

According to the property of the superposition of the functions and of the Sobolev embedding, we have

$$\|f(v(x,\tau),\nabla v(x,\tau))\|_{H^{s-1}(\mathbb{R}^n)} \le C_s(\|v(x,\tau)\|_{H^s(\mathbb{R}^n)})\,.$$

Hence,

$$\begin{aligned}
E_s(t) &\le C_0E_s(0) + C(\|v\|_X)\int_0^t \|f(v(x,\tau),\nabla v(x,\tau))\|_{H^{s-1}(\mathbb{R}^n)}d\tau\\
&\le C_0E_s(0) + C(\|v\|_X)\int_0^t C_s(\|v(x,\tau)\|_{H^s(\mathbb{R}^n)})d\tau\\
&\le C_0E_s(0) + TC(\|v\|_X)C_s(\sup_{0\le\tau\le T}\|v(x,\tau)\|_{H^s(\mathbb{R}^n)}), \qquad t\in[0,T]\,.
\end{aligned}$$

Now we choose $M=2C_0E_s(0)$ and then T small enough, so that

$$C_0E_s(0)+TC(\|v\|_X)C_s(\|v\|_X)\le M$$

and we can conclude that

$$\|u\|_X \le M\,.$$

Step 2. Second, we prove that L is contraction on X_M. Indeed, for $v_1, v_2 \in X_M$, let $u_1, u_2 \in X_M$ be the corresponding solutions of the Cauchy problem. Then,

$$\begin{aligned}
&(u_1-u_2)_{tt} - \sum_{i,j=1,\dots,n} a_{ij}(v_1,\nabla v_1)(u_1-u_2)_{ij}\\
&= f(v_1,\nabla v_1)-f(v_2,\nabla v_2) + \sum_{i,j=1,\dots,n}\left(a_{ij}(v_1,\nabla v_1)-a_{ij}(v_2,\nabla v_2)\right)(u_2)_{ij}, \quad (x,t)\in\mathbb{R}^n\times[0,T],
\end{aligned}$$

while

$$(u_1-u_2)(x,0)=0, \qquad (u_1-u_2)_t(x,0)=0, \qquad x\in\mathbb{R}^n\,.$$

From the energy estimate we have

$$\begin{aligned}
& \rho(u_1,u_2) \\
\le\ & C(\|v\|_X)\int_0^t \Big\{ \|f(v_1,\nabla v_1)-f(v_2,\nabla v_2) \\
& + \sum_{i,j=1,\dots,n} (a_{ij}(v_1,\nabla v_1)-a_{ij}(v_2,\nabla v_2))\,(u_2)_{ij}\|_{H^{s-1}(\mathbb{R}^n)}\Big\} d\tau \\
\le\ & C(\|v\|_X)\int_0^t \|f(v_1,\nabla v_1)-f(v_2,\nabla v_2)\|_{H^{s-1}(\mathbb{R}^n)} \\
& + C(\|v\|_X)\int_0^t \|\sum_{i,j=1,\dots,n} (a_{ij}(v_1,\nabla v_1)-a_{ij}(v_2,\nabla v_2))\,(u_2)_{ij}\|_{H^{s-1}(\mathbb{R}^n)} d\tau\,.
\end{aligned}$$

To make the estimates more transparent we take case of

$$f(v,\nabla v)=|\nabla v|^2\,, \quad a_{ij}(v,\nabla v)=a_{ij}$$

with the constants a_{ij}. Then, since $s-1>n/2$, we derive by product property

$$\begin{aligned}
\|f(v_1,\nabla v_1)-f(v_2,\nabla v_2)\|_{H^{s-1}(\mathbb{R}^n)} &= \||\nabla v_1|^2-|\nabla v_2|^2\|_{H^{s-1}(\mathbb{R}^n)} \\
&= \|(\nabla v_1-\nabla v_2)\cdot(\nabla v_1+\nabla v_2)\|_{H^{s-1}(\mathbb{R}^n)} \\
&\le \|\nabla v_1-\nabla v_2\|_{H^{s-1}(\mathbb{R}^n)}\|\nabla v_1+\nabla v_2\|_{H^{s-1}(\mathbb{R}^n)} \\
&\le \|v_1-v_2\|_{H^s(\mathbb{R}^n)}\|v_1+v_2\|_{H^s(\mathbb{R}^n)} \\
&\le 2M\|v_1-v_2\|_{H^s(\mathbb{R}^n)} \\
&\le 2M\sup_{[0,T]}\|v_1-v_2\|_{H^s(\mathbb{R}^n)} \\
&\le 2M\rho(v_1,v_2)\,.
\end{aligned}$$

Hence,

$$\rho(u_1,u_2) \ \le\ TC(\|v\|_X)2M\rho(v_1,v_2)\,.$$

Again, for T sufficiently small we obtain

$$\rho(u_1,u_2) \ \le\ \frac{1}{2}\rho(v_1,v_2)\,.$$

By the contraction mapping principle, L has a unique fixed point that depends continuously on the data.
Exercise. Complete the proof by discussion of the case of the general $a_{ij}(v,\nabla v)$ and $f(v,\nabla v)$. (Hint: Provide the set X_M with the metric $\rho(u,v)$ defined by

$$\rho_1(u,v) := \sup_{0\le t\le T}\left(\|u-v\|_{H^1}+\|\partial_t u-\partial_t v\|_{L^2}\right)\,.$$

Prove that (X_M,ρ) is a complete metric space. In order to estimate

$$\sum_{i,j} \Big\| \Big(a_{ij}(v_1,\nabla v_1) - a_{ij}(v_2,\nabla v_2) \Big) (u_2)_{ij} \Big\|_{L^2((\mathbb{R}^n)}$$

use condition on s, namely, $s > \frac{n}{2} + 2 \Longrightarrow s-2 > \frac{n}{2}$. Then Sobolev embedding implies

$$\|\partial_{x_i}\partial_{x_j} u_2\|_{L^\infty(\mathbb{R}^n)} \le C\|\partial_{x_i}\partial_{x_j} u_2\|_{H^{s-2}(\mathbb{R}^n)} \le C\|u_2\|_{H^s(\mathbb{R}^n)} \le C\|u_2\|_X \le CM\,).$$

Theorem is proven. □

6 Counterexamples to the Global Existence

6.1 Nirenberg's Example. Method of Representation Formula

The following one is an example of an equation without global classical solution $u \in C^2(\mathbb{R}\times \mathbb{R}^3)$ to the Cauchy problem with large initial data in $\mathbb{R}^3$:

$$\begin{aligned} &u_{tt} - \Delta u = |\nabla u|^2 - |u_t|^2\,, \\ &u(0,x) = 0, \quad u_t(0,x) = \varphi_1(x)\,, \qquad x \in \mathbb{R}^3\,, \end{aligned}$$

where further conditions on φ_1 are given below. If $u = u(t,x)$ is a solution to this Cauchy problem, then the function $v = v(t,x)$ defined by

$$v(t,x) = e^{u(t,x)}$$

solves the following Cauchy problem for the wave equation

$$\begin{aligned} &v_{tt} - \Delta v = 0\,, \\ &v(0,x) = 1, \quad v_t(0,x) = \varphi_1(x)\,, \qquad x \in \mathbb{R}^3\,. \end{aligned}$$

The solution v is unique and there is an explicit representation for v:

$$v(t,x) = 1 + \frac{t}{4\pi}\int_{|y|=1} \varphi_1(x+ty)dS_y\,.$$

If $v > 0$, then one obtains

$$u(t,x) = \log\left(1 + \frac{t}{4\pi}\int_{|y|=1} \varphi_1(x+ty)dS_y\right)\,.$$

For every given positive t_0 and point $x_0 \in \mathbb{R}^3$ one can find a function $\varphi_1(x) \in C_0^\infty(\mathbb{R}^3)$ such that solution $u(t,x)$ develops singularity not later than at time t_0 at the point x_0. Indeed, φ_1 has only to satisfy the following relation:

$$\frac{t_0}{4\pi}\int_{|y|=1} \varphi_1(x_0+t_0 y)dS_y = -1\,.$$

On the other hand, it is easily seen from the next proposition that for sufficiently small data solution exists for all t.

Proposition 6.1 *If* $\varphi_1(z) = \mathcal{O}(|z|^{-1})$ *as* $|z| \to \infty$, $\|\nabla\varphi_1\|_{L^1(\mathbb{R}^3)} < 4\pi$, $\quad \|\varphi_1\|_{L^\infty(\mathbb{R}^3)} < 1$, *then* u *is defined globally, that is, for all* $x \in \mathbb{R}^3$ *and all* $t \in \mathbb{R}$ *.*

Proof. We note that for the "large time", $t > 1$, there is some decay of the varying part $v(t,x) - 1$ of the solution $v = v(t,x)$ to the wave equation. Namely the L^∞-norm of that part satisfies the following estimates:

$$\begin{aligned}
|v(t,x) - 1| &= \frac{t}{4\pi}\left|\int_{|y|=1}\int_t^\infty \frac{d}{d\tau}\varphi_1(x+\tau y)d\tau dS_y\right| \\
&= \frac{t}{4\pi}\left|\int_{|y|=1}\int_t^\infty y\cdot(\bigtriangledown_x\varphi_1)(x+\tau y)d\tau dS_y\right| \\
&= \frac{t}{4\pi}\left|\int_t^\infty d\tau\int_{|z|=\tau}\left(\frac{z}{\tau}\right)\cdot(\nabla\varphi_1)(x+z)\frac{dS_z}{\tau^2}\right| \\
&\le \frac{t}{4\pi}\int_t^\infty d\tau\int_{|z|=\tau}\frac{1}{\tau^2}|(\nabla\varphi_1)(x+z)|dS_z \\
&\le \frac{1}{4\pi t}\int_{|z|\ge t}|\nabla\varphi_1(x+z)|dz \\
&\le \frac{1}{4\pi t}\|\nabla\varphi_1\|_1 \le \frac{1}{t} < 1\,.
\end{aligned}$$

The same estimate holds for $t < -1$. For the small time t, $|t| \le 1$, that difference vanishes at $t = 0$, and we have

$$|v(t,x) - 1| = \left|\frac{t}{4\pi}\int_{|y|=1}\varphi_1(x+ty)ds\right| \le |t|\|\varphi_1\|_\infty < 1\,.$$

Therefore, $v(t,x)$ is positive for all $(t,x) \in \mathbb{R}\times\mathbb{R}^3$ and $u = u(t,x)$ is defined globally in $\mathbb{R}\times\mathbb{R}^3$. □

From now on through this lectures "*blow-up*" means the nonexistence of the global in time classical solution $u \in C^2([0,\infty)\times\mathbb{R}^n)$ to the Cauchy problem under consideration.

Exercise: Prove that for every given positive t_0 and point $x_0 \in \mathbb{R}^3$ one can find a function $\varphi_1(x) \in C_0^\infty(\mathbb{R}^3)$ such that the solution $u(t,x)$ of the Cauchy problem

$$u_{tt} - t^l \bigtriangleup u + (u_t)^2 - t^l\sum_{j=1}^n (u_{x_j})^2 = 0, \quad u(0,x) = 0, \quad u_t(0,x) = \varphi_1(x)\,, \qquad x \in \mathbb{R}^n\,,$$

develops a singularity not later than at time t_0 at the point x_0.

6.2 Parametric resonance breaks down the small data solution

We give an example of the influence of the behavior of a time-dependent coefficient $a = a(t)$, and in particular its oscillating behavior, on the global existence of solutions to nonlinear hyperbolic equation. More precisely, for arbitrary small initial data we will construct blowing up solution. To this end we consider in $\mathbb{R}\times\mathbb{R}^n$ the equation

$$u_{tt} - a^2(t,x)\bigtriangleup u + u_t^2 - a^2(t,x)|\bigtriangledown_x u|^2 = 0\,, \qquad (6.52)$$

where $u = u(t,x)$ is a real-valued unknown function. We restrict ourselves to the case of $a(t,x) = b(t)$:

$$u_{tt} - b^2(t)\bigtriangleup u + (u_t)^2 - b^2(t)\sum_{j=1}^{n}(u_{x_j})^2 = 0\,. \qquad (6.53)$$

The next theorem shows that if the function $b = b(t)$ differs from the constant, for instance, oscillates, the situation with the global existence for small data changes dramatically. It was first detected in [16]. In the next theorem $\|\varphi\|_{(s)}$ denotes the norm of the function $\varphi = \varphi(x)$ from the Sobolev space $H^s(\mathbb{R}^n)$.

Theorem 6.2 (K.Y [16], J. of Math. Anal. and Appl. (2001))
Let $b = b(t)$ be a defined on $\mathbb{R}$, a periodic, non-constant, smooth, and positive function. Then for every n, s, and for every positive δ there are data $u_0 \in C_0^\infty(\mathbb{R}^n)$ and $u_1 \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|u_0\|_{(s+1)} + \|u_1\|_{(s)} \le \delta \qquad (6.54)$$

but a solution $u \in C^2(\mathbb{R}_+\times\mathbb{R}^n)$ to the problem with data

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x)\,, \qquad x \in \mathbb{R}^n\,, \qquad (6.55)$$

does not exist.

Proof. We skip proof of the theorem. □

For a simplest example of the equation (6.53) one can take

$$b(t) = 1 + \varepsilon\sin(2\pi t), \qquad \varepsilon \in (-1/2, 1/2),\ \varepsilon \neq 0.$$

The oscillations are responsible for the blow-up of the solutions. Indeed, if we switch off them, that is, if we set $\varepsilon = 0$, then for all $n \ge 3$ the problem has small data global in time solution. (See, e.g. [8].) The proof of Theorem 6.2 is based on the well-known in Physics *parametric resonance* phenomena. The main feature of the parametric resonance is the exponentially increasing amplitudes of oscillatory system, whereas in ordinary resonance they increase with a power law.

6.3 Coefficient stabilizing to a periodic one. Parametric resonance dominates.

After studying the periodic case the next question is that for corresponding results for equations with a coefficient stabilizing to a periodic one. Equations with the coefficient being a product of periodic function and a function stabilizing to a constant, are considered in many papers and books (see, e.g., Ch.4 [4]). Following this let us restrict to the model equation

$$u_{tt} - \exp(2t^\alpha)b^2(t) \triangle u + (u_t)^2 - \exp(2t^\alpha)b^2(t)|\nabla u|^2 = 0\,, \tag{6.56}$$

$t \in [1,\infty)$, $x \in \mathbb{R}^n$, for $\alpha \in \mathbb{R}$, $\alpha < 0$. Here $b = b(t)$ is a real-valued, defined on $\mathbb{R}$, a periodic, non-constant, smooth, and positive function. We study the global solvability of the Cauchy problem with the data prescribed on $t = 1$:

$$u(1,x) = u_0(x), \quad u_t(1,x) = u_1(x)\,. \tag{6.57}$$

In Section 11 [18] we proved the global existence for small data for the equation (6.56) with the *fast oscillating coefficients*, that is, for the case $\alpha \in [1/2,\infty)$.

The next theorem shows that for equation with a coefficient stabilizing to a periodic one, the oscillations, which approach for the large time the pure periodic behavior, in general break the global existence. (See also [16]).

Theorem 6.3 (K.Y. [18] Oper. Theory Adv. Appl., **159**, 2005)
Let $\alpha \in (-\infty,-1)$ while $b = b(t)$ be a defined on $\mathbb{R}$, a periodic, non-constant, smooth, and positive function. Then for every n, s, and for every positive δ there are data $u_0 \in C_0^\infty(\mathbb{R}^n)$ and $u_1 \in C_0^\infty(\mathbb{R}^n)$ such that the inequality

$$\|u_0\|_{(s+1)} + \|u_0\|_{(s)} \le \delta \tag{6.58}$$

is fulfilled, but a solution $u \in C^2([1,\infty) \times \mathbb{R}^n)$ to the problem (6.56), (6.57) *does not exist.*

Open Problem: Consider the case of $\alpha \in [-1,0) \cup (0,1/2)$.

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6.4 Functional method. Second order differential inequalities

Second order differential inequalities with power decreasing kernel play a key role in proving blow-up of the solutions to semilinear equations. Kato's lemma [9] allows us to derive from the second order differential inequality the boundedness of the life-span of solution with the property $w_t \ge a > 0$.

Lemma 6.4 ([9] Kato, Comm. Pure Appl. Math.1980)
If $p > 1$, $b > 0$, there is no global solution to the differential inequality

$$\ddot{w} \ge bt^{-1-p}w^p, \qquad t \ge R > 0 \text{ large}$$

such that $w_t \ge a > 0$ and $w \ge at > 0$.

Proof. We skip proof of the lemma. □

The next lemma (Lemma 4[12]) is a version of Kato's lemma.

Lemma 6.5 ([12] Sideris, T.C. J. Diff. Equations (1984))
Suppose $F(t) \in C^2([a,b))$, *and for* $a \le t < b$,

$$F(t) \ge C_0(k_0+t)^r\,,$$
$$\ddot F(t) \ge C_1(k_0+t)^{-q}F(t)^p\,,$$

where C_0, C_1, *and* k_0 *are positive numbers. If* $p>1$, $r \ge 1$, *and* $(p-1)r > q-2$, *then* b *must be finite.*

Proof. We skip proof of the lemma. □

For the equation in de Sitter spacetime

$$u_{tt} - e^{-2t} \triangle u + M^2 u = 0,$$

the kernel of the corresponding ordinary differential inequality decreases exponentially:

$$\ddot w \ge b e^{-Mt} w^p, \qquad p>1,\ b>0,\ M>0, \quad t \text{ large.}$$

There is a global solution to the last differential inequality. Hence, to generalize Kato's lemma we have to look for proper supplementary conditions on the involving functions. It is done in the following lemma.

Lemma 6.6 (K.Y. [26] Discrete Contin. Dyn. Syst. Ser. S 2 (2009))
Suppose $F(t) \in C^2([a,b))$, *and*

$$F(t) \ge c_0 A(t), \quad \dot F(t) \ge 0, \quad \ddot F(t) \ge \gamma(t)A(t)^{-p}F(t)^p \quad \text{for all } t \in [a,b), \tag{6.59}$$

where $A, \gamma \in C^1([a,\infty))$ *are non-negative functions and* $p>1$, $c_0>0$. *Assume that*

$$\lim_{t\to\infty} A(t) = \infty\,, \tag{6.60}$$

and that

$$\frac{d}{dt}\left(\gamma(t)A(t)^{-p}\right) \le 0 \quad \text{for all} \quad t \in [a,b)\,. \tag{6.61}$$

If there exist $\varepsilon>0$ *and* $c>0$ *such that*

$$\gamma(t) \ge cA(t)(\ln A(t))^{2+\varepsilon} \quad \text{for all} \quad t \in [a,b), \tag{6.62}$$

then b *must be finite.*

Proof. We skip proof of the lemma. □

We note here that the equation

$$\ddot F(t) = e^{-dt}F(t)^p\,, \quad d>0,$$

has a global solution $F(t) = c_F e^{\frac{d}{p-1}t}$, where $c_F = (d/(p-1))^{2/(p-1)}$, while the corresponding $A(t) = c_A e^{at}$, $a>0$, and $\gamma(t) = c_\gamma e^{(pa-d)t}$. The condition (6.62) implies $a > d/(p-1)$. On the other hand, the first inequality of (6.59) holds only if $a \le d/(p-1)$.

6.5 Functional method. Nonexistence of global solution for the semilinear Tricomi-type equation

The next example illustrate the so-called Functional Method to prove blow-up phenomena.
Example. (K.Y. [18]) Assume that $u = u(x,t)$ is a smooth solution of

$$
\begin{aligned}
& u_{tt} - a^2(t)\Delta u = m(t)|u|^p \quad \text{on} \quad [0,T]\times\mathbb{R}^n\,, \quad 0 < T \le \infty\,,\\
& u(x,0) = \varphi_0(x)\,,\; u_t(x,0) = \varphi_1(x) \quad \text{on} \quad \mathbb{R}^n\,,
\end{aligned}
$$

where for the speed of propagation $a = a(t)$ and for the function $m = m(t)$ with some positive constants C and k the following conditions are fulfilled:

$$
\begin{aligned}
& 0 < a_0 \le a(t)\,, \quad a(t), m(t) \in C^\infty(\mathbb{R}_+)\,, \quad A(t) := \int_0^t \max_{\tau\le s} a(\tau)\,ds\,,\\
& m(t) \ge C(k+t)^{-c}(k+A(t))^{n(p-1)} \quad \text{for large } t \quad \text{with} \quad c < p+1\,.
\end{aligned}
$$

Suppose that

$$
\operatorname{supp}\varphi_0, \operatorname{supp}\varphi_1 \subset \{\, x\in\mathbb{R}^n\,;\, |x|\le R\,\}.
$$

By the domain of dependence property,

$$
\operatorname{supp} u(t,\cdot) \subset \{\, x\in\mathbb{R}^n\,;\, |x|\le R + A(t)\,\}\,.
$$

By integrating the equation with respect to spatial variables we obtain

$$
\int_{\mathbb{R}^n}\big(u_{tt}(t,x) - a^2(t)\Delta u(t,x)\big)\,dx = \int_{\mathbb{R}^n} m(t)|u(t,x)|^p\,dx\,.
$$

On the other hand the divergence theorem gives

$$
\int_{\mathbb{R}^n} a^2(t)\Delta u(t,x)\,dx = a^2(t)\int_{\mathbb{R}^n}\Delta u(t,x)\,dx = 0\,,
$$

while

$$
\frac{d^2}{dt^2}\int_{\mathbb{R}^n} u(t,x)\,dx = \int_{\mathbb{R}^n}\frac{\partial^2 u(t,x)}{\partial t^2}\,dx\,.
$$

Hence,

$$
\frac{d^2}{dt^2}\int_{\mathbb{R}^n} u(t,x)\,dx = \int_{\mathbb{R}^n} m(t)|u(t,x)|^p\,dx\,.
$$

Now we introduce a functional

$$
F(t) = \int_{\mathbb{R}^n} u(t,x)\,dx\,,
$$

then

$$
\ddot F(t) = \int_{\mathbb{R}^n} m(t)|u(t,x)|^p\,dx\,.
$$

Using the compact support of $u(t,\cdot)$ and Hölder's inequality we get with $1/p+1/q=1$, $p/q=p-1$, τ_n the volume of the unit ball in $\mathbb{R}^n$,

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} u(t,x)\,dx\right|^p &= \left|\int_{|x|\le R+A(t)} u(t,x)\,dx\right|^p \\
&\le \left(\int_{|x|\le R+A(t)} |u(t,x)|^p\,dx\right)\left(\int_{|x|\le R+A(t)} 1\,dx\right)^{p/q} \\
&= \left(\int_{\mathbb{R}^n} m(t)|u(t,x)|^p\,dx\right)\tau_n \frac{1}{m(t)}(R+A(t))^{np/q} \\
&= \ddot{F}(t)\tau_n \frac{1}{m(t)}(R+A(t))^{np/q}\,.
\end{aligned}$$

Thus

$$\ddot{F}(t) \ge \tau_n^{-1} m(t)(R+A(t))^{-n(p-1)}|F(t)|^p$$

for all t in the interval $[0,T]$. In particular, $\ddot{F}(t)\ge 0$, and so $F(t)\ge \dot{F}(0)t+F(0)$. Now

$$\dot{F}(0) = \int_{\mathbb{R}^n} u_t(0,x)\,dx = \int_{\mathbb{R}^n} \varphi_1(x)\,dx \equiv C_{\varphi_1}\,.$$

If $C_{\varphi_1}>0$, then

$$F(t) \ge (\text{pos. const.})t\,, \qquad t \quad \text{large}\,.$$

The Lemma 6.5 with $r=1$ and $c=q$ satisfying $(p-1)r>q-2$, shows that the function $F(t)$ cannot remain finite. Hence, $T<\infty$.

Example. (Tricomi-type linear part.) If we consider the Cauchy problem for the equation

$$\begin{aligned}
&u_{tt} - t^{2k}\Delta u = m(t)|u|^p \quad \text{on} \quad [0,T]\times\mathbb{R}^n\,, \quad 0<T\le\infty\,,\\
&u(x,0)=\varphi_0(x)\,,\ u_t(x,0)=\varphi_1(x) \quad \text{on} \quad \mathbb{R}^n\,,
\end{aligned}$$

that is with $a(t)=t^k$ and $m(t)=t^m$ for large t, then the assumptions on these functions imply

$$p < \frac{n(k+1)+1+m}{n(k+1)-1}\,.$$

Example. (Semilinear wave equation in Minkowski spacetime.) If we set $k=0$, $n=3$, and $m=0$ in the previous example, then the condition on p is

$$p<2\,.$$

It is known (see,e.g, [18]) that for the semilinear wave equation in Minkowski spacetime

$$u_{tt} - \Delta u = |u|^p \quad \text{on} \quad [0,T]\times\mathbb{R}^n\,, \quad 0<T\le\infty\,,$$

the last condition can be improved

$$1<p<1+\sqrt{2}\,.$$

For the general $n \geq 2$ denote by p_n the positive root of the equation

$$(n-1)p_n^2 - (n+1)p_n - 2 = 0\,.$$

Conjecture (W. Strauss [15]): *For the semilinear wave equation in Minkowski spacetime for $n \geq 2$ blow-up for all data if $p < p_n$ and global existence for all small data if $p > p_n$.*

Open Problem: **Prove Conjecture [19]:** *For the Cauchy problem*

$$\begin{aligned} &u_{tt} - t^{2k}\Delta u = |u|^p \quad \text{on,} \quad [0,T]\times\mathbb{R}^n\,, \quad 0 < T \leq \infty\,,\\ &u(x,0) = \varphi_0(x)\,,\ u_t(x,0) = \varphi_1(x) \quad \text{on} \quad \mathbb{R}^n\,, \end{aligned}$$

if $n = 3$, then for p, given by

$$p > \frac{3k+4}{3k+2}\,,$$

the small data solution exists globally, that is $T = \infty$.

7 Global Existence Theorem

7.1 Invariance of the wave operator under Lorentz group and homotheties.

The 4-dimensional spacetime (ct, x, y, z) in which we find ourselves is called a *Minkowski space*, and a point in such a space is termed an *event*. The equation

$$s^2 \equiv c^2t^2 - x^2 - y^2 - z^2 = 0$$

defines the *light cone*. Events that have $s^2 > 0$ are called *timelike*; events that have $s^2 < 0$ are called *spacelike*; events on the light cone ($s^2 = 0$) are called *lightlike*. Timelikes events can be causally connected to the origin with signals for which $v < c$, but spacelike events cannot be connected to the origin except by signals for which $v > c$.

The *special theory of relativity* is based on the two following postulates:
(1) the velocity of light in vacuum is constant in all inertial systems (an inertial system is a system of reference in which Newton's first law of motion holds),
(2) the laws of physics are invariant under transformations between inertial systems (*covariance*).

Thus *Lorentz transformation* of the variables x_0, x_1, x_2, x_3 ($x_0 = t$) is any linear homogeneous transformation of these variables

$$y_i = \sum_{j=0}^{3} a_{ij}x_j\,, \qquad i = 0,1,2,3\,,$$

with real coefficients a_{ij} which leaves invariant the quadratic form

$$x_0^2 - x_1^2 - x_2^2 - x_3^2\,.$$

That is

$$y_0^2 - y_1^2 - y_2^2 - y_3^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \,.$$

More general *inhomogeneous Lorentz transformations (Poincaré transformation)* are defined by

$$y_i = \sum_{j=0}^{3} a_{ij} x_j + b_i \,, \qquad i = 0, 1, 2, 3 \,.$$

The homogeneous Lorentz transformations form a group, called *homogeneous Lorentz group* (often just termed the Lorentz group). This group is a 6-parameter Lie group in four dimensions. The 3-dimensional orthogonal group of rotations $O(3)$ is a subgroup of the homogeneous Lorentz group.

We now write down a formula for a special class of Lorentz transformations which have the property of leaving invariant two of the last three (spatial) coordinates. Such transformation has the form

$$\begin{cases} y_0 = \alpha x_0 + \beta x_1 \,, \\ y_1 = \gamma x_0 + \delta x_1 \,, \\ y_2 = \qquad\qquad x_2 \,, \\ y_3 = \qquad\qquad\quad x_3 \,. \end{cases}$$

For such transformations the identity

$$y_0^2 - y_1^2 = x_0^2 - x_1^2$$

must hold. Substituting in this identity the expression for y_0 and y_1, we get

$$(\alpha x_0 + \beta x_1)^2 - (\gamma x_0 + \delta x_1)^2 = x_0^2 - x_1^2 \,.$$

Whence

$$\begin{cases} \alpha^2 - \gamma^2 = 1 \,, \\ \beta^2 - \delta^2 = -1 \,, \\ \alpha\beta - \gamma\delta = 0 \,. \end{cases}$$

In particular, these equations are satisfied if we put

$$\alpha = \delta = \cosh\psi \,,$$
$$\beta = \gamma = \sinh\psi \,,$$

where ψ is an arbitrary number. Then

$$\begin{cases} y_0 = (\cosh\psi) x_0 + (\sinh\psi) x_1 \,, \\ y_1 = (\sinh\psi) x_0 + (\cosh\psi) x_1 \,, \\ y_2 = \qquad\qquad\qquad\qquad x_2 \,, \\ y_3 = \qquad\qquad\qquad\qquad\quad x_3 \,. \end{cases}$$

If we denote $\tanh\psi = v/c$ ($|v/c| < 1$ for any ψ), then we obtain the usual formulas for the class of Lorentz transformations of this special case:

$$(*) \qquad \begin{cases} y_0 = \frac{1}{\sqrt{1-(v/c)^2}} x_0 + \frac{v/c}{\sqrt{1-(v/c)^2}} x_1 \,, \\ y_1 = \frac{v/c}{\sqrt{1-(v/c)^2}} x_0 + \frac{1}{\sqrt{1-(v/c)^2}} x_1 \,, \\ y_2 = \qquad\qquad\qquad\qquad\qquad x_2 \,, \\ y_3 = \qquad\qquad\qquad\qquad\qquad\quad x_3 \,. \end{cases}$$

Any Lorentz transformation is a combination of an orthogonal transformation of the variables x_1, x_2, x_3 which leaves x_0 fixed, a transformation of the form $(*)$, and a possible change of sign of one of the variables (*a reflection*).

Theorem 7.1 *Every non-singular linear transformation of the variables t, x_1, x_2, x_3 with real constant coefficients which does not change the form of the wave equation is a combination of a Lorentz transformation, a translation of the origin in the space (t, x_1, x_2, x_3), and a similarity transformation in that space.*

By the statement that some transformation "does not change the form of the wave equation" we mean that any function $u(t, x_1, x_2, x_3)$ which satisfies the equation

$$\frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} = 0$$

goes over, after transformation of the x_i into the y_i, into a function $u(y_0, y_1, y_2, y_3)$ satisfying the equation

$$\frac{\partial^2 u}{\partial y_0^2} - \frac{\partial^2 u}{\partial y_1^2} - \frac{\partial^2 u}{\partial y_2^2} - \frac{\partial^2 u}{\partial y_3^2} = 0\,.$$

Homotheties transformations are the transformations given for every positive number α by

$$\begin{cases} y_0 = \alpha x_0\,, \\ y_1 = \alpha x_1\,, \\ y_2 = \alpha x_2\,, \\ y_3 = \alpha x_3\,, \end{cases}$$

with a generator coinciding with the identity matrix.

7.2 Homogeneous fields

By $Z = Z(x, \partial_x)$ we shall denote any one of the vector fields

$$Z_{jk}(x, \partial_x) = g^{jj}(0) x_j \frac{\partial}{\partial x_k} - g^{kk}(0) x_k \frac{\partial}{\partial x_j}\,, \qquad j, k = 0, \dots, , n\,,$$

$$Z_0(x, \partial_x) = \sum_{j=0}^{n} x_j \frac{\partial}{\partial x_j} \quad \text{(radial vector field)}\,,$$

where $g^{jk}(0) = diag(1, -1, \dots, -1)$ are the coefficients of the d'Alembertian. We shall call these vector fields the *homogeneous fields* since their coefficients are homogeneous of degree one. We often omit (x, ∂_x) and write shortly Z_0, Z_{jk}.

Z_{jk} with $0 < j < k \le n$ are *spatial angular momentum* operators. Z_0 is a generator of the representation of the homotheties transformations.

Generators of the representation of the Poincaré (inhomogeneous Lorentz) group are given by

$$\partial_0, \partial_1, \ldots, \partial_n, \quad Z_{jk} = g^{jj}(0) x_j \partial/\partial x_k - g^{kk}(0) x_k \partial/\partial x_j \,, \qquad j,k = 0, \ldots, , n \,.$$

For example Z_{12} is the infinitesimal generator of the representation of the group of rotations $\{T_{12}(h)\}$ given by

$$\begin{cases} y_0 = x_0 \,, \\ y_1 = (\cos h) x_1 + (\sin h) x_2 \,, \\ y_2 = -(\sin h) x_1 + (\cos h) x_2 \,, \\ y_3 = \qquad\qquad\qquad\qquad x_3 \,. \end{cases}$$

We denote by Z^I any product of $|I|$ such vector fields.

Exercise: Check the following identities

$$\begin{aligned}
&[\Box, Z_{jk}] = \Box Z_{jk} - Z_{jk} \Box = 0 \,, \quad j,k = 0, \ldots, , n \,, \\
&[\Box, Z_0] = \Box Z_0 - Z_0 \Box = 2\Box \,, \\
&[Z_0, Z_{jk}] = 0 \,, \\
&[Z_0, \partial_j] = -\partial_j \,, \quad j = 0, 1, \ldots, n \,, \\
&[Z_{ij}, Z_{kl}] = -g^{ik}(0) Z_{jl} - g^{jl}(0) Z_{ik} + g^{il}(0) Z_{jk} + g^{jk}(0) Z_{il} \,, \quad i,j,k,l = 0, \ldots, n \,, \\
&[Z_{ij}, \partial_k] = -g^{ik}(0) \partial_j + g^{jk}(0) \partial_i \,, \quad i,j,k = 0, \ldots, n \,.
\end{aligned}$$

In what follows, we shall denote

$$\partial_0, \partial_1, \ldots, \partial_n, \quad Z_0, \quad Z_{01}, \ldots, Z_{n-1,n}$$

respectively by Γ_i, $i = 0, \ldots, m(n)$.

Exercise: Find number $m(n)$.

At times we shall suppress the subscript. Also, if $(\alpha_0, \ldots, \alpha_m)$, $m = m(n)$, is a multi-index, we shall write

$$\Gamma^\alpha = \Gamma_0^{\alpha_0} \Gamma_1^{\alpha_1} \ldots \Gamma_m^{\alpha_m} \,, \qquad m = m(n) \,.$$

Exercise: Prove that: (i) for homogeneous fields (h.v.f.) we have the commutativity relations

$$[\Gamma_i, \Gamma_j] = \sum_{\Gamma_k : h.f.} c_{ijk} \Gamma_k \,, \qquad \Gamma_i, \Gamma_j : \text{h.v.f.}$$

for certain fixed constants, where the sum just involves homogeneous vector fields; (ii) the commutator of two homogeneous vector fields is a linear combination of homogeneous vector fields; (iii) commutator of ∂_j with a homogeneous vector field is a translation invariant vector field:

$$[\Gamma_k, \partial_j] = \sum_{i=0}^{n} a_{ijk} \partial_i \,, \qquad \Gamma_k : \text{h.v.f.} \tag{7.63}$$

since the above calculated commutators:

$$(**) \quad \begin{cases} [Z_0, \partial_k] = -\partial_k \,, \quad k = 0, 1, \ldots, n \,, \\ [\partial_k, Z_{0j}] = \delta_{0k} \partial_j + \delta_{jk} \partial_0 \,, \\ [\partial_k, Z_{ij}] = \delta_{jk} \partial_i - \delta_{ik} \partial_j \,, \quad 0 < i < j \leq n \,. \end{cases}$$

If we introduce the three families of first-order operators

$$\Omega := \Big(Z_{ij}\Big)_{1\le i<j\le n}\,,$$
$$\overline{\Omega} := \Big(Z_{ij}\Big)_{0\le i<j\le n}\,,$$
$$\overline{\gamma} := \Big(Z_0, \overline{\Omega}, \partial_0, \dots, \partial_n\Big)\,,$$

then commutator relations imply that the $\mathbb{R}$-linear span of each of the families is a Lie-algebra with bracket $[\cdot,\cdot]$. (Check that statement.)

Proposition 7.2 *Let $r = |x|$ and $\partial_r = r^{-1}\sum_{i=1}^n x_i\partial_i$. Then in $\mathbb{R}^{1+n}_+ \setminus 0$ we can write*

$$(t-r)\partial_r = a_0(t,x)Z_0 + \sum_{i=1}^n a_i(t,x)Z_{0i}\,, \tag{7.64}$$

where the coefficients

$$a_0(t,x) = -\frac{|x|}{|x|+t}\,, \qquad a_i(t,x) = \frac{t}{|x|+t}\,\frac{x_i}{|x|}\,,$$

are smooth, homogeneous of degree zero and satisfy bounds of the form

$$|\partial^\alpha a_j(t,x)| \le C_\alpha (t+|x|)^{-|\alpha|} \quad \text{for all} \quad |x| > \delta t\,,$$

and all α with $\delta > 0$ fixed. Also,

$$(t-r)^2 \sum_{i=0}^n |\partial_i u_i(t,x)|^2 \le 2\left(|Z_0 u(t,x)|^2 + \sum_{0\le j<k\le n} |(Z_{jk}u)(t,x)|^2\right). \tag{7.65}$$

Proof. We skip proof of the lemma. $\square$

There are some useful formulas:

$$(t-r)\partial_r = \frac{1}{r+t}(t^2-r^2)\partial_r = \frac{1}{r+t}\left(t\sum_{i=1}^n \frac{x_i}{|x|}Z_{0i} - rZ_0\right)\,, \tag{7.66}$$

$$(t-r)\partial_t = \frac{1}{r+t}\left(tZ_0 - \sum_{i=1}^n x_i Z_{0i}\right)\,, \tag{7.67}$$

$$(t^2+|x|^2)\sum_{j=2}^n |\partial_i u(t,x)|^2 = \sum_{0<j<k} |(Z_{jk}u)(t,x)|^2 + \sum_{k=2}^n |(Z_{0k}u)(t,x)|^2\,, \tag{7.68}$$

$$
\begin{aligned}
&\text{if} \quad \Box u = 0 \quad \text{then} \quad \Box Z^I u = 0 \quad \text{for every} \quad I\,,\\
&Z_{jk}(|x|-t) = 0 \quad j,k \neq 0\,,\\
&Z_{0k}(|x|-t) = (|x|-t)x_k/|x|\,, \quad k \neq 0\,,\\
&Z_0(|x|-t) = |x|-t\,.
\end{aligned}
$$

Lemma 7.3 *The vector fields Z_{jk}, Z_0 form a basis for all vector fields when $t^2 \neq |x|^2$, that is outside the light cone $\Lambda = \{(t,x) \in \mathbb{R}^{1+n}\,,\ |t| = |x|\}$:*

$$
\partial_{x_j} = \frac{g^{jj}x_j}{\sum_{k=0}^n g^{kk}x_k^2} Z_0 - \sum_{l=0}^n \frac{x_l}{\sum_{k=0}^n g^{kk}x_k^2} Z_{jl}\,, \qquad j = 0,\ldots,n\,,
$$

$$
\partial_t = \frac{t}{t^2-r^2}Z_0 - \sum_{k=1}^n \frac{x_k}{t^2-r^2}Z_{0k}\,, \quad \partial_{x_j} = -\frac{x_j}{t^2-r^2}Z_0 - \sum_{k=0}^n \frac{x_k}{t^2-r^2}Z_{jk},\ j = 0,\ldots,n\,.
$$

Thus,

$$
\partial_j = \sum_\nu a_{j\nu}(t,x) Z_\nu\,, \qquad j = 0,\ldots,n\,,
$$

where Z_ν is any labeling of the vector fields Z_{jk}, Z_0, and $a_{j\nu}$ is in $C^\infty(\mathbb{R}^{1+n} \setminus \Lambda)$ and homogeneous of degree -1.

Proof. We skip proof of the lemma. □

Lemma 7.4 *Span of the homogeneous vector fields at the point $(t,x) \in \mathbb{R}^{1+n} \setminus 0$:*
a) If $t^2 \neq |x|^2$, that is if (t,x) is not on the light cone, then homogeneous vector fields span the full tangent space above (t,x);
b) If $t^2 = |x|^2$, then they only span the tangent space to the light cone;
c) If $t^2 = |x|^2$, then the missing normal component vanishes only to first order.

Proof. We skip proof of the lemma. □

7.3 Variant of Sobolev's theorem

The proof of the estimates we shall require will use this proposition as well as the following variant of Sobolev's theorem for $\mathbb{R}^n$.

We shall use the energy integral method to estimate $\|Z^I u(t,\cdot)\|$ and deduce maximum norm estimates from the next proposition.

Proposition 7.5 ([8]) *There is a constant C such that*

$$(1+|t|+|x|)^{n-1}(1+||t|-|x||)|u(t,x)|^2 \le C \sum_{|I|\le(n+2)/2} \|Z^I u(t,\cdot)\|^2_{\mathbb{R}^n_x} \tag{7.69}$$

if $u \in \mathcal{S}$ in $(t-1,t+1)\times\mathbb{R}^n$, say. Here by Z it is denoted any one of the vector fields Z_0, Z_{jk} or ∂_0, $\partial_1,\ldots,$ ∂_n.

Proof. We skip proof of the proposition. □

7.4 L^∞-weighted estimate for the solution to the wave equation

Proposition 7.6 ([8]) *If u is solution of the homogeneous unperturbed wave equation with Cauchy data $\varphi_0,\varphi_1 \in C_0^\infty(\mathbb{R}^n)$, then*

$$\sup_{\mathbb{R}^{n+1}} (1+|t|)^{\frac{n-1}{2}}(1+||t|-|x||)^{-\frac{1}{2}}|u(t,x)| \le C_{\varphi_0,\varphi_1} < \infty\,,$$

where constant C_{φ_0,φ_1} depends also on the diameters of the supports of data.

From Section 6.2 [8] we know that these estimates have the right order of magnitude near the boundary of the light cone, that is, when $||t|-|x||$ is bounded. However, the best bounds are

$$\sup\,(1+|t|)^{\frac{n-1}{2}}(1+||t|-|x||)^{\frac{n-1}{2}}(|u(t,x)|+|u'(t,x)|) < \infty\,,$$

and they cannot be obtained from L^2 estimates of $Z^I u$.

7.5 Global existence theorem

In this section we shall study the Cauchy problem in $\mathbb{R}^{1+n}$

$$\sum_{j,k=0}^{n} g^{jk}(u')\partial_j\partial_k u = f(u')\,, \tag{7.70}$$

$$u(0,x) = \varepsilon\varphi_0(x)\,, \quad \partial_0 u(0,x) = \varepsilon\varphi_1(x)\,. \tag{7.71}$$

We assume that $u \equiv 0$ is a solution of (7.70) and that the linearization at this solution is the wave operator, that is,

$$\textit{we assume that} \quad \sum_{j,k=0}^{n} g^{jk}(0)\partial_j\partial_k = \Box \quad \textit{and that } f \textit{ vanishes of second order at } 0.$$

Theorem 7.7 ([11] S. Klainerman, Comm. Pure Appl. Math. (1980))
The Cauchy problem (7.70), (7.71) with $\varphi_j \in C_0^\infty(\mathbb{R}^n)$, $j=0,1$, has a C^∞ solution for $t \ge 0$ if $n \ge 4$ and ε is sufficiently small.

Proof. We skip proof of the theorem. □

The graphics in this lecture notes were made by Mathematica 7.0.0.

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