Armenian Journal of Mathematics
Volume 3, Number 3, 2010, 105–126

A Good λ Estimate for Multilinear Commutator of Singular Integral on Spaces of Homogeneous Type

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Abstract

In this paper, a good λ estimate for the multilinear commutator associated to the singular integral operator on the spaces of homogeneous type is obtained. Under this result, we get the$(L^p(X), L^q(X))$-boundedness of the multilinear commutator.

Key Words: Singular Integral; Multilinear Commutator; Lipschitz Space; BMO; Space of Homogeneous Type; Good λ Inequality.
Mathematics Subject Classification 2000: 42B20, 42B25

1 Introduction

Let T be the Calderón-Zygmund operator, Coifman, Rochberg and Weiss [9] proves that the commutator $[b, T](f) = bT(f) - T(bf)$(where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo [5] proves a similiar result when T is replaced by the fractional operators. In [15], [18], Janson and Paluszynski study these results for the Triebel-Lizorkin spaces and the case $b \in Lip_\beta(R^n)$, where $Lip_\beta(R^n)$ is the homogeneous Lipschitz space. The main purpose of this paper is to establish the good λ estimate for the multilinear commutator associated to the singular integral operator on the spaces of homogeneous type, where $b \in Lip_\beta(X)$ or $b \in BMO(X)$. Under this result, we get $(L^p(X), L^q(X))$-boundedness of the multilinear commutator.

2 Preliminaries and Theorems

Give a set X, a function $d : X \times X \to R^+$ is called a quasi-distance on X if the following conditions are satisfied:
(i) for every x and y in X, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) for every x and y in X, $d(x, y) = d(y, x)$,
(iii) there exists a constant $k \geq 1$ such that

$$d(x, y) \leq k(d(x, z) + d(z, y)) \tag{1}$$

for every x, y and z in X.

Let μ be a positive measure on the σ-algebra of subsets of X which contains the r-balls $B(x, r) = \{y : d(x, y) < r\}$. We assume that μ satisfies a doubling condition, that is, there exists a constant A such that

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty \tag{2}$$

holds for all $x \in X$ and $r > 0$.

A structure (X, d, μ), with d and μ as above, is called a space of homogeneous type. The constants k and A in (1) and (2) will be called the constants of the space.

Then let us introduce some notations [3], [13], [18]. Throughout this paper, B will denote a ball of X , and for a ball B let $f_B = \mu(B)^{-1} \int_B f(x) d\mu(x)$ and the sharp function of f is defined by

$$f^{\#}(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y) - f_B| d\mu(y).$$

It is well-known that [13]

$$f^{\#}(x) \approx \sup_{B \ni x} \inf_{c \in C} \frac{1}{\mu(B)} \int_B |f(y) - C| d\mu(y).$$

We say that f belongs to $BMO(X)$ if $f^{\#}$ belongs to $L^\infty(X)$ and define $||f||_{BMO} = ||f^{\#}||_{L^\infty}$. It has been known that[13]

$$||f - f_{2^k B}||_{BMO} \leq Ck||f||_{BMO}.$$

For $1 \leq p < \infty$ and $0 \leq \gamma < 1$, let

$$M_{\gamma,p}(f)(x) = \sup_{x \in B} \left(\frac{1}{\mu(B)^{1-p\gamma}} \int_B |f(y)|^p d\mu(y) \right)^{1/p}.$$

If $\gamma = 0$, $M_{p,\gamma}(f) = M_p(f)$ which is the Hardy-Littlewood maximal function when $p = 1$. For $0 < \beta < 1$, the Lipschitz space $\dot{\Lambda}_\beta(X)$ is the space of functions f such that

$$||f||_{\dot{\Lambda}_\beta} = \sup_{\substack{x, h \in X \\ h \neq 0}} \rho\left(\Delta_h^{[\beta]+1} f(x) \right) / \rho(x + h, x)^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator [18] and the existence of ρ is guaranteed by the following Lemma 1.

In this paper, we will study some multilinear commutators as follows.

Definition. Suppose b_j $(j = 1, \cdots, m)$ are the fixed locally integrable functions on X. Let T be the singular integral operator as

$$T(f)(x) = \int_X K(x, y)f(y)d\mu(y),$$

where K is a locally integrable function on $X \times X \setminus \{(x, y) : x = y\}$ and satisfies the following properties:
(1) $|K(x, y)| \leq \dfrac{C}{\mu(B(x, d(x, y)))}$,
(2) $|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)| \leq C \dfrac{(d(y, y'))^\delta}{\mu(B(y, d(x, y)))(d(x, y))^\delta}$,
when $d(x, y) \geq 2d(y, y')$, with some $\delta \in (0, 1]$.

The multilinear commutator of the singular integral operator is defined by

$$T_{\vec{b}}(f)(x) = \int_X \prod_{j=1}^m (b_j(x) - b_j(y))K(x, y)f(y)d\mu(y).$$

and $T_\star^{\vec{b}}(f)(x) = \sup_{\varepsilon>0} |T_\varepsilon^{\vec{b}}(f)(x)|$, where

$$T_\varepsilon^{\vec{b}}(f)(x) = \int_{\rho(x,y)>\varepsilon} \prod_{j=1}^m (b_j(x) - b_j(y))K(x, y)f(y)d\mu(y)$$

Note that when $b_1 = \cdots = b_m$, $T_{\vec{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors$[1]-[6], [12]-[14], [19]-[21]$. Our main purpose is to find the good λ estimate for the multilinear commutator $T_\star^{\vec{b}}$, and with this result to find $(L^p(X), L^q(X))$-boundedness for the multilinear commutator $T_{\vec{b}}$.

Given some functions b_j $(j = 1, \cdots, m)$ and a positive integer m and $1 \leq j \leq m$, we set $||\vec{b}||_{Lip_\beta} = \prod_{j=1}^m ||b_j||_{Lip_\beta}$, $||\vec{b}||_{BMO} = \prod_{j=1}^m ||b_j||_{BMO}$ and denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements, $|\sigma| = j$ is the element number of σ. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$, $||\vec{b}_\sigma||_{\dot{\Lambda}_\beta} = ||b_{\sigma(1)}||_{\dot{\Lambda}_\beta} \cdots ||b_{\sigma(j)}||_{\dot{\Lambda}_\beta}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

In what follows, $C > 0$ always denotes a constant that is independent of main parameters involved but whose value may differ from line to line. For any index $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$.

Now we state our results as following.

Theorem 1. . *Let $0 < \beta < 1$ and $b_j \in \dot{\Lambda}_\beta(X)$ for $j = 1, \cdots, m$.*

(a). *Suppose $1 < r < p < \infty$. Then there exists $\xi_0 > 0$ such that, for any $0 < \xi < \xi_0$ and $\lambda > 0$,*

$$\mu\left(\left\{x \in X : T_\star^{\vec{b}}(f)(x) > 3\lambda, ||\vec{b}||_{\dot{\Lambda}_\beta} M_{m\beta,p}(f)(x) \le \xi\lambda\right\}\right)$$
$$\le C\xi^r \mu(\{x \in X : T_\star^{\vec{b}}(f)(x) > \lambda\})$$

(b). $T_{\vec{b}}$ is bounded from $L^p(X)$ to $L^q(X)$ for $1 < p < 1/m\beta$ and $1/q = 1/p - m\beta$.

Theorem 2. . *Let $b_j \in BMO(X)$ for $j = 1, \cdots, m$.*

(a). *Suppose $1 < r < p < \infty$. Then there exists $\xi_0 > 0$ such that, for any $0 < \xi < \xi_0$ and $\lambda > 0$,*

$$\mu\left(\left\{x \in X : T_\star^{\vec{b}}(f)(x) > 3\lambda, ||\vec{b}||_{BMO} M_p(f)(x) \le \xi\lambda\right\}\right)$$
$$\le C\xi^r \mu(\{x \in X : T_\star^{\vec{b}}(f)(x) > \lambda\})$$

(b). *$T_{\vec{b}}$ is bounded on $L^p(X)$ for $1 < p < \infty$.*

3 Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1. *[16] Let d be a quasi-distance on a set X. Then there exists a quasi-distance d' on X, a finite constant C and a number $0 < \alpha < 1$, such that d' is equivalent to d and, for every x, y and z in X*

$$|d'(x, y) - d'(z, y)| \le Cd'(x, z)^\alpha (d'(x, y) + d'(z, y))^{1-\alpha}.$$

Lemma 2. *if $x \in B^* \subset B \subset \Omega \subset X$, then*

$$||f_B - f_{B^*}||_{L^\infty(B^*)} \le C|B|^\beta \inf_{u \in B^*} f_\beta^\#(u),$$

where $f_\beta^\#(x) = \sup_B \frac{1}{\mu(B)^{1+\beta}} \int_B |f(x) - f_B| d\mu(x)$.

Proof. Since $||f_B||_{L^\infty(B)} \le C\frac{1}{\mu(B)}\int_B |f(x)|d\mu(x)$, so we have

$$\begin{aligned} ||f_B - f_{B^*}||_{L^\infty(B^*)} &\le ||(f - f_B)_{B^*}||_{L^\infty(B^*)} \le \frac{C}{\mu(B^*)}\int_{B^*} |f - f_B| d\mu(x) \\ &\le \frac{C}{\mu(B)}\int_B |f - f_B| d\mu(x) \le C\mu(B)^\beta \inf_{u \in B^*} f_\beta^\#(u). \end{aligned}$$

□

Lemma 3. *[11],[17] Let* $0<\beta<1, 1\le p\le\infty$, *then*

$$\begin{aligned}||f||_{\dot\Lambda_\beta} &\approx \sup_B \frac{1}{\mu(B)^{1+\beta}}\int_B |f(x)-f_B|d\mu(x)\\ &\approx \sup_B \frac{1}{\mu(B)^{\beta}}\left(\frac{1}{\mu(B)}\int_B |f(x)-f_B|^p d\mu(x)\right)^{1/p}\\ &\approx \sup_B \inf_c \frac{1}{\mu(B)^{1+\beta}}\int_B |f(x)-c|d\mu(x)\\ &\approx \sup_B \inf_c \frac{1}{\mu(B)^{\beta}}\left(\frac{1}{\mu(B)}\int_B |f(x)-c|^p d\mu(x)\right)^{1/p}\end{aligned}$$

Proof. **Step 1.** We first show

$$||f||_{\dot\Lambda_\beta}\le C\sup_B \frac{1}{\mu(B)^{1+\beta}}\int_B |f(x)-f_B|d\mu(x)$$

The difference $\triangle_h^k$ is defined for each x such that $x,\cdots,x+kh\in\Omega\subset X$. Let Ω_h be the set of all points x such that there is a ball $B_x\subset\Omega$ with $x+ih\in B_x,\ i=0,1,\cdots,k$. Fix h and set $\tilde\Omega_h=\{x\in\Omega_h: x,\cdots,x+kh$ are Lebesgue points of $f\}$, then $\Omega_h\setminus\tilde\omega_h$ has measure zero. If $x\in\tilde\omega_h$ is fixed, set $y_i=x+ih$ with $i=0,1,\cdots,k$. Choose B as the smallest ball with $\{y_0,y_1,\cdots,y_k\}\subset B\subset\Omega$. Since each y_i is a Lebesgue point of f, if we choose balls $B^*\downarrow\{y_i\}$, then $f_{B^*}(y_i)\to f(y_i)$ and so according to Lemma 2,

$$|f_B(y_i)-f(y_i)|=\lim_{B^*\downarrow\{y_i\}}|f_B(y_i)-f_{B^*}(y_i)|\le Cf_\beta^{\#}(y_i)\mu(B)^\beta.$$

Since $\triangle_h^k(f_B)=0$, we have

$$|\triangle_h^k(f,x)|=|\triangle_h^k(f-f_B,x)|\le C\max_{1\le i\le k}|f(y_i)-f_B(y_i)\le C\max_{1\le i\le k}f_\beta^{\#}(y_i)|h|^\beta.$$

Therefore

$$|\triangle_h^k(f,x)|\le C\sum_{i=0}^k f_\beta^{\#}(x+ih)|h|^\beta \quad a.e.\ \ x\in\Omega_h.$$

Step 2. Then we prove

$$\sup_B \frac{1}{\mu(B)^{1+\beta}}\int_B |f(x)-f_B|d\mu(x)\le C\sup_B \frac{1}{\mu(B)^{\beta}}\left(\frac{1}{\mu(B)}\int_B |f(x)-f_B|^p d\mu(x)\right)^{1/p}$$

By Hölder's inequality, we have

$$\begin{aligned}\frac{1}{\mu(B)^{1+\beta}}\int_B |f(x)-f_B|d\mu(x) &\le C\frac{1}{\mu(B)^{1+\beta}}\left(\int_B |f(x)-f_B|^p d\mu(x)\right)^{1/p}\mu(B)^{1-1/p}\\ &= C\frac{1}{\mu(B)^{\beta}}\left(\frac{1}{\mu(B)}\int_B |f(x)-f_B|^p d\mu(x)\right)^{1/p}.\end{aligned}$$

Taking a sup over $B\ni x$ in both sides of the above inequality, we finish the step 2.

Step 3. Last we get

$$\sup_B \frac{1}{\mu(B)^\beta}\left(\frac{1}{\mu(B)}\int_B |f(x)-f_B|^p d\mu(x)\right)^{1/p} \le C||f||_{\dot{\Lambda}_\beta}.$$

Set $\omega_k(f,t)=\sup_{|h|\le t}||\triangle_h^k(f)||_{L^\infty(\Omega_h)}$, then $||f||_{\dot{\Lambda}_\beta}=\sup_{t>0} t^{-\beta}\omega_k(f,t)$. We claim that

$$||f-f_B||_{L^\infty(B)} \le C\mu(B)^\beta ||f||_{\dot{\Lambda}_\beta}. \tag{3}$$

It is enough to verify (3) for the unit ball B_0 since the case of arbitrary B then follows from a linear change of variables. Now suppose (3) does not hold for B_0. In this case, there is a sequence of functions (f_m) such that

$$\inf_B ||f_m - f_B||_{L^\infty(B_0)} \ge m||f_m||_{\dot{\Lambda}_\beta}.$$

If we let $(f_B)_m$ denote best $L^\infty(B_0)$ approximate to f_m, $m=1,2,\cdots$, then by rescaling if necessary, we find functions $g_m=\lambda_m(f_m-(f_B)_m)$ such that

$$1=\inf_B ||g_m-f_B||_{L^\infty(B_0)} = ||g_m||_{L^\infty(B_0)} \ge m||g_m||_{\dot{\Lambda}_\beta}.$$

Thus $\{g_m\}_1^\infty$ is precompact in $L^\infty(B_0)$ and for an appropriate subsequence, $g_{m_j}\to g$ with $g\in L^\infty(B_0)$. It follows that

$$||g||_{\dot{\Lambda}_\beta(B_0)} = \lim_{j\to\infty} ||g_{m_j}||_{\dot{\Lambda}_\beta(B_0)} = 0.$$

On the other hand, $\inf_B ||g-f_m||_{L^\infty(B_0)}=1$ and so we have a contradiction.

For the case of general ball B, we note that if f is defined on B and A is the linear transformation which maps B_0 onto B, then the function $\tilde{f}=f\circ A$ has a modulus of smoothness which satisfies

$$\omega_k(\tilde{f},t)=l^{-1/p}\omega_k(f,lt)$$

with l the radius of B. Thus $||\tilde{f}||_{\dot{\Lambda}_\beta(B_0)} = l^{\beta-1/p}||f||_{\dot{\Lambda}_\beta(B)}$.

For any ball $B\ni x$, we have

$$\left(\frac{1}{\mu(B)}\int_B |f-f_B|^p d\mu(x)\right)^{1/p} \le C||f-f_B||_{L^\infty(B)},$$

therefore

$$\sup_B \frac{1}{\mu(B)^\beta}\left(\frac{1}{\mu(B)}\int_B |b(x)-b_B|^p d\mu(x)\right)^{1/p} \le C||b||_{\dot{\Lambda}_\beta}.$$

The proofs of the third and fourth parts of Lemma 3 are similar, so we omit the details. And this completes the Lemma 3. □

Lemma 4. *[10] Let $f \in L^1(X)$ and $\alpha > 0$, and assume that $\mu(X) > \alpha^{-1}||f||_{L^1}$. Then f may be decomposed as $f = g + b$ where*

(1) $||g||_{L^2}^2 \leq C\alpha||f||_{L^1}$,

(2)$b = \sum b_j$, where each b_j is supported on some ball $B(x_j, r_j)$,

(3)$\int b_j d\mu = 0$,

(4)$||b_j||_{L^1} \leq C\alpha\mu(B(x_j, r_j))$,

(5) $\sum_j \mu(B(x_j, r_j)) \leq C\alpha^{-1}||f||_{L^1}$.

Lemma 5. *[5] Let $0 \leq \nu < 1$, $1 \leq r < p < 1/\nu$ and $1/q = 1/p - \nu$, then*

$$||M_{\nu,r}(f)||_{L^q} \leq C||f||_{L^p}.$$

Proof. We first show that

$$\mu(\{x : M_{\nu,r}(x) > \lambda\}) \leq \left(\frac{C}{\lambda}||f||_{L^r}\right)^{r/(1-\nu r)}.$$

Let us consider the set E, where

$$E = \{x : M_{\nu,r}(x) > \lambda\}.$$

By the Lemma 4, it follows that there exists a sequence of balls B_j, with bounded overlap so that $E \subseteq \cup_{j=1}^{\infty} B_j$ and so that on each B_j, we have

$$\frac{1}{\mu(B_j)^{1-\nu r}} \int_{B_j} |f|^r d\mu(y) \geq \lambda^r.$$

Now $\mu(E)^{r/q} \leq \left(\sum \mu(B_j)\right)^{r/q} \leq \sum \mu(B_j)^{r/q}$, $q = r/(1-\nu r)$; the last inequality is true because $r/q \leq 1$.

Now $\mu(B_j)^{1-\nu r} \leq \frac{1}{\lambda^r}\int_{B_j} |f|^r d\mu(y)$ and $r/q = 1 - \nu r$, so

$$\sum \mu(B_j)^{r/q} \leq \frac{1}{\lambda^r}\int_{B_j} |f|^r \left(\sum \chi_{B_j}\right) d\mu(y).$$

Hence

$$\sum \mu(B_j)^{r/q} \leq \frac{C}{\lambda^r}||f||_{L^r}^r$$

and

$$\mu(E) \leq \frac{C}{\lambda^q}||f||_{L^r}^q.$$

Note now that if $r < p < 1/\nu$, then using Hölder's inequality

$$M_{\nu,r}(x) \leq M_{\nu,p}(x).$$

Therefore, by the preceeding arguments, we have

$$\mu(E) \leq \left(\frac{C}{\lambda}||f||_{L^p}\right)^{p/(1-\nu p)}.$$

The Lemma 5 follows by the Marcinkiewicz interpolation theorem in homogeneous spaces. □

Proof. **Theorem 1(a).** When $\mathbf{m}=\mathbf{1}$, by the Whitney decomposition, $\{x\in X: T_\star^{b_1}(f)(x)>\lambda\}$ may be written as a union of balls $\{B_k\}$ with mutually disjoint interiors and with distance from each to $X\setminus\bigcup_k B_k$ comparable to the diameter of B_k. It suffices to prove the good λ estimate for each B_k. There exists a constant $C=C(n)$ such that for each k, the cube $\tilde{B}_k$ intersects $X\setminus\bigcup_k B_k$, where $\tilde{B}_k$ denotes the ball with the same center as B_k and with the diam $\tilde{B}_k = C$ diam B_k. Then, for each k, there exists a point $x_0=x_0(k)\in\tilde{B}_k$ such that

$$T_\star^{b_1}(f)(x_0)\le\lambda.$$

Now, we fix a ball B_k. Without loss of generality, we may assume there exists a point $z=z(k)$ with

$$||b_1||_{\dot{\Lambda}_\beta}M_{\beta,p}(f)(z)\le\xi\lambda.$$

Set $\overline{B}_k=\tilde{\tilde{B}}_k$ and write $f=f_1+f_2$ for $f_1=f\chi_{\overline{B}_k}$ and $f_2=f\chi_{X\setminus\overline{B}_k}$. We turn to the estimates on f_1 and f_2.

The estimates on f_1**.** For $x\in B_k$, $1/r=1/p+1/q<1$

$$\begin{aligned}
||T_\star^{b_1}(f_1)||_{L^r} &\le C\left(\int_{\rho(x,y)>\varepsilon}|(b_1(x)-b_1(y))K(x,y)f_1(y)|^r\,d\mu(y)\right)^{1/r}\\
&\le C\mu(\overline{B}_k)^{-1}\left(\int_{\overline{B}_k}|b_1(x)-b_1(y)|^q d\mu(y)\right)^{1/q}\left(\int_{\overline{B}_k}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\le C\mu(\overline{B}_k)^{-1}\mu(\overline{B}_k)^{\beta+1/q}||b_1||_{\dot{\Lambda}_\beta}\mu(\overline{B}_k)^{1/p-\beta}\left(\frac{1}{\mu(\overline{B}_k)^{1-\beta p}}\int_{\overline{B}_k}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\le C\mu(\overline{B}_k)^{1/r-1}||b_1||_{\dot{\Lambda}_\beta}M_{\beta,p}(f)(z).
\end{aligned}$$

Let $\eta>0$, we have

$$\begin{aligned}
\mu(\{x\in X: T_\star^{b_1}(f_1)(x)>\eta\lambda\}) &\le C(\eta\lambda)^{-r}||T_\star^{b_1}(f_1)||_{L^r}^r\\
&\le C(\eta\lambda)^{-r}[||b_1||_{\dot{\Lambda}_\beta}M_{\beta,p}(f)(z)]^r\mu(\overline{B}_k)^{1-1/r}\\
&\le C(\eta\lambda)^{-r}(\xi\lambda)^r\mu(\overline{B}_k)\\
&\le C(\xi/\eta)^r\mu(B_k).
\end{aligned}$$

The estimates on f_2**.** Let $H=H(X)$ be a large positive integer depending only on X. We consider the following two cases:

Case 1. $\mathrm{diam}(\tilde{B}_k)\le\varepsilon\le H\mathrm{diam}(\tilde{B}_k)$. Set $V(x,y)=(b_1(x)-b_1(y))K(x,y)f(y)$. Choose $x_0\in\tilde{B}_k$ such that $x_0\in X\setminus\bigcup_k B_k$. For $x\in B_k$, following [8], we have

$$\begin{aligned}
|T_\varepsilon^{b_1}(f_2)(x)| &\le |\int_{\rho(x,y)>\varepsilon}[V(x,y)-V(x_0,y)]f_2(y)d\mu(y)|+\int_{R(x)}|V(x_0,y)f(y)|d\mu(y)\\
&\quad+\int_{R(x_0)}|V(x_0,y)f(y)|d\mu(y)+|T_\varepsilon^{b_1}(f)(x_0)|\\
&= I+II+III+IV,
\end{aligned}$$

where $R(u) = \{y \in X : diam(\tilde{B}_k) < \rho(u, y) \leq H diam(\tilde{B}_k)\}$. Now let us treat I, II and III respectively. For I, we write

$$\begin{aligned}
|V(x,y) - V(x_0,y)| &= |(b_1(x) - b_1(y))K(x,y)f_2(y) - (b_1(x_0) - b_1(y))K(x_0,y)f_2(y)| \\
&= |b_1(x)K(x,y)f_2(y) - b_1(y)K(x,y)f_2(y) - b_1(x_0)K(x_0,y)f_2(y) \\
&\quad + b_1(y)K(x_0,y)f_2(y) - b_1(x_0)K(x,y)f_2(y) + b_1(x_0)K(x,y)f_2(y)| \\
&\leq |(b_1(x) - b_1(x_0))K(x,y)f_2(y)| \\
&\quad + |(b_1(x_0) - b_1(y))(K(x,y) - K(x_0,y))f_2(y)| \\
&= I_1 + I_2.
\end{aligned}$$

For I_1, by Lemma 3, Hölder's inequality and the following inequality, for $b \in \dot{\Lambda}_\beta$

$$|b(x) - b_B| \leq \frac{1}{\mu(B)} \int_B ||b||_{\dot{\Lambda}_\beta} |x-y|^\beta d\mu(y) \leq ||b||_{\dot{\Lambda}_\beta} \mu(B)^\beta,$$

we have

$$\begin{aligned}
\int_{\rho(x,y)>\varepsilon} I_1 d\mu(y) &\leq C \sum_{v=1}^{\infty} \int_{B_{2^{v+1}\varepsilon}(x_0) \setminus B_{2^v\varepsilon}(x_0)} |b_1(x) - b_1(x_0)||K(x,y)||f(y)| d\mu(y) \\
&\leq C||b_1||_{\dot{\Lambda}_\beta} \mu(\tilde{B}_k)^\beta \sum_{v=1}^{\infty} \mu(B_{2^v\varepsilon}(x_0))^{-1} \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \mu(B_{2^{v+1}\varepsilon}(x_0))^{1-1/p} \\
&\leq C||b_1||_{\dot{\Lambda}_\beta} \mu(\tilde{B}_k)^\beta \sum_{v=1}^{\infty} \mu(B_{2^{v+1}\varepsilon}(x_0))^{-1+1-1/p+1/p-\beta} \\
&\quad \times \left(\frac{1}{\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-\beta p}} \int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \\
&\leq C||b_1||_{\dot{\Lambda}_\beta} \mu(\tilde{B}_k)^\beta \sum_{v=1}^{\infty} \mu(B_{2^{v+1}\varepsilon}(x_0))^{-\beta} M_{\beta,p}(f)(z) \\
&\leq C||b_1||_{\dot{\Lambda}_\beta} \sum_{v=1}^{\infty} 2^{-v\beta} M_{\beta,p}(f)(z) \\
&\leq C||b_1||_{\dot{\Lambda}_\beta} M_{\beta,p}(f)(z) \\
&\leq C\xi\lambda.
\end{aligned}$$

For I_2, by K's properties, μ's doubling condition, Hölder's inequality and Lemma 3 , we obtain

$$\begin{aligned}\int_{\rho(x,y)>\varepsilon} I_2 d\mu(y) &\leq C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)\backslash B_{2^v\varepsilon}(x_0)} |(b_1(x_0)-b_1(y))(K(x,y)-K(x_0,y))f(y)|d\mu(y)\\
&\leq C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)} |b_1(x_0)-b_1(y)|\left[\frac{|d(x_0,x)|}{|d(x,y)|}\right]^{\delta}\frac{1}{|\mu(B(x_0,d(x_0,y)))|}|f(y)|d\mu(y)\\
&\leq C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)} |b_1(x_0)-b_1(y)|\frac{d(x_0,x)^{\delta}}{d(x_0,y)^{1+\delta}}|f(y)|d\mu(y)\\
&\leq C\sum_{v=1}^{\infty}\frac{\mu(\tilde{B}_k)^{\delta}}{\mu(B_{2^{v+1}\varepsilon}(x_0)^{1+\delta}}\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|b_1(x_0)-b_1(y)|^{p'}d\mu(y)\right)^{1/p'}\\
&\quad\times\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leq C\sum_{v=1}^{\infty}\mu(\tilde{B}_k)^{\delta}\mu(B_{2^{v+1}\varepsilon})(x_0)^{-1-\delta+\beta+1/p'+1/p-\beta}||b_1||_{\dot{\Lambda}_\beta}M_{\beta,p}(f)(z)\\
&\leq C\sum_{v=1}^{\infty}2^{-v\delta}||b_1||_{\dot{\Lambda}_\beta}M_{\beta,p}(f)(z)\\
&\leq C||b_1||_{\dot{\Lambda}_\beta}M_{\beta,p}(f)(z)\\
&\leq C\xi\lambda.\end{aligned}$$

Therefore $I \leq C\xi\lambda$.

For II and III, note that, for $y \in R(x)$,

$$\rho(x,y) \leq Hdiam(\tilde{B}_k),$$

we get, by Lemma 3 and Hölder's inequality,

$$\begin{aligned}II &\leq C\int_{H\tilde{B}_k}|b_1(x_0)-b_1(y)||K(x,y)||f(y)|d\mu(y)\\
&\leq C\mu(H\tilde{B}_k)^{-1}\left(\int_{H\tilde{B}_k}|b_1(x_0)-b_1(y)|^{p'}d\mu(y)\right)^{1/p'}\left(\int_{H\tilde{B}_k}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leq C\mu(H\tilde{B}_k)^{-1+\beta+1/p'+1/p-\beta}||b_1||_{\dot{\Lambda}_\beta}\left(\frac{1}{\mu(H\tilde{B}_k)^{1-\beta p}}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leq C||b_1||_{\dot{\Lambda}_\beta}M_{\beta,p}(f)(z)\\
&\leq C\xi\lambda.\end{aligned}$$

Similar $III \leq C\xi\lambda$.

Thus $I + II + III \leq C\xi\lambda$.

For IV, since $x \notin \bigcup_k B_k$, then $|T_\varepsilon^{b_1}(f)(x_0)| \leq \lambda$. For $x \in B_k$,

$$\sup_{\varepsilon\approx diam(\tilde{B}_k)}|T_\varepsilon^{b_1}(f_2)(x)| \leq C\xi\lambda + \lambda.$$

Case 2. $\varepsilon > H\text{diam}(\tilde{B}_k)$. Let B_k^ε denote the ball with the same center as B_k and with the diam $B_k^\varepsilon = \varepsilon$. Similar to the proof of **Case 1**, we get

$$\sup_{\varepsilon > diam(\tilde{B}_k)} |T_\varepsilon^{b_1}(f_2)(x)| \leq C\xi\lambda + \lambda.$$

Thus, we have shown that for $x \in B_k$,

$$T_\star^{b_1}(f_2)(x) \leq C\xi\lambda + \lambda.$$

Now, choose ξ_0 such that $C\xi_0 < 1$, let $\eta = 1$ and combine the estimates on f_1 with f_2, we get

$$\begin{aligned} &\mu\left(\left\{x \in B_k : T_\star^{b_1}(f)(x) > 3\lambda, ||b_1||_{\dot{\Lambda}_\beta} M_{\beta,p}(f)(x) \leq \xi\lambda\right\}\right) \\ \leq\ &\mu(\{x \in B_k : T_\star^{b_1}(f_1)(x) > 2\lambda - C\xi\lambda\}) + \mu(\{x \in X : T_\star^{b_1}(f_2)(x) > \lambda + C\xi\lambda\}) \\ \leq\ &\mu(\{x \in B_k : T_\star^{b_1}(f_1)(x) > \lambda\}) \leq \xi^r \mu(B_k). \end{aligned}$$

When $\mathbf{m > 1}$, similar to the case $\mathbf{m = 1}$, there exists a point $x_0 = x_0(k) \in \tilde{B}_k$ such that

$$T_\star^{\vec{b}}(f)(x_0) \leq \lambda.$$

Now, we fix a ball B_k. Without loss of generality, we may assume there exists a point $z = z(k)$ with

$$||\vec{b}||_{\dot{\Lambda}_\beta} M_{m\beta,p}(f)(z) \leq \xi\lambda.$$

Set $\overline{B}_k = \tilde{\tilde{B}}_k$ and write $f = f_1 + f_2$ for $f_1 = f\chi_{\overline{B}_k}$ and $f_2 = f\chi_{X\setminus\overline{B}_k}$. We turn to the estimates on f_1 and f_2.

The estimates on f_1**.** For $x \in B_k$, $1/r = 1/p + 1/q < 1$

$$\begin{aligned} ||T_\star^{\vec{b}}(f_1)||_{L^r} &\leq C\left(\int_{\rho(x,y)>\varepsilon} \left|\prod_{j=1}^m (b_j(x) - b_j(y))K(x,y)f_1(y)\right|^r d\mu(y)\right)^{1/r} \\ &\leq C\mu(\overline{B}_k)^{-1}\left(\int_{\overline{B}_k} |\prod_{j=1}^m b_j(x) - b_j(y)|^q d\mu(y)\right)^{1/q}\left(\int_{\overline{B}_k} |f(y)|^p d\mu(y)\right)^{1/p} \\ &\leq C\mu(\overline{B}_k)^{-1}\mu(\overline{B}_k)^{m\beta+1/q}||\vec{b}||_{\dot{\Lambda}_\beta}\mu(\overline{B}_k)^{1/p-m\beta}\left(\frac{1}{\mu(\overline{B}_k)^{1-m\beta p}}\int_{\overline{B}_k}|f(y)|^p d\mu(y)\right)^{1/p} \\ &\leq C\mu(\overline{B}_k)^{1/r-1}||\vec{b}||_{\dot{\Lambda}_\beta} M_{m\beta,p}(f)(z). \end{aligned}$$

Let $\eta > 0$, we have

$$\begin{aligned} &\mu(\{x \in X : T_\star^{\vec{b}}(f_1)(x) > \eta\lambda\}) \\ \leq\ &C(\eta\lambda)^{-r}||T_\star^{\vec{b}}(f_1)||_{L^r}^r \\ \leq\ &C(\eta\lambda)^{-r}[||\vec{b}||_{\dot{\Lambda}_\beta} M_{m\beta,p}(f)(z)]^r \mu(\overline{B}_k)^{1-1/r} \\ \leq\ &C(\eta\lambda)^{-r}(\xi\lambda)^r\mu(\overline{B}_k) \\ \leq\ &C(\xi/\eta)^r\mu(B_k). \end{aligned}$$

The estimates on f_2. Let $H = H(X)$ be a large positive integer depending only on X. We consider the following two cases:

Case 1. $\mathrm{diam}(\tilde{B}_k) \le \varepsilon \le H\mathrm{diam}(\tilde{B}_k)$. Set $U(x,y) = \prod_{j=1}^m (b_j(x) - b_j(y))K(x,y)f(y)$. Choose $x_0 \in \tilde{B}_k$ such that $x_0 \in X \setminus \bigcup_k B_k$. For $x \in B_k$, following [8], we have

$$\begin{aligned}
|T_\varepsilon^{\vec{b}}(f_2)(x)| &\le |\int_{\rho(x,y)>\varepsilon} [U(x,y) - U(x_0,y)]f_2(y)d\mu(y)| + \int_{R(x)} |U(x_0,y)f(y)|d\mu(y) \\
&\quad + \int_{R(x_0)} |U(x_0,y)f(y)|d\mu(y) + |T_\varepsilon^{\vec{b}}(f)(x_0)| \\
&= J + JJ + JJJ + JJJJ,
\end{aligned}$$

where $R(u) = \{y \in X : diam(\tilde{B}_k) < \rho(u,y) \le Hdiam(\tilde{B}_k)\}$. Now let us treat J, JJ and JJJ respectively. For J, we write

$$\begin{aligned}
J &\le |(b_1(x) - b_1(x_0)) \cdots (b_m(x) - b_m(x_0)) \int_{\rho(x,y)>\varepsilon} K(x,y)f_2(y)d\mu(y)| \\
&\quad + \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c} \int_{\rho(x,y)>\varepsilon} (\vec{b}(y) - \vec{b}(x_0))_\sigma (K(x,y) - K(x_0,y)f_2(y)d\mu(y)| \\
&= J_1 + J_2.
\end{aligned}$$

For J_1, we have

$$\begin{aligned}
J_1 &\le \prod_{j=1}^m |b_j(x) - b_j(x_0)| \sum_{v=1}^\infty \int_{B_{2^{v+1}\varepsilon}(x_0) \setminus B_{2^v\varepsilon}(x_0)} |K(x,y)||f(y)|d\mu(y) \\
&\le C||\vec{b}||_{\dot{\Lambda}_\beta} \mu(\tilde{B}_k)^{m\beta} \sum_{v=1}^\infty \mu(B_{2^v\varepsilon}(x_0))^{-1} \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \mu(B_{2^{v+1}\varepsilon}(x_0))^{1-1/p} \\
&\le C||\vec{b}||_{\dot{\Lambda}_\beta} \mu(\tilde{B}_k)^{m\beta} \sum_{v=1}^\infty \mu(B_{2^{v+1}\varepsilon}(x_0))^{-1+1-1/p+1/p-m\beta} \\
&\quad \times \left(\frac{1}{\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-m\beta p}} \int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \\
&\le C||\vec{b}||_{\dot{\Lambda}_\beta} \mu(\tilde{B}_k)^{m\beta} \sum_{v=1}^\infty \mu(B_{2^{v+1}\varepsilon}(x_0))^{-m\beta} M_{m\beta,p}(f)(z) \\
&\le C||\vec{b}||_{\dot{\Lambda}_\beta} \sum_{v=1}^\infty 2^{-vm\beta} M_{m\beta,p}(f)(z) \\
&\le C||\vec{b}||_{\dot{\Lambda}_\beta} M_{m\beta,p}(f)(z) \\
&\le C\xi\lambda.
\end{aligned}$$

For J_2, let $\tau, \tau' \in \mathbf{N}$ such that $\tau + \tau' = m$, and $\tau' \neq 0$, we get

$$
\begin{aligned}
J_2 &\leq \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c} \int_{B_{2^{v+1}\varepsilon}(x_0) \setminus B_{2^v\varepsilon}(x_0)} (\vec{b}(y) - \vec{b}(x_0))_{\sigma} (K(x,y) - K(x_0,y) f(y) d\mu(y)| \\
&\leq C \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c}| \sum_{v=1}^{\infty} \frac{\mu(\tilde{B}_k)^{\delta}}{\mu(B_{2^v\varepsilon}(x_0))^{1+\delta}} \int_{B_{2^{v+1}\varepsilon}(x_0)} |(\vec{b}(y) - \vec{b}(x_0))_{\sigma} f(y) d\mu(y)| \\
&\leq C \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c}| \sum_{v=1}^{\infty} \frac{\mu(\tilde{B}_k)^{\delta}}{\mu(B_{2^v\varepsilon}(x_0))^{1+\delta}} \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |(\vec{b}(y) - \vec{b}(x_0))_{\sigma}|^{p'} d\mu(y) \right)^{1/p'} \\
&\quad \times \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \\
&\leq C \sum_{\tau+\tau'=m} ||\vec{b}_{\sigma^c}||_{\dot{\Lambda}_\beta} \mu(\tilde{B}_k)^{\tau\beta+\delta} \sum_{v=1}^{\infty} \mu(B_{2^{v+1}\varepsilon}(x_0))^{-1-\delta+\tau'\beta+1/p'+1/p-m\beta} ||\vec{b}_{\sigma}||_{\dot{\Lambda}_\beta} M_{m\beta,p}(f)(z) \\
&\leq C \sum_{\tau+\tau'=m} ||\vec{b}||_{\dot{\Lambda}_\beta} \mu(\tilde{B}_k)^{\tau\beta+\delta} \sum_{v=1}^{\infty} \mu(B_{2^{v+1}\varepsilon}(x_0))^{-\delta-\tau\beta} M_{m\beta,p}(f)(z) \\
&\leq C ||\vec{b}||_{\dot{\Lambda}_\beta} \sum_{v=1}^{\infty} 2^{-v(\delta+\tau\beta)} M_{m\beta,p}(f)(z) \\
&\leq C ||\vec{b}||_{\dot{\Lambda}_\beta} M_{m\beta,p}(f)(z) \\
&\leq C\xi\lambda.
\end{aligned}
$$

Therefore $J \leq C\xi\lambda$. For JJ and JJJ, note that, for $y \in R(x)$,

$$\rho(x,y) \leq H diam(\tilde{B}_k),$$

we get, by Lemma 3 and Hölder's inequality,

$$
\begin{aligned}
JJ &\leq C \int_{H\tilde{B}_k} |\prod_{j=1}^{m} b_j(x_0) - b_j(y)||K(x,y)||f(y)| d\mu(y) \\
&\leq C\mu(H\tilde{B}_k)^{-1} \left(\int_{H\tilde{B}_k} |\prod_{j=1}^{m} b_j(x_0) - b_j(y)|^{p'} d\mu(y) \right)^{1/p'} \left(\int_{H\tilde{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\
&\leq C\mu(H\tilde{B}_k)^{-1+m\beta+1/p'+1/p-m\beta} ||\vec{b}||_{\dot{\Lambda}_\beta} \left(\frac{1}{\mu(H\tilde{B}_k)^{1-m\beta p}} |f(y)|^p d\mu(y) \right)^{1/p} \\
&\leq C ||\vec{b}||_{\dot{\Lambda}_\beta} M_{m\beta,p}(f)(z) \\
&\leq C\xi\lambda.
\end{aligned}
$$

Similar $JJJ \leq C\xi\lambda$.

Thus $J + JJ + JJJ \leq C\xi\lambda$.

For $JJJJ$, since $x \notin \bigcup_k B_k$, then $|T_\varepsilon^{\vec{b}}(f)(x_0)| \leq \lambda$. For $x \in B_k$,

$$\sup_{\varepsilon \approx diam(\tilde{B}_k)} |T_\varepsilon^{\vec{b}}(f_2)(x)| \leq C\xi\lambda + \lambda.$$

Case 2. $\varepsilon > H\text{diam}(\tilde{B}_k)$. Let B_k^ε denote the ball with the same center as B_k and with the diam $B_k^\varepsilon = \varepsilon$. Similar to the proof of **Case 1**, we get

$$\sup_{\varepsilon > diam(\tilde{B}_k)} |T_\varepsilon^{\vec{b}}(f_2)(x)| \leq C\xi\lambda + \lambda.$$

Thus, we have shown that for $x \in B_k$,

$$T_\star^{\vec{b}}(f_2)(x) \leq C\xi\lambda + \lambda.$$

Now, choose ξ_0 such that $C\xi_0 < 1$, let $\eta = 1$ and combine the estimates on f_1 with f_2, we get

$$\begin{aligned} & \mu\left(\left\{x \in B_k : T_\star^{\vec{b}}(f)(x) > 3\lambda, ||\vec{b}||_{\dot{\Lambda}_\beta} M_{m\beta,p}(f)(x) \leq \xi\lambda\right\}\right) \\ \leq\ & \mu(\{x \in B_k : T_\star^{\vec{b}}(f_1)(x) > 2\lambda - C\xi\lambda\}) + \mu(\{x \in X : T_\star^{\vec{b}}(f_2)(x) > \lambda + C\xi\lambda\}) \\ \leq\ & \mu(\{x \in B_k : T_\star^{\vec{b}}(f_1)(x) > \lambda\}) \leq \xi^r \mu(B_k). \end{aligned}$$

This completes the proof of Theorem 1(a). (b) follows from (a) and Lemma 5. □

Proof. **Theorem 2(a).** When $\mathbf{m = 1}$, by the Whitney decomposition, $\{x \in X : T_\star^{b_1}(f)(x) > \lambda\}$ may be written as a union of balls $\{B_k\}$ with mutually disjoint interiors and with distance from each to $X \setminus \bigcup_k B_k$ comparable to the diameter of B_k. It suffices to prove the good λ estimate for each B_k. There exists a constant $C = C(n)$ such that for each k, the cube $\tilde{B}_k$ intersects $X \setminus \bigcup_k B_k$, where $\tilde{B}_k$ denotes the ball with the same center as B_k and with the diam $\tilde{B}_k = C$ diam B_k. Then, for each k, there exists a point $x_0 = x_0(k) \in \tilde{B}_k$ such that

$$T_\star^{b_1}(f)(x_0) \leq \lambda.$$

Now, we fix a ball B_k. Without loss of generality, we may assume there exists a point $z = z(k)$ with

$$||b_1||_{BMO} M_p(f)(z) \leq \xi\lambda.$$

Set $\overline{B}_k = \tilde{\tilde{B}}_k$ and write $f = f_1 + f_2$ for $f_1 = f\chi_{\overline{B}_k}$ and $f_2 = f\chi_{X \setminus \overline{B}_k}$. We turn to the estimates on f_1 and f_2.

The estimates on f_1**.** For $x \in B_k$, $1/r = 1/p + 1/q < 1$, we have

$$\begin{aligned} ||T_\star^{b_1}(f_1)||_{L^r} &\leq C\left(\int_{\rho(x,y)>\varepsilon} |(b_1(x) - b_1(y))K(x,y)f_1(y)|^r \, d\mu(y)\right)^{1/r} \\ &\leq C\mu(\overline{B}_k)^{-1}\left(\int_{\overline{B}_k} |b_1(x) - b_1(y)|^q d\mu(y)\right)^{1/q}\left(\int_{\overline{B}_k} |f(y)|^p d\mu(y)\right)^{1/p} \\ &\leq C\mu(\overline{B}_k)^{-1}\mu(\overline{B}_k)^{1/q}||b_1||_{BMO}\mu(\overline{B}_k)^{1/p}\left(\frac{1}{\mu(\overline{B}_k)}\int_{\overline{B}_k} |f(y)|^p d\mu(y)\right)^{1/p} \\ &\leq C\mu(\overline{B}_k)^{1/r-1}||b_1||_{BMO} M_p(f)(z). \end{aligned}$$

Let $\eta > 0$, we have

$$\begin{aligned} & \mu(\{x \in X : T_{\star}^{b_1}(f_1)(x) > \eta\lambda\}) \\ \leq\ & C(\eta\lambda)^{-r}||T_{\star}^{b_1}(f_1)||_{L^r}^r \\ \leq\ & C(\eta\lambda)^{-r}[||b_1||_{BMO}M_p(f)(z)]^r\mu(\overline{B}_k)^{1-1/r} \\ \leq\ & C(\eta\lambda)^{-r}(\xi\lambda)^r\mu(\overline{B}_k) \\ \leq\ & C(\xi/\eta)^r\mu(B_k). \end{aligned}$$

The estimates on f_2. Let $H = H(X)$ be a large positive integer depending only on X. We consider the following two cases:

Case 1. $\mathrm{diam}(\tilde{B}_k) \leq \varepsilon \leq H\mathrm{diam}(\tilde{B}_k)$. Set $V(x,y) = (b_1(x) - b_1(y))K(x,y)f(y)$. Choose $x_0 \in \tilde{B}_k$ such that $x_0 \in X \setminus \bigcup_k B_k$. For $x \in B_k$, following [8], we have

$$\begin{aligned} |T_{\varepsilon}^{b_1}(f_2)(x)| &\leq |\int_{\rho(x,y)>\varepsilon}[V(x,y) - V(x_0,y)]f_2(y)d\mu(y)| + \int_{R(x)}|V(x_0,y)f(y)|d\mu(y) \\ &\quad + \int_{R(x_0)}|V(x_0,y)f(y)|d\mu(y) + |T_{\varepsilon}^{b_1}(f)(x_0)| \\ &= I' + II' + III' + IV', \end{aligned}$$

where $R(u) = \{y \in X : diam(\tilde{B}_k) < \rho(u,y) \leq Hdiam(\tilde{B}_k)\}$. Now let us treat I', II' and III' respectively. For I', we write

$$\begin{aligned} & |V(x,y) - V(x_0,y)| \\ =\ & |(b_1(x) - b_1(y))K(x,y)f_2(y) - (b_1(x_0) - b_1(y))K(x_0,y)f_2(y)| \\ =\ & |b_1(x)K(x,y)f_2(y) - b_1(y)K(x,y)f_2(y) - b_1(x_0)K(x_0,y)f_2(y) \\ & +b_1(y)K(x_0,y)f_2(y) - b_1(x_0)K(x,y)f_2(y) + b_1(x_0)K(x,y)f_2(y)| \\ \leq\ & |(b_1(x) - b_1(x_0))K(x,y)f_2(y)| + |(b_1(x_0) - b_1(y))(K(x,y) - K(x_0,y))f_2(y)| \\ =\ & I_1' + I_2'. \end{aligned}$$

For I_1', by Hölder's inequality and the following inequality, for $b \in BMO(X)$

$$|b(x) - b_B| \leq \frac{1}{\mu(B)}\int_B |b(x) - b_B|d\mu(x) \leq ||b||_{BMO},$$

we have

$$\begin{aligned}
\int_{\rho(x,y)>\varepsilon} I_1 d\mu(y) &\leq C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)\backslash B_{2^v\varepsilon}(x_0)} |b_1(x)-b_1(x_0)||K(x,y)||f(y)|d\mu(y) \\
&\leq C||b_1||_{BMO}\sum_{v=1}^{\infty}\mu(B_{2^v\varepsilon}(x_0))^{-1}\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-1/p} \\
&\leq C||b_1||_{BMO}\sum_{v=1}^{\infty}\mu(B_{2^{v+1}\varepsilon}(x_0))^{-1+1-1/p+1/p} \\
&\quad\times\left(\frac{1}{\mu(B_{2^{v+1}\varepsilon}(x_0))}\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p} \\
&\leq C||b_1||_{BMO}M_p(f)(z) \\
&\leq C\xi\lambda.
\end{aligned}$$

For I_2', similar to I_2, we obtain

$$\begin{aligned}
\int_{\rho(x,y)>\varepsilon} I_2 d\mu(y) &\leq C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)\backslash B_{2^v\varepsilon}(x_0)} |(b_1(x_0)-b_1(y))(K(x,y)-K(x_0,y))f(y)|d\mu(y) \\
&\leq C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)}|b_1(x_0)-b_1(y)|\left[\frac{|d(x_0,x)|}{|d(x,y)|}\right]^{\delta}\frac{1}{|\mu(B(x_0,d(x_0,y)))|}|f(y)|d\mu(y) \\
&\leq C\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)}|b_1(x_0)-b_1(y)|\frac{d(x_0,x)^{\delta}}{d(x_0,y)^{1+\delta}}|f(y)|d\mu(y) \\
&\leq C\sum_{v=1}^{\infty}\frac{\mu(\tilde{B}_k)^{\delta}}{\mu(B_{2^{v+1}\varepsilon}(x_0)^{1+\delta}}\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|b_1(x_0)-b_1(y)|^{p'}d\mu(y)\right)^{1/p'} \\
&\quad\times\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p} \\
&\leq C\sum_{v=1}^{\infty}\mu(\tilde{B}_k)^{\delta}\mu(B_{2^{v+1}\varepsilon})(x_0)^{-1-\delta+1/p'+1/p}||b_1||_{BMO}M_p(f)(z) \\
&\leq C\sum_{v=1}^{\infty}2^{-v\delta}||b_1||_{BMO}M_p(f)(z) \\
&\leq C||b_1||_{BMO}M_p(f)(z) \\
&\leq C\xi\lambda.
\end{aligned}$$

Therefore $I' \leq C\xi\lambda$.

For II' and III', note that, for $y \in R(x)$,

$$\rho(x,y) \leq H diam(\tilde{B}_k),$$

we get, by Hölder's inequality,

$$
\begin{aligned}
II' &\le C\int_{H\tilde{B}_k}|b_1(x_0)-b_1(y)||K(x,y)||f(y)|d\mu(y)\\
&\le C\mu(H\tilde{B}_k)^{-1}\left(\int_{H\tilde{B}_k}|b_1(x_0)-b_1(y)|^{p'}d\mu(y)\right)^{1/p'}\left(\int_{H\tilde{B}_k}|f(y)|^{p}d\mu(y)\right)^{1/p}\\
&\le C\mu(H\tilde{B}_k)^{-1+1/p'+1/p}||b_1||_{BMO}\left(\frac{1}{\mu(H\tilde{B}_k)}|f(y)|^{p}d\mu(y)\right)^{1/p}\\
&\le C||b_1||_{BMO}M_p(f)(z)\\
&\le C\xi\lambda.
\end{aligned}
$$

Similar $III' \le C\xi\lambda$.

Thus $I'+II'+III' \le C\xi\lambda$.

For IV', since $x \notin \bigcup_k B_k$, then $|T_\varepsilon^{b_1}(f)(x_0)| \le \lambda$. For $x\in B_k$,

$$\sup_{\varepsilon\approx diam(\tilde{B}_k)}|T_\varepsilon^{b_1}(f_2)(x)| \le C\xi\lambda+\lambda.$$

Case 2. $\varepsilon > H\mathrm{diam}(\tilde{B}_k)$. Let B_k^ε denote the ball with the same center as B_k and with the diam $B_k^\varepsilon=\varepsilon$. Similar to the proof of **Case 1**, we get

$$\sup_{\varepsilon> diam(\tilde{B}_k)}|T_\varepsilon^{b_1}(f_2)(x)| \le C\xi\lambda+\lambda.$$

Thus, we have shown that for $x\in B_k$,

$$T_\star^{b_1}(f_2)(x)\le C\xi\lambda+\lambda.$$

Now, choose ξ_0 such that $C\xi_0<1$, let $\eta=1$ and combine the estimates on f_1 with f_2, we get

$$
\begin{aligned}
&\mu\left(\left\{x\in B_k : T_\star^{b_1}(f)(x)>3\lambda, ||b_1||_{BMO}M_p(f)(x)\le\xi\lambda\right\}\right)\\
\le\ &\mu(\{x\in B_k : T_\star^{b_1}(f_1)(x)>2\lambda-C\xi\lambda\})+\mu(\{x\in X : T_\star^{b_1}(f_2)(x)>\lambda+C\xi\lambda\})\\
\le\ &\mu(\{x\in B_k : T_\star^{b_1}(f_1)(x)>\lambda\})\le\xi^r\mu(B_k).
\end{aligned}
$$

When $\mathbf{m}>\mathbf{1}$, similar to the case $\mathbf{m}=\mathbf{1}$, there exists a point $x_0=x_0(k)\in\tilde{B}_k$ such that

$$T_\star^{\vec{b}}(f)(x_0)\le\lambda.$$

Now, we fix a ball B_k. Without loss of generality, we may assume there exists a point $z=z(k)$ with

$$||\vec{b}||_{BMO}M_p(f)(z)\le\xi\lambda.$$

Set $\overline{B}_k=\tilde{\tilde{B}}_k$ and write $f=f_1+f_2$ for $f_1=f\chi_{\overline{B}_k}$ and $f_2=f\chi_{X\setminus\overline{B}_k}$. We turn to the estimates on f_1 and f_2.

The estimates on f_1**.** For $x \in B_k$, $1/r = 1/p + 1/q_j < 1$, $j = 1, \cdots, m$

$$\begin{aligned}
||T_\star^{\vec{b}}(f_1)||_{L^r} &\leq C\left(\int_{\rho(x,y)>\varepsilon}\left|\prod_{j=1}^m (b_j(x)-b_j(y))K(x,y)f_1(y)\right|^r d\mu(y)\right)^{1/r} \\
&\leq C\mu(\overline{B}_k)^{-1}\left(\int_{\overline{B}_k}|\prod_{j=1}^m b_j(x)-b_j(y)|^{q_j}d\mu(y)\right)^{1/q_j}\left(\int_{\overline{B}_k}|f(y)|^p d\mu(y)\right)^{1/p} \\
&\leq C\mu(\overline{B}_k)^{-1}\mu(\overline{B}_k)^{\sum_{j=1}^m 1/q_j}||\vec{b}||_{BMO}\mu(\overline{B}_k)^{1/p}\left(\frac{1}{\mu(\overline{B}_k)}\int_{\overline{B}_k}|f(y)|^p d\mu(y)\right)^{1/p} \\
&\leq C\mu(\overline{B}_k)^{1/r-1}||\vec{b}||_{BMO}M_p(f)(z).
\end{aligned}$$

Let $\eta > 0$, we have

$$\begin{aligned}
&\mu(\{x \in X : T_\star^{\vec{b}}(f_1)(x) > \eta\lambda\}) \\
\leq\ & C(\eta\lambda)^{-r}||T_\star^{\vec{b}}(f_1)||_{L^r}^r \\
\leq\ & C(\eta\lambda)^{-r}[||\vec{b}||_{BMO}M_p(f)(z)]^r\mu(\overline{B}_k)^{1-1/r} \\
\leq\ & C(\eta\lambda)^{-r}(\xi\lambda)^r\mu(\overline{B}_k) \\
\leq\ & C(\xi/\eta)^r\mu(B_k).
\end{aligned}$$

The estimates on f_2**.** Let $H = H(X)$ be a large positive integer depending only on X. We consider the following two cases:

Case 1. $\mathrm{diam}(\tilde{B}_k) \leq \varepsilon \leq H\mathrm{diam}(\tilde{B}_k)$. Set $U(x,y) = \prod_{j=1}^m (b_j(x)-b_j(y))K(x,y)f(y)$. Choose $x_0 \in \tilde{B}_k$ such that $x_0 \in X \setminus \bigcup_k B_k$. For $x \in B_k$, following [8], we have

$$\begin{aligned}
|T_\varepsilon^{\vec{b}}(f_2)(x)| &\leq |\int_{\rho(x,y)>\varepsilon}[U(x,y)-U(x_0,y)]f_2(y)d\mu(y)| + \int_{R(x)}|U(x_0,y)f(y)|d\mu(y) \\
&\quad + \int_{R(x_0)}|U(x_0,y)f(y)|d\mu(y) + |T_\varepsilon^{\vec{b}}(f)(x_0)| \\
&= J' + JJ' + JJJ' + JJJJ',
\end{aligned}$$

where $R(u) = \{y \in X : diam(\tilde{B}_k) < \rho(u,y) \leq Hdiam(\tilde{B}_k)\}$. Now let us treat J', JJ' and JJJ' respectively. For J', we write

$$\begin{aligned}
J' &\leq |(b_1(x)-b_1(x_0))\cdots(b_m(x)-b_m(x_0))\int_{\rho(x,y)>\varepsilon}K(x,y)f_2(y)d\mu(y)| \\
&\quad + \sum_{j=1}^m\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_{\sigma^c}\int_{\rho(x,y)>\varepsilon}(\vec{b}(y)-\vec{b}(x_0))_\sigma(K(x,y)-K(x_0,y)f_2(y)d\mu(y)| \\
&= J_1' + J_2'.
\end{aligned}$$

For J_1', we have

$$\begin{aligned}
J_1' &\leq \prod_{j=1}^{m}|b_j(x)-b_j(x_0)|\sum_{v=1}^{\infty}\int_{B_{2^{v+1}\varepsilon}(x_0)\setminus B_{2^v\varepsilon}(x_0)}|K(x,y)||f(y)|d\mu(y)\\
&\leq C||\vec{b}||_{BMO}\sum_{v=1}^{\infty}\mu(B_{2^v\varepsilon}(x_0))^{-1}\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-1/p}\\
&\leq C||\vec{b}||_{BMO}\sum_{v=1}^{\infty}\mu(B_{2^{v+1}\varepsilon}(x_0))^{-1+1-1/p+1/p}\\
&\quad\times\left(\frac{1}{\mu(B_{2^{v+1}\varepsilon}(x_0))}\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leq C||\vec{b}||_{BMO}M_p(f)(z)\\
&\leq C\xi\lambda.
\end{aligned}$$

For J_2', we get

$$\begin{aligned}
J_2' &\leq \sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_{\sigma^c}\int_{B_{2^{v+1}\varepsilon}(x_0)\setminus B_{2^v\varepsilon}(x_0)}(\vec{b}(y)-\vec{b}(x_0))_\sigma(K(x,y)-K(x_0,y)f(y)d\mu(y)|\\
&\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_{\sigma^c}|\sum_{v=1}^{\infty}\frac{\mu(\tilde{B}_k)^\delta}{\mu(B_{2^v\varepsilon}(x_0))^{1+\delta}}\int_{B_{2^{v+1}\varepsilon}(x_0)}|(\vec{b}(y)-\vec{b}(x_0))_\sigma f(y)d\mu(y)|\\
&\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_{\sigma^c}|\sum_{v=1}^{\infty}\frac{\mu(\tilde{B}_k)^\delta}{\mu(B_{2^v\varepsilon}(x_0))^{1+\delta}}\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|(\vec{b}(y)-\vec{b}(x_0))_\sigma|^{p'}d\mu(y)\right)^{1/p'}\\
&\quad\times\left(\int_{B_{2^{v+1}\varepsilon}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leq C||\vec{b}_{\sigma^c}||_{BMO}\mu(\tilde{B}_k)^\delta\sum_{v=1}^{\infty}\mu(B_{2^{v+1}\varepsilon}(x_0))^{-1-\delta+1/p'+1/p}||\vec{b}_\sigma||_{BMO}M_{\gamma,p}(f)(z)\\
&\leq C||\vec{b}||_{BMO}\mu(\tilde{B}_k)^\delta\sum_{v=1}^{\infty}\mu(B_{2^{v+1}\varepsilon}(x_0))^{-\delta}M_p(f)(z)\\
&\leq C||\vec{b}||_{BMO}\sum_{v=1}^{\infty}2^{-v\delta}M_p(f)(z)\\
&\leq C||\vec{b}||_{BMO}M_p(f)(z)\\
&\leq C\xi\lambda.
\end{aligned}$$

Therefore $J'\leq C\xi\lambda$. For JJ' and JJJ', note that, for $y\in R(x)$,

$$\rho(x,y)\leq H diam(\tilde{B}_k),$$

we get, by Hölder's inequality, for $1/p_1' + \cdots 1/p_m' + 1/p = 1$

$$\begin{aligned} JJ' &\leq C \int_{H\tilde{B}_k} |\prod_{j=1}^m b_j(x_0) - b_j(y)||K(x,y)||f(y)|d\mu(y) \\ &\leq C\mu(H\tilde{B}_k)^{-1} \left(\int_{H\tilde{B}_k} |\prod_{j=1}^m b_j(x_0) - b_j(y)|^{p_j'} d\mu(y) \right)^{1/p_j'} \left(\int_{H\tilde{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C\mu(H\tilde{B}_k)^{-1+1/p_1'+\cdots 1/p_m'+1/p} ||\vec{b}||_{BMO} \left(\frac{1}{\mu(H\tilde{B}_k)} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C||\vec{b}||_{BMO} M_p(f)(z) \\ &\leq C\xi\lambda. \end{aligned}$$

Similar $JJJ' \leq C\xi\lambda$.

Thus $J' + JJ' + JJJ' \leq C\xi\lambda$.

For $JJJJ'$, since $x \notin \bigcup_k B_k$, then $|T_\varepsilon^{\vec{b}}(f)(x_0)| \leq \lambda$. For $x \in B_k$,

$$\sup_{\varepsilon \approx diam(\tilde{B}_k)} |T_\varepsilon^{\vec{b}}(f_2)(x)| \leq C\xi\lambda + \lambda.$$

Case 2. $\varepsilon > H\text{diam}(\tilde{B}_k)$. Let B_k^ε denote the ball with the same center as B_k and with the diam $B_k^\varepsilon = \varepsilon$. Similar to the proof of **Case 1**, we get

$$\sup_{\varepsilon > diam(\tilde{B}_k)} |T_\varepsilon^{\vec{b}}(f_2)(x)| \leq C\xi\lambda + \lambda.$$

Thus, we have shown that for $x \in B_k$,

$$T_\star^{\vec{b}}(f_2)(x) \leq C\xi\lambda + \lambda.$$

Now, choose ξ_0 such that $C\xi_0 < 1$, let $\eta = 1$ and combine the estimates on f_1 with f_2, we get

$$\begin{aligned} &\mu\left(\left\{x \in B_k : T_\star^{\vec{b}}(f)(x) > 3\lambda, ||\vec{b}||_{BMO} M_p(f)(x) \leq \xi\lambda\right\}\right) \\ \leq\ & \mu(\{x \in B_k : T_\star^{\vec{b}}(f_1)(x) > 2\lambda - C\xi\lambda\}) + \mu(\{x \in X : T_\star^{\vec{b}}(f_2)(x) > \lambda + C\xi\lambda\}) \\ \leq\ & \mu(\{x \in B_k : T_\star^{\vec{b}}(f_1)(x) > \lambda\}) \leq \xi^r \mu(B_k). \end{aligned}$$

This completes the proof of Theorem 2(a). (b) follows from (a) and Lemma 5. □

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