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Difference sequence spaces in n-normed spaces defined by Musielak-Orlicz function

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Abstract

In the present paper we introduced generalized difference sequence spaces combining lacunary sequences and Musielak-Orlicz function $\mathcal{M} = (M_k)$ over n-normed spaces and examine some properties of the resulting sequence spaces.

Key Words: Orlicz function, Musielak-Orlicz function, n-normed space, Paranorm space, Difference sequence, Lacunary sequence.
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1 Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler[1] in the mid of 1960's, while that of n-normed spaces one can see in Misiak[12]. Since then, many others have studied this concept and obtained various results, see Gunawan ([2],[3]) and Gunawan and Mashadi [4]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$ of dimension d, where $d \geq n \geq 2$ and $\mathbb{K}$ is the field of real or complex numbers. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$; and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$;

is called a n-norm on X and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_n||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_n|| = 0 \text{ for every } z_1, \cdots, z_n \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,l \to \infty} ||x_k - x_l, z_1, \cdots, z_n|| = 0 \text{ for every } z_1, \cdots, z_n \in X.$$

If every Cauchy sequence in X is convergent then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm if it satisfies the following :

1. $p(x) \geq 0$ for all $x \in X$;
2. $p(-x) = p(x)$ for all $x \in X$;
3. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$ and
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [16], Theorem 10.4.2, pp. 183).

Let ℓ_∞, c and c_0 denotes the sequence spaces of bounded, convergent and null sequences respectively. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent if all Banach limits of (x_k) coincide. In [6] it was shown that

$$\hat{c} = \Big\{x = (x_k) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s\Big\}.$$

In ([7],[8]) Maddox defined strongly almost convergent sequences. Recall that a sequence $x = (x_k)$ is strongly almost convergent if there is a number L such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L| = 0, \text{ uniformly in } s.$$

By a lacunary sequence $\theta = (i_r)$, $r = 0, 1, 2, \cdots$, where $i_0 = 0$, we shall mean an increasing sequence of non-negative integers $g_r = (i_r - i_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio i_r/i_{r-1} will be denoted by q_r. The space N_θ of lacunary strongly convergent sequences was defined by Freedman [15] as follows:

$$N_\theta = \Big\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} |x_k - L| = 0 \text{ for some } L \Big\}.$$

In [5] Kizmaz defined the sequence spaces

$$Z(\Delta) = \Big\{ x = (x_k) : (\Delta x_k) \in Z \Big\} \text{ for } Z = \ell_\infty, c \text{ and } c_0,$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. Et and Colak [14] generalized the difference sequence spaces to the sequence spaces

$$Z(\Delta^m) = \Big\{ x = (x_k) : (\Delta^m x_k) \in Z \Big\} \text{ for } Z = \ell_\infty, c \text{ and } c_0,$$

where $m \in N$, $\Delta^0_x = (x_k)$, $\Delta x = (x_k - x_{k+1})$,

$$\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}).$$

The generalized difference sequence has the following binomial representation

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+v}.$$

An orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{ x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \le 1 \Big\}.$$

It is shown in [10] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \ge 1)$. The Δ_2-condition is equivalent to $M(Lx) \le kLM(x)$ for all values of $x \ge 0$, and for $L > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see ([11],[13]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\},\ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for all } \ c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty}(M_k)(x_k), \quad x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let M be an Orlicz function and $p = (p_k)$ be any sequence of strictly positive real numbers. Gungor and Et [9] defined the following sequence spaces

$$[c, M, p](\Delta^m) = \Big\{x = (x_k) : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\Big[M\Big(\frac{|\Delta^m x_{k+s} - L|}{\rho}\Big)\Big]^{p_k} = 0,$$
$$\text{uniformly in } \ s, \ \text{ for some } \ \rho > 0 \ \text{ and } \ L > 0\Big\},$$

$$[c, M, p]_0(\Delta^m) = \Big\{x = (x_k) : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\Big[M\Big(\frac{|\Delta^m x_{k+s}|}{\rho}\Big)\Big]^{p_k} = 0,$$
$$\text{uniformly in } \ s, \ \text{ for some } \ \rho > 0 \ \Big\},$$

$$[c, M, p]_\infty(\Delta^m) = \Big\{x = (x_k) : \sup_{n,s} \frac{1}{n}\sum_{k=1}^{n}\Big[M\Big(\frac{|\Delta^m x_{k+s}|}{\rho}\Big)\Big]^{p_k} < \infty \ \text{ for some } \ \rho > 0\Big\}.$$

Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space and $w(n-X)$ denotes the space of X-valued sequences. Let $p = (p_k)$ be any bounded sequence of positive real numbers and $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. In this paper, we define the following sequence spaces

$$[\ c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\ ]^\theta(\Delta^m)$$
$$= \Big\{x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0,$$
$$\text{uniformly in } \ s, \ \ z_1, \cdots, z_{n-1} \in X \ \text{ for some } \ \rho > 0 \ \text{ and } \ L > 0\Big\},$$

$$[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^{\theta}(\Delta^m) = \Big\{ x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0, \text{ uniformly in } s, \ z_1, \cdots, z_{n-1} \in X \text{ for some } \rho > 0 \Big\},$$

$$[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^{\theta}(\Delta^m) = \Big\{ x = (x_k) \in w(n-X) : \sup_{r,s} \frac{1}{g_r} \sum_{k=1}^{n} \Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \infty, z_1, \cdots, z_{n-1} \in X \text{ for some } \rho > 0 \Big\}.$$

When, $\mathcal{M}(x) = x$, we get

$$[\, c, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta^m) = \Big\{ x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big(||\frac{\Delta^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} = 0, \text{ uniformly in } s, \ z_1, \cdots, z_{n-1} \in X \text{ for some } \rho > 0 \text{ and } L > 0 \Big\},$$

$$[\, c, p, ||\cdot, \cdots, \cdot||\,]_0^{\theta}(\Delta^m) = \Big\{ x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big(||\frac{\Delta^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} = 0, \text{uniformly in } s, z_1, \cdots, z_{n-1} \in X \text{ for some } \rho > 0 \Big\},$$

$$[\, c, p, ||\cdot, \cdots, \cdot||\,]_\infty^{\theta}(\Delta^m) = \Big\{ x = (x_k) \in w(n-X) : \sup_{r,s} \frac{1}{g_r} \sum_{k=1}^{n} \Big(||\frac{\Delta^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} < \infty, z_1, \cdots, z_{n-1} \in X \text{ for some } \rho > 0 \Big\}.$$

If we take $p_k = 1$ for all k, then we get

$$[\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta^m) = \Big\{ x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Delta^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0, \text{ uniformly in } s, \ z_1, \cdots, z_{n-1} \in X \text{ for some } \rho > 0 \text{ and } L > 0 \Big\},$$

$$[\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]_0^{\theta}(\Delta^m) = \Big\{ x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0, \text{uniformly in } s, \ z_1, \cdots, z_{n-1} \in X \text{ for some } \rho > 0 \Big\},$$

$$[\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]^{\theta}_{\infty}(\Delta^m) = \Big\{ x = (x_k) \in w(n-X) : \sup_{r,s} \frac{1}{g_r} \sum_{k=1}^{n} \Big[M_k \Big(|| \frac{\Delta^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}|| \Big) \Big] < \infty, z_1, \cdots, z_{n-1} \in X \text{ for some } \rho > 0 \Big\}.$$

The following inequality will be used throughout the paper

$$|x_k + y_k|^{p_k} \leq K(|x_k|^{p_k} + |y_k|^{p_k}), \tag{1}$$

where x_k's and y_k's are complex numbers, $K = \max(1, 2^{H-1})$ and $H = \sup_k p_k < \infty$.

2 Some properties of difference sequence spaces

In this section we prove some results on difference sequence spaces defined in the present paper.

Theorem 1. *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of strictly real numbers. Then $[\, c, \mathcal{M}, p, ||., \cdots, \cdot||\,]^{\theta}(\Delta^m)$, $[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}_0(\Delta^m)$ and $[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}_{\infty}(\Delta^m)$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Proof. Let $x = (x_k), y = (y_k) \in [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}_0(\Delta^m)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{r \longrightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k \Big(|| \frac{\Delta^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}|| \Big) \Big]^{p_k} = 0, \text{ uniformly in } s,$$

and

$$\lim_{r \longrightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k \Big(|| \frac{\Delta^m x_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1}|| \Big) \Big]^{p_k} = 0, \text{ uniformly in } s.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M_k's are non-decreasing convex functions, by using inequality (1), we have

$$\begin{aligned}
&\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m(\alpha x_{k+s}+\beta y_{k+s})}{\rho_3},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad\le\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\alpha\Delta^m(x_{k+s})}{\rho_3},z_1,\cdots,z_{n-1}||+||\frac{\beta\Delta^m(y_{k+s})}{\rho_3},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad\le K\frac{1}{g_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\Big[M_k\Big(||\frac{\Delta^m(x_{k+s})}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad\quad+K\frac{1}{g_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\Big[M_k\Big(||\frac{\Delta^m(y_{k+s})}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad\le K\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m(x_{k+s})}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad\quad+K\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m(y_{k+s})}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad\longrightarrow 0 \ \text{ as } \ r\longrightarrow\infty, \ \text{ uniformly in } \ s.
\end{aligned}$$

Thus, we have $\alpha x+\beta y\in[\,c,\mathcal{M},p,||\cdot,\cdots,\cdot||\,]_0^\theta(\Delta^m)$. Hence $[\,c,\mathcal{M},p,||\cdot,\cdots,\cdot||\,]_0^\theta(\Delta^m)$ is a linear space.

Similarly, we can prove that $[\,c,\mathcal{M},p,||\cdot,\cdots,\cdot||\,]^\theta(\Delta^m)$ and $[\,c,\mathcal{M},p,||\cdot,\cdots,\cdot||\,]_\infty^\theta(\Delta^m)$ are linear spaces. □

Theorem 2. *For any Musielak-Orlicz function $\mathcal{M}=(M_k)$ and a bounded sequence $p=(p_k)$ of strictly positive real numbers, $[\,c,\mathcal{M},p,||\cdot,\cdots,\cdot||\,]_0^\theta(\Delta^m)$ is a topological linear space paranormed by*

$$g(x)=\inf\Big\{\rho^{\frac{p_r}{H}}:\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k(||\frac{\Delta^m x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||)\Big]^{p_k}\Big)^{\frac{1}{H}}\le 1,r,s\in\mathbb{N}\Big\},$$

where $H=\max(1,\sup_k p_k)<\infty$.

Proof. Clearly $g(x)\ge 0$ for $x=(x_k)\in[\,c,\mathcal{M},||\cdot,\cdots,\cdot||\,]_0^\theta(\Delta^m)$. Since $M_k(0)=0$, we get $g(0)=0$. Again, if $g(x)=0$, then

$$\inf\Big\{\rho^{\frac{p_r}{H}}:\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\le 1,r,s\in\mathbb{N}\Big\}=0.$$

This implies that for a given $\epsilon>0$ there exists some $\rho_\epsilon(0<\rho_\epsilon<\epsilon)$ such that

$$\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\rho_\epsilon},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\le 1.$$

Thus

$$\begin{aligned}
\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\epsilon},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}&\le\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\rho_\epsilon},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\\
&\le 1,
\end{aligned}$$

for each r and s. Suppose that $x_k \neq 0$ for each $k \in N$. This implies that $\Delta^m x_{k+s} \neq 0$, for each $k, s \in N$. Let $\epsilon \longrightarrow 0$, then $||\frac{\Delta^m x_{k+s}}{\epsilon}, z_1, \cdots, z_{n-1}|| \longrightarrow \infty$. It follows that

$$\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \longrightarrow \infty$$

which is a contradiction. Therefore, $\Delta^m x_{k+s} = 0$ for each k and s and thus $x_k = 0$ for each $k \in N$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

and

$$\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

for each r and s. Let $\rho = \rho_1 + \rho_2$. Then, by Minkowski's inequality, we have

$$\begin{aligned}
&\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m (x_{k+s}+y_{k+s})}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\quad\leq \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m (x_{k+s})+\Delta^m (y_{k+s})}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\quad\leq \Big(\sum_{k\in I_r}\Big[\frac{\rho_1}{\rho_1+\rho_2}M_k\Big(||\frac{\Delta^m (x_{k+s})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big) \\
&\qquad + \frac{\rho_2}{\rho_1+\rho_2}M_k\Big(||\frac{\Delta^m (y_{k+s})}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p^k}\Big)^{\frac{1}{H}} \\
&\quad\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m (x_{k+s})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\qquad + \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m (y_{k+s})}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\quad\leq 1
\end{aligned}$$

Since ρ , ρ_1 and ρ_2 are non-negative, so we have

$$\begin{aligned}
g(x+y) &= \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m (x_{k+s}+y_{k+s})}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, s \in \mathbb{N}\Big\}, \\
&\leq \inf\Big\{\rho_1^{\frac{p_r}{H}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m (x_{k+s})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, s \in \mathbb{N}\Big\} \\
&+ \inf\Big\{\rho_2^{\frac{p_r}{H}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m (y_{k+s})}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, s \in \mathbb{N}\Big\}.
\end{aligned}$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m \lambda x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1, r, s \in \mathbb{N}\Big\}.$$

Then

$$g(\lambda x) = \inf\Big\{(|\lambda| t)^{\frac{p_r}{H}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1, r, s \in \mathbb{N}\Big\},$$

where $t = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_r} \le \max(1, |\lambda|^{\sup p_r})$, we have

$$g(\lambda x) \le \max(1, |\lambda|^{\sup p_r}) \inf\Big\{t^{\frac{p_r}{H}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1, r, s \in \mathbb{N}\Big\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality. □

Theorem 3. *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function. If $\sup_k [M_k(x)]^{p_k} < \infty$ for all fixed $x > 0$, then $[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta^m) \subset [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta^m)$.*

Proof. Let $x = (x_k) \in [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta_m)$. There exists positive number ρ_1 such that

$$\lim_{r\to\infty} \frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0, \text{ uniformly in } s.$$

Define $\rho = 2\rho_1$. Since M_k's are non-decreasing and convex, by using inequality(1), we have

$$\begin{aligned}
&\sup_{r,s} \frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&= \sup_{r,s} \frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s} - L + L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\le K \sup_{r,s} \frac{1}{g_r}\sum_{k\in I_r}\Big[\frac{1}{2^{p_k}} M_k\Big(||\frac{\Delta^m x_{k+s} - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&+ K \sup_{r,s} \frac{1}{g_r}\sum_{k\in I_r}\Big[\frac{1}{2^{p_k}} M_k\Big(||\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\le K \sup_{r,s} \frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s} - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&+ K \sup_{r,s} \frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&< \infty.
\end{aligned}$$

Hence $x = (x_k) \in [\, c, M_k, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta$. □

Theorem 4. *Let $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$ and $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M'_k)$ be Musielak-Orlicz functions satisfying Δ_2-condition, then we have*

$(i)[\ c, \mathcal{M}', p, ||\cdot, \cdots, \cdot||\ ]_0^{\theta}(\Delta^m) \subset [\ c, \mathcal{M} \circ \mathcal{M}', p, ||\cdot, \cdots, \cdot||\ ]_0^{\theta}(\Delta^m)$,

$(ii)[\ c, \mathcal{M}', p, ||\cdot, \cdots, \cdot||\ ]^{\theta}(\Delta^m) \subset [\ c, \mathcal{M} \circ \mathcal{M}', p, ||\cdot, \cdots, \cdot||\ ]^{\theta}(\Delta^m)$,

$(iii)[\ c, \mathcal{M}', p, ||\cdot, \cdots, \cdot||\ ]_{\infty}^{\theta}(\Delta^m) \subset [\ c, \mathcal{M} \circ \mathcal{M}', p, ||\cdot, \cdots, \cdot||\ ]_{\infty}^{\theta}(\Delta^m)$.

Proof. Let $x = (x_k) \in [c, \mathcal{M}', p, ||\cdot, \cdots, \cdot||]_0^{\theta}(\Delta_m)$. Then we have

$$\lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M'_k \Big(|| \frac{\Delta^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1} || \Big) \Big]^{p_k} = 0, \text{ uniformly in } s \text{ for some } L.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \leq t \leq \delta$. Let

$$y_{k,s} = M'_k \Big(|| \frac{\Delta^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1} || \Big) \text{ for all } k, s \in \mathbb{N}.$$

We can write

$$\frac{1}{g_r} \sum_{k\in I_r} [M_k(y_{k,s})]^{p_k} = \frac{1}{g_r} \sum_{k\in I_r\ \&\ y_{k,s\leq\delta}} [M_k(y_{k,s})]^{p_k} + \frac{1}{g_r} \sum_{k\in I_r\ \&\ y_{k,s>\delta}} [M_k(y_{k,s})]^{p_k}.$$

So,we have

$$\begin{aligned} \frac{1}{g_r} \sum_{k\in I_r\ \&\ y_{k,s}\leq\delta} [M_k(y_{k,s})]^{p_k} &\leq [M_k(1)]^H \frac{1}{g_r} \sum_{k\in I_r\ \&\ y_{k,s}\leq\delta} [M_k(y_{k,s})]^{p_k} \\ &\leq [M_k(2)]^H \frac{1}{g_r} \sum_{k\in I_r\ \&\ y_{k,s}\leq\delta} [M_k(y_{k,s})]^{p_k} \end{aligned} \tag{2}$$

For $y_{k,s} > \delta$

$$y_{k,s} < \frac{y_{k,s}}{\delta} < 1 + \frac{y_{k,s}}{\delta}.$$

Since M_k's are non-decreasing and convex, it follows that

$$M_k(y_{k,s}) < M_k\Big(1 + \frac{y_{k,s}}{\delta}\Big) < \frac{1}{2} M_k(2) + \frac{1}{2} M_k\Big(\frac{2y_{k,s}}{\delta}\Big).$$

Since $\mathcal{M} = (M_k)$ satisfies Δ_2-condition, we can write

$$\begin{aligned} M_k(y_{k,s}) &< \frac{1}{2} T \frac{y_{k,s}}{\delta} M_k(2) + \frac{1}{2} T \frac{y_{k,s}}{\delta} M_k(2) + \frac{1}{2} T \frac{y_{k,s}}{\delta} M_k(2) \\ &= T \frac{y_{k,s}}{\delta} M_k(2). \end{aligned}$$

Hence,

$$\frac{1}{g_r} \sum_{k\in I_r\ \&\ y_{k,s}>\delta} [M_k(y_{k,s})]^{p_k} \leq \max\Big(1, \Big(\frac{TM_k(2)}{\delta}\Big)^H\Big) \frac{1}{g_r} \sum_{k\in I_r\ \&\ y_{k,s}>\delta} [(y_{k,s})]^{p_k} \tag{3}$$

from equations (2) and (3), we have

$$x = (x_k) \in [\, c, \mathcal{M} \circ \mathcal{M}', p, ||\cdot, \cdots, \cdot||]_0^\theta(\Delta^m).$$

This completes the proof of (i).
Similarly, we can prove that

$$[\, c, \mathcal{M}', p, ||\cdot, \cdots, \cdot||]_0^\theta(\Delta^m) \subset [\, c, \mathcal{M} \circ \mathcal{M}', ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta^m)$$

and

$$[\, c, \mathcal{M}', p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta^m) \subset [\, c, \mathcal{M} \circ \mathcal{M}', p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta^m).$$

□

Corollary 1. *Let* $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$ *and* $\mathcal{M} = (M_k)$ *be Musielak-Orlicz function satisfying* Δ_2*-condition, then we have*

$$[\, c, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta^m) \subset [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta^m)$$

and

$$[\, c, \mathcal{M}', p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta \subset [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta^m).$$

Proof. Taking $\mathcal{M}'(x) = x$ in the above theorem, we get the required result. □

Theorem 5. *Let* $\mathcal{M} = (M_k)$ *be the Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $[\, c, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta^m) \subset [c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta^m)$,
(ii) $[\, c, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta^m) \subset [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta^m)$,
(iii) $\sup_r \frac{1}{g_r} \sum_{k \in I_r} [M_k(\frac{t}{\rho})]^{p_k} < \infty \ \ (t, \rho > 0)$.

Proof. (i) ⇒ (ii) The proof is obvious in view of the fact that $[\, c, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta^m) \subset [\, c, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta^m)$.

(ii) ⇒(iii) Let $[\, c, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta^m) \subset [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta^m)$. Suppose that (iii) does not hold. Then for some $t,\ \rho > 0$

$$\sup_r \frac{1}{g_r} \sum_{k \in I_r} [M_k(\frac{t}{\rho})]^{p_k} = \infty$$

and therefore we can find a subinterval $I_{r(j)}$ of the set of interval I_r such that

$$\frac{1}{g_r} \sum_{k \in I_{r(j)}} \Big[M_k\Big(\frac{j^{-1}}{\rho}\Big)\Big]^{p_k} > j, \quad j = 1, 2, \cdots . \tag{4}$$

Define the sequence $x = (x_k)$ by

$$\Delta^m x_{k+s} = \begin{cases} j^{-1}, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \quad \text{for all } \ s \in \mathbb{N}. \end{cases}$$

Then $x = (x_k) \in [\ c, p, ||\cdot, \cdots, \cdot||\ ]_0^\theta(\Delta^m)$ but by equation(4),
$x = (x_k) \notin [\ c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\ ]_\infty^\theta(\Delta^m)$, which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i) Let (iii) hold and $x = (x_k) \in [\ c, p, ||\cdot, \cdots, \cdot||\ ]_\infty^\theta(\Delta^m)$. Suppose that
$x = (x_k) \notin [\ c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\ ]_\infty^\theta(\Delta^m)$.
Then

$$\sup_{r,s} \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k \Big(|| \frac{\Delta^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1} || \Big) \Big]^{p_k} = \infty. \tag{5}$$

Let $t = ||\Delta^m_{x_{k+s}}, z_1, \cdots, z_{n-1}||$ for each k and fixed s, then by equation(5)

$$\sup_r \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k \Big(\frac{t}{\rho} \Big) \Big] = \infty,$$

which contradicts (iii). Hence (i) must hold. □

Theorem 6. *Let $1 \leq p_k \leq \sup p_k < \infty$ and $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $[\ c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\ ]_0^\theta(\Delta^m) \subset [\ c, p, ||\cdot, \cdots, \cdot||\ ]_0^\theta(\Delta^m)$,
(ii) $[\ c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\ ]_0^\theta(\Delta^m) \subset [\ c, p, ||\cdot, \cdots, \cdot||\ ]_\infty^\theta(\Delta^m)$,
(iii) $\inf_r \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k \Big(\frac{t}{\rho} \Big) \Big]^{p_k} > 0 \ \ (t, \rho > 0)$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) Suppose that (iii) does not hold. Then

$$\inf_r \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k \Big(\frac{t}{\rho} \Big) \Big]^{p_k} = 0 \ \ (t, \rho > 0),$$

so we can find a subinterval $I_{r(j)}$ of the set of interval I_r such that

$$\frac{1}{g_r} \sum_{k \in I_{r(j)}} \Big[M_k \Big(\frac{j}{\rho} \Big) \Big]^{p_k} < j^{-1}, \quad j = 1, 2, \cdots . \tag{6}$$

Define a sequence $x = (x_k)$ by

$$\Delta^m x_{k+s} = \begin{cases} j, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \quad \text{for all } \ s \in \mathbb{N}. \end{cases}$$

Thus by equation(6), $x=(x_k)\in [\,c,\mathcal{M},p,||\cdot,\cdots,\cdot||\,]_0^{\theta}(\Delta^m)$, but by equation(4), $x=(x_k)\notin [\,c,p,||\cdot,\cdots,\cdot||\,]_{\infty}^{\theta}(\Delta^m)$, which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i) Let (iii) hold and suppose that $x=(x_k)\in [\,c,\mathcal{M},p,||\cdot,\cdots,\cdot||\,]_0^{\theta}(\Delta^m)$, i.e,

$$\lim_{r\to\infty}\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}=0,\ \text{ uniformly in } s, \text{ for some } \rho>0. \qquad (7)$$

Again, suppose that $x=(x_k)\notin [\,c,p,||\cdot,\cdots,\cdot||\,]_0^{\theta}(\Delta^m)$. Then, for some number $\epsilon>0$ and a subinterval $I_{r(j)}$ of the set of interval I_r, we have $||\Delta^m x_{k+s},z_1,\cdots,z_{n-1}||\geq\epsilon$ for all $k\in\mathbb{N}$ and some $s\geq s_0$. Then, from the properties of the Orlicz function, we can write

$$M_k\Big(||\frac{\Delta^m x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\Big)^p_k\geq M_k\Big(\frac{\epsilon}{\rho}\Big)^{p_k}$$

and consequently by (7)

$$\lim_{r\to\infty}\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(\frac{\epsilon}{\rho}\Big)\Big]^{p_k}=0,$$

which contradicts (iii). Hence (i) must hold. □

Theorem 7. *Let $0<p_k\leq q_k$ for all $k\in\mathbb{N}$ and $\left(\frac{q_k}{p_k}\right)$ be bounded. Then,* $[\,c,\mathcal{M},q,||\cdot,\cdots,\cdot||\,]^{\theta}(\Delta^m)\subset[\,c,\mathcal{M},p,||\cdot,\cdots,\cdot||\,]^{\theta}(\Delta^m)$.

Proof. Let $x\in[\,c,\mathcal{M},q,||\cdot,\cdots,\cdot||\,]^{\theta}(\Delta^m)$. Write

$$t_k=\Big[M_k\Big(||\frac{\Delta^m x_{k+s}-L}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{q_k}$$

and $\mu_k=\frac{p_k}{q_k}$ for all $k\in\mathbb{N}$. Then $0<\mu_k\leq 1$ *for* $k\in\mathbb{N}$. Take $0<\mu<\mu_k$ for $k\in\mathbb{N}$. Define sequences (u_k) and (v_k) as follows: For $t_k\geq 1$, let $u_k=t_k$ and $v_k=0$ and for $t_k<1$, let $u_k=0$ and $v_k=t_k$. Then clearly for all $k\in\mathbb{N}$, we have

$$t_k=u_k+v_k,\qquad t_k^{\mu_k}=u_k^{\mu_k}+v_k^{\mu_k}$$

Now it follows that $u_k^{\mu_k}\leq u_k\leq t_k$ and $v_k^{\mu_k}\leq v_k^{\mu}$. Therefore,

$$\begin{aligned}\frac{1}{g_r}\sum_{k\in I_r}t_k^{\mu_k}&=\frac{1}{g_r}\sum_{k\in I_r}(u_k^{\mu_k}+v_k^{\mu_k})\\&\leq\frac{1}{g_r}\sum_{k\in I_r}t_k+\frac{1}{g_r}\sum_{k\in I_r}v_k^{\mu}.\end{aligned}$$

Now for each k,

$$\begin{aligned}\frac{1}{g_r}\sum_{k\in I_r}v_k^{\mu}&=\sum_{k\in I_r}\Big(\frac{1}{g_r}v_k\Big)^{\mu}\Big(\frac{1}{g_r}\Big)^{1-\mu}\\&\leq\Big(\sum_{k\in I_r}\Big[\Big(\frac{1}{g_r}v_k\Big)^{\mu}\Big]^{\frac{1}{\mu}}\Big)^{\mu}\Big(\sum_{k\in I_r}\Big[\Big(\frac{1}{g_r}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}}\Big)^{1-\mu}\\&=\Big(\frac{1}{g_r}\sum_{k\in I_r}v_k\Big)^{\mu}\end{aligned}$$

and so

$$\frac{1}{g_r}\sum_{k\in I_r} t_k^{\mu_k} \le \frac{1}{g_r}\sum_{k\in I_r} t_k + \Big(\frac{1}{g_r}\sum_{k\in I_r} v_k\Big)^{\mu}.$$

Hence $x \in [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta^m)$. □

Theorem 8. *(a) If* $0 < \inf p_k \le p_k \le 1$ *and for all* $k \in \mathbb{N}$, *then*

$$[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta^m) \subset [\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta^m).$$

(b) If $1 \le p_k \le \sup p_k < \infty$ *and for all* $k \in \mathbb{N}$. *Then*

$$[\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta^m) \subset [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta^m).$$

Proof. (a) Let $x \in [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta^m)$, then

$$\lim_{r\to\infty}\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0.$$

Since $0 < \inf p_k \le p_k \le 1$. This implies that

$$\lim_{r\to\infty}\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m_{x_{k+s}}-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] \le \lim_{r\to\infty}\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k},$$

therefore, $\lim_{r\to\infty}\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0.$
This shows that $x \in [\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta^m)\cdot$
Therefore,

$$[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta^m) \subset [\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta^m).$$

This completes the proof.

(b) Let $p_k \ge 1$ for each k and $\sup p_k < \infty$. Let $x \in [\, c, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta^m)$. Then for each $\rho > 0$, we have

$$\lim_{r\to\infty}\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0 < 1.$$

Since $1 \le p_k \le \sup p_k < \infty$, we have

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} &\le \lim_{r\to\infty}\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Delta^m x_{k+s}-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]\\ &= 0\\ &< 1.\end{aligned}$$

Therefore $x \in [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta^m)$. □

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