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# Groups whose Derived Subgroup is not Supplemented by Any Proper Subgroup

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**Abstract.** In this paper, we introduce two new classes of groups that are described as weakly nilpotent and weakly solvable groups. A group $G$ is weakly nilpotent if its derived subgroup does not have a supplement except $G$ and a group $G$ is weakly solvable if its derived subgroup does not have a normal supplement except $G$. We present some examples and counter-examples for these groups and characterize a finitely generated weakly nilpotent group. Moreover, we characterize the nilpotent and solvable groups in terms of weakly nilpotent and weakly solvable groups. Finally, we prove that if $F$ is a free group of rank $n$ such that every normal subgroup of $F$ has rank $n$, then $F$ is weakly solvable.

*Key Words:* Derived Subgroup, Supplement, Frat(G), nFrat(G), Weakly Nilpotent Groups, Weakly Solvable Groups, Free Groups
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# Introduction

Let $G$ be a group and $G'$ denote the derived subgroup of $G$. A subgroup $H$ of $G$ is called *supplemented* in $G$ if there exists a proper subgroup $K$ of $G$ such that $G = HK$. The subgroup $K$ is then called a *supplement* of $H$. In addition, if $H \cap K = \{e\}$, then $K$ is called a *complement* of $H$ and $H$ is called *complemented* in $G$. Similiarly, if there exists a proper normal subgroup $N$ of $G$ such that $G = HN$, then we say that $H$ has a *normal supplement*. If every nontrivial subgroup of $G$ is supplemented, then $G$ is termed as an *aS-group*, whereas $G$ is called an *aC-group* if every nontrivial subgroup of $G$ is complemented. A group $G$ is called an *nS-group* if every normal subgroup of $G$ has a supplement.

Many researchers have been interested in studying groups that have some specific conditions on subgroups (see, for example, [2, 3, 5]). The $aS$-groups and $aC$-groups have been extensively studied by Kappe and Kirtland in [4]. Several authors have studied supplementation and complementation to depict many important properties about the structure of groups. Malan'ina [7] investigated those groups which have a unique non-complemented subgroup and proved that all these groups are finite $p$-groups. Tyutyanov and Bychkov [9] studied the structure of those finite groups which have exactly two non-complemented subgroups.

Motivated by the work of [9], in this paper, we investigate those groups in which the derived subgroup is not supplemented by any proper subgroup. We also investigate those groups in which the derived subgroup is not supplemented by any proper normal subgroup. We now present the formal definitions of these groups, i.e., the groups in which the derived subgroup is not supplemented by any proper subgroup and the groups in which the derived subgroup is not supplemented by any proper normal subgroup.

A group in which the derived subgroup is not supplemented by any proper subgroup is termed as *weakly nilpotent*. More precisely, a group $G$ is called weakly nilpotent if $G^{'}$ does not have a proper supplement, i.e., if $G = HG^{'}$ for some subgroup $H$ of $G$, then $G = H$. Similarly, a group $G$ is called *weakly solvable* if $G^{'}$ does not have a proper normal supplement, i.e., if $G = HG^{'}$ for some normal subgroup $H$ of $G$, then $G = H$. Since the derived subgroup of abelian groups is trivial, abelian groups are groups which are nilpotent as well as weakly nilpotent. Similarly, abelian groups are solvable as well as weakly solvable. On the other hand, a perfect group is neither weakly nilpotent nor weakly solvable as the derived subgroup of such group is supplemented by the proper subgroup $\{e\}$.

Weakly nilpotent and weakly solvable groups exhibit very close connection with nilpotent and solvable groups. There are a number of properties of finite groups which are equivalent to nilpotent groups (see Theorems 5.2.4 and 5.2.16, [8]). But these properties usually get weaker in the case of infinite groups giving rise to generalizations of nilpotent groups. The situation is similar for solvable groups. In this purview, weakly nilpotent and weakly solvable groups are considered generalizations of nilpotent and solvable groups, respectively. We show that the class of nilpotent groups forms a proper subclass of the class of weakly nilpotent groups. To assert this, an example of a group which is weakly nilpotent but not nilpotent is constructed. However, Theorem 4 shows that these two classes coincides when we consider only finite groups. Similarly, Theorem 6 shows that the class of solvable groups forms a proper subclass of the class of weakly solvable groups. We present some examples and counter-examples for these groups and also present the characterization of a finitely generated weakly nilpotent group. Moreover, we characterize the nilpotent and solvable groups in term of weakly nilpotent and weakly solvable groups. A sufficent condition is also obtained for a free group to be a weakly solvable group.

This paper is organized as follows. We discuss some notations and preliminaries required in our work in Section 1. In Sections 2 and 3, we study our main results related to certain properties of weakly nilpotent groups and weakly solvable groups, respectively. The Section 4 contains results on free groups and characterization of weakly nilpotent and weakly solvable groups. The final section includes some concluding remarks.

# 1 Notations and Basic result

In this section, we recall standard notations and definitions that are used in our main results. For details, we refer to [8].

Subgroups in the lower central series of $G$ are written as $\gamma_n(G)$, where $n$ runs over all strictly positive integers. They are inductively defined by

$$\gamma_1(G) = G \text{ and } \gamma_{n+1}(G) = [\gamma_n(G), G] \text{ for } n \geq 1.$$

We denote the derived subgroup by $G' = [G, G]$, which is $\gamma_2(G)$ in the lower central series of $G$. A descending series $\{\delta_i(G)\}_{i\geq 0}$ of subgroups of $G$, defined inductively as

$$\delta_0(G) = G, \quad \delta_i(G) = \delta(\delta_{i-1}(G)) = [\delta_{i-1}(G), \delta_{i-1}(G)],$$

is called the *derived series* of $G$.

**Definition 1** *A group $G$ is called nilpotent if and only if it has a finite lower central series, i.e., $\gamma_m(G) = \{e\}$ for some positive integer $m$.*

**Definition 2** *A group $G$ is called solvable if and only if it has a finite derived series, i.e., $\delta_m(G) = \{e\}$ for some positive integer $m$.*

**Definition 3** *Let $X$ be a non empty subset of the group $G$. Then $X^G$ is the normal closure of $X$ in $G$ and it is a subgroup generated by all elements of the form $g^{-1}xg$ for any $g \in G$ and $x \in X$. $\langle X\rangle^G$ is the normal closure of the subgroup generated by $X$, whereas $\langle X^G\rangle$ is the subgroup of $G$ generated by the normal closure $X^G$.*

**Definition 4** *A subset $X$ of a group $G$ is called a normal generating set of $G$ if $G = \langle X\rangle^G = \langle X^G\rangle$.*

**Definition 5** *For a group $G$, let $\zeta$ and $\eta$ denote the collection of all maximal subgroups of $G$ and all maximal normal subgroups of $G$, respectively. Then*
*(i) $Frat(G) = \bigcap_{M\in\zeta} M$ if $\zeta \neq \phi$ and $Frat(G) = G$ if $\zeta = \phi$.*
*(ii) $nFrat(G) = \bigcap_{M\in\eta} M$ if $\eta \neq \phi$, and $nFrat(G) = G$ if $\eta = \phi$.*

Throughout the paper, we denote $Frat(G)$ and $nFrat(G)$ by $\Phi(G)$ and $\Psi(G)$, respectively. In group theory, the Frattini subgroup of a group has remarkable role in determining the structural properties of that group. For example, if $G$ is a non abelian group such that $\{e\} \neq Z(G) \subseteq \Phi(G)$, then $G$ can not have an abelian direct factor. Moreover, if $G$ is a finite group with $\Phi(G) = \{e\}$, then $G$ is an $nS$-group [5].

It is well known that $\Phi(G)$ is precisely the set of non-generators of $G$. For $\Psi(G)$, we have the following lemmas.

**Lemma 1** *([6, Theorem 2.6]) Let $G$ be a finite group. Then $\Psi(G)$ is precisely the set of those elements of $G$ which can be omitted from any normal generating set for $G$.*

***Proof.*** Let $x \in G$ be such that $x \notin \Psi(G)$. Then there exists a maximal normal subgroup $H$ of $G$ which does not contain $x$. Since $x \notin H$, $G = \langle H, x^G \rangle$. Clearly, we have $G \neq H$. Thus, $x$ can not be omitted from a normal generating set of $G$. From this it follows that if $x \in \Psi(G)$, then it can be omitted from any normal generating set of $G$. Conversely, suppose that $x \in \Psi(G)$. Let $S$ be a subset of $G$ such that $G = \langle S, x \rangle^G = \langle S^G, x^G \rangle$. We claim that $G = \langle S^G \rangle$. If $G \neq \langle S^G \rangle$, then there exists a maximal normal subgroup $H$ of $G$ such that $S^G \subseteq H$. Next, we note that $G = \langle S^G, x^G \rangle \subseteq H\Psi(G)$, i.e., $G = H\Psi(G)$. Since $\Psi(G) \subseteq H$, we have that $G = H$. But this is a contradiction. Hence, $G = \langle S^G \rangle = \langle S \rangle^G$, and the result holds. □

**Lemma 2** *([6, Theorem 4.2]) If $G$ is a finite group, then $\Phi(G) \subseteq \Psi(G)$.*

***Proof.*** Suppose $\Phi(G)$ is not contained in $\Psi(G)$, then there exists a maximal normal subgroup $H$ of $G$ such that $\Phi(G)$ is not contained in $H$. Since $H$ is a maximal normal subgroup of $G$, it follows that $G = H\Phi(G)$. But then $G = H$, which is a contradiction. Hence, $\Phi(G) \subseteq \Psi(G)$. □

# 2 Weakly nilpotent groups

In this section, we discuss some general properties of weakly nilpotent groups along with some examples as well as counter-examples.

**Theorem 1** *The quotient group of a weakly nilpotent group is weakly nilpotent.*

***Proof.*** Let $G$ be a weakly nilpotent group and $K$ be a normal subgroup of $G$. Consider the quotient group $G/K$, then $(G/K)' = G'K/K$. Suppose $H/K$ is a subgroup of $G/K$ such that $G/K = H/K(G/K)'$. Then $G/K = (H/K)(G'K/K) = HG'/K$. From this it follows that $G = HG'$. Since $G$ is weakly nilpotent, we have $G = H$. Consequently, $H/K = G/K$, and hence, $G/K$ is weakly nilpotent. □

**Example 1** *In this example, we show that converse of Theorem 1 is not true. Let* $G = \langle a, b \mid b^2 = e, bab = a^{-1} \rangle$ *be the infinite dihedral group and* $N$ *be its normal subgroup generated by* $a$*, i.e.,* $N = \langle a \rangle$*. Then both* $N$ *and* $G/N$*, being abelian, are weakly nilpotent. We take* $H = \langle a^3, b \rangle$*. Since* $G' = \langle a^2 \rangle$*, we have that* $G = HG'$ *but* $G \neq H$*. This shows that* $G$ *is not weakly nilpotent. It is worth to note that* $G$ *is a solvable group.*

We now present a result from [8] which will be useful to prove Theorem 3.

**Theorem 2** *For a group* $G$*, the following holds:*
*(i) Let* $x, y, z \in G$*. Then* $[xy, z] = [x, z]^y [y, z]$ *and* $[x, yz] = [x, z][x, y]^z$*.*
*(ii) If* $A$ *and* $B$ *are subgroups of* $G$ *such that* $G = \langle A \cup B \rangle = \langle A, B \rangle$*, then the commutator subgroup* $[A, B]$ *is a normal subgroup of* $G$*.*

**Theorem 3** *Every nilpotent group is weakly nilpotent.*

***Proof.*** Let $G$ be a nilpotent group of class $c$ and let

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \gamma_3(G) \supseteq \ldots \gamma_c(G) \supseteq \gamma_{c+1}(G) = 1$$

be a lower central series for $G$. If $H$ is a subgroup of $G$ such that $G = HG'$, then using Theorem 2, we have $G' = H'[H, G']G'' \subseteq H'\gamma_3(G)$, and hence, $G = HG' \subseteq HH'\gamma_3(G) = H\gamma_3(G)$. Consequently, $G = H\gamma_3(G)$. An easy induction shows that $G = H\gamma_k(G)$ for all $k \geq 1$. Thus, if $k = c + 1$, then $G = H$ and therefore, $G$ is weakly nilpotent. □

**Theorem 4** *A finite group* $G$ *is nilpotent if and only if it is weakly nilpotent.*

***Proof.*** The proof follows immediately from the fact that $\Phi(G)$ is the intersection of all maximal subgroups of $G$ and a finite group is nilpotent if and only if $G' \subseteq \Phi(G)$. □

From the above discussion, it is clear that finite weakly nilpotent groups are precisely finite nilpotent groups. Since finite nilpotent group have a well-known structural properties, it is interesting to study the infinite weakly nilpotent groups. Any weakly nilpotent group which is not nilpotent must be an infinite group. In this direction, we obtain the following counter-examples.

**Example 2** *A free group need not be weakly nilpotent. Let* $F$ *be a free group of rank* 2*. Suppose that* $F = \langle x, y \rangle$*. We take* $R = \langle x^{-1}yx, y^{-1}xy \rangle$*. Since*

$$x = y^{-1}xy(y^{-1}x^{-1}yx) \quad \textit{and} \quad y = x^{-1}yx(x^{-1}y^{-1}xy),$$

*we have that* $x, y \in RF'$. *Hence,* $F = RF'$. *Let, if possible,* $R = F$. *Then, we must have* $x, y \in R$. *Put* $a_1 = x^{-1}yx$ *and* $a_2 = y^{-1}xy$. *Then*

$$a_1^{\epsilon_1} a_2^{\epsilon_2} \neq 1, \quad \textit{where } \epsilon_i = \pm 1.$$

*Since* $x \in R$, *it folows that*

$$x = c_1^{\epsilon_1} c_2^{\epsilon_2} \cdots c_k^{\epsilon_k}, \quad c_i \in \{a_1, a_2\}.$$

*Clearly, the right hand side of obtained relation is a reduced word of length* $3k$, *but* $x$ *itself is a word of length* 1. *This is a contradiction. Thus,* $x$ *does not belong to* $R$ *and* $R \neq F$. *Hence,* $F$ *is not a weakly nilpotent group.*

**Example 3** *A weakly nilpotent group need not be nilpotent. Let* $F$ *be a free group of rank* $\geq 2$ *and let* $F_n$ *denote the* $n$*-th term of the lower central series of* $F$. *Define* $G^{(n)} = F/F_{n+1}$. *Then, each* $G^{(n)}$ *is nilpotent group of class* $n$ *and hence, weakly nilpotent. If we put* $G = \prod_{n \geq 1} G^{(n)}$, *then* $G$ *is an arbitrary product of weakly nilpotent groups and hence,* $G$ *is weakly nilpotent. As each* $G^{(n)}$ *is a subgroup of* $G$, *nilpotency class of* $G$ *is greater than or equal to* $n$. *Since* $n$ *is an arbitrary natural number, it follows that* $G$ *can not be nilpotent.*

**Remark 1** *The above results show that the class of nilpotent groups forms a proper subclass of the class of weakly nilpotent groups. However, for finite groups these two classes coincide.*

**Theorem 5** *A finite direct product of groups is weakly nilpotent if and only if each direct factor is weakly nilpotent.*

***Proof.*** Let $A$ and $B$ be weakly nilpotent groups, and let $G = A \times B$. We have that $G' = A'B'$. Suppose $H$ is a subgroup of $G$ such that $G = HG'$, then $G = HA'B'$. This means

$$\begin{aligned} A &= A \cap HA'B' \\ &= (A \cap HB')A'. \end{aligned}$$

Thus, $A = (A \cap HB')A'$. Since $A$ is weakly nilpotent, $A = A \cap HB'$. This shows that $A \subseteq HB'$ and hence, $G = HA'B' \subseteq HB'$. Therefore $G = HB'$. Next,

$$\begin{aligned} B &= B \cap HB' \\ &= (H \cap B)B'. \end{aligned}$$

Since $B$ is weakly nilpotent, $B = H \cap B$ and thus, $B \subseteq H$. Similarly, $A \subseteq H$. Hence, $G = AB \subseteq H$, i.e., $G = H$. This shows that $G$ is weakly nilpotent. Further, using induction, the result can be extended to any finite number of groups.

Conversely, suppose that $G = G_1 \times G_2 \times \cdots \times G_n$ and $G$ is weakly nilpotent. Put $N = \prod_{j \neq i} G_j$, then the quotient group $G/N \cong G_i$. Since $G$ is weakly nilpotent, $G/N$ is weakly nilpotent by Theorem 1. Hence, each $G_i$ is also weakly nilpotent. This completes the proof. □

# 3 Weakly Solvable Groups

In this section, we present some general properties of weakly solvable groups along with some examples and counter-examples.

**Theorem 6** *Every solvable group is weakly solvable.*

***Proof.*** Let $G$ be a solvable group and $H$ be a normal subgroup of $G$ such that $G = HG^{'}$. Assume that $G \neq H$. Since $G$ is solvable, $G/H$ is also solvable and therefore, $G/H \neq (G/H)^{'}$. But $G = HG^{'}$ implies that $G/H = (G/H)^{'}$, which is a contradiction to our assumption. This shows that $H = G$ and hence, $G$ is weakly solvable. □

It is worth to mention that the converse of Theorem 6 need not be true. To see this, we take the symmetric group of degree $n$, i.e., $G = S_n$ for $n \geq 5$, with $G^{'} = A_n$. Since $A_n$ is the only proper nontrivial normal subgroup of $G$, if $G = HG^{'}$ for any normal subgroup $H$ of $G$, we must have $H = G$. This shows that $G$ is weakly solvable but not solvable.

**Theorem 7** *A weakly nilpotent group is weakly solvable.*

***Proof.*** Let $G$ be a weakly nilpotent group. Then $G^{'}$ does not have any supplement except $G$ and therefore, $G^{'}$ does not have any normal supplement except $G$. Hence, $G$ is a weakly solvable group. □

The converse of Theorem 7 is not true as if we take $G = S_n, n \geq 5$, the symmetric group of degree $n$, then $G$ is weakly solvable. Since $G^{'} = A_n$, if $H$ is a subgroup generated by the transposition $(1\ 2)$, then $G = HG^{'}$ but $G \neq H$. This shows that $G$ is not weakly nilpotent.

**Theorem 8** *The quotient group of a weakly solvable group is weakly solvable but a subgroup need not be weakly solvable.*

***Proof.*** Let $G$ be a weakly solvable group and $G/K$ be the quotient group of $G$ induced by some normal subgroup $K$. Then $(G/K)^{'} = KG^{'}/K$. Let $H/K$ be a normal subgroup of $G/K$ such that $G/K = H/K(G/K)^{'}$. Then

$$G/K = H/K(KG^{'}/K) = HG^{'}/K.$$

From this it follows that $G = HG^{'}$. Now $G$ is weakly solvable, which implies that $G = H$ and $G/K = H/K$. Hence, $G/K$ is weakly solvable. For the second part of the theorem, we take $G = S_n, n \geq 5$, the symmetric group of degree $n$. Clearly, $G$ is weakly solvable. But the subgroup $H = A_n$ is not weakly solvable. This completes the proof. □

**Theorem 9** *Let $N$ be a normal subgroup of a group $G$ such that both $G/N$ and $N$ are weakly solvable. Then $G$ is weakly solvable.*

***Proof.*** Let $H$ be normal subgroup of $G$ such that $G = HG^{'}$. Then $HN$ is a normal subgroup of $G$ and $G/N = HG^{'}/N = (HN/N)(G/N)^{'}$. Since $G/N$ is weakly solvable, $G/N = HN/N$. But then $G = HN$. Using $G^{'} = H^{'}[H, N]N^{'}$, we have

$$G = HG^{'} = H(H^{'}[H, N]N^{'}) = HN^{'}.$$

Also $G = HN^{'}$ implies that $N = (H \cap N)N^{'}$. Since $H \cap N$ is a normal subgroup of $N$ and $N$ is weakly solvable, therefore, $N = H \cap N$. This shows that $N \subseteq H$. Finally, $G = HN^{'}$ implies that $G = H$. Hence, $G$ is weakly solvable. □

Keeping in view Theorem 5, the following result can be proved similarly.

**Theorem 10** *A finite direct product of groups is weakly solvable if and only if each direct factor is weakly solvable.*

It is well known that a finite group is nilpotent if and only if $G^{'} \subseteq \mathrm{Frat}(G)$ ([8, Theorem 5.2.16]). For weakly solvable groups, we have the following analogue.

**Theorem 11** *If a group $G$ admits maximal normal subgroups, then $G$ is weakly solvable if and only if $G^{'} \subseteq nFrat(G) = \Psi(G)$.*

***Proof.*** Suppose that $G$ is weakly solvable. Then $G^{'} \subseteq M$ for every maximal normal subgroup $M$ of $G$. From this it follows that $G^{'} \subseteq \Psi(G)$. Conversely, suppose that $G^{'} \subseteq \Psi(G)$. Let $H$ be a normal subgroup of $G$ such that $G = HG^{'}$. Then $G = H\Psi(G)$. But elements of $\Psi(G)$ are normal non-generators of $G$, it follows that $G = H$. Hence, $G$ is weakly solvable. □

# 4 Main Results

In this section, we present certain characterizations of weakly nilpotent and weakly solvable groups. In addition to this, we obtain a sufficient condition for a free group to be weakly solvable.

**Theorem 12** *A finite group $G$ is nilpotent if and only if $G$ is weakly solvable with $\Phi(G) = \Psi(G)$.*

***Proof.*** Let $G$ be a finite group. First, suppose that $G$ is nilpotent. Then $G$ is weakly nilpotent and hence, $G$ is weakly solvable group. Since in a nilpotent group every maximal subgroup is a normal subgroup, $\Phi(G) = \Psi(G)$.

Conversely, suppose that $G$ is weakly solvable and $\Phi(G) = \Psi(G)$. Let $M$ be a maximal normal subgroup of $G$. Since $G$ is weakly solvable, $G^{'} \subseteq M$. This shows that $G^{'}$ is contained in every maximal subgroup of $G$. Therefore, every maximal subgroup of $G$ is normal. Consequently, $G$ is nilpotent. □

**Theorem 13** *A group $G$ is solvable if and only if $G$ is weakly solvable and $\Psi(G)$ is solvable.*

***Proof.*** Since a solvable group is weakly solvable and a subgroup of a solvable group is solvable, only if part follows immediately. Conversely, suppose that $G$ is weakly solvable. Then $G^{'} \subseteq H$ for any maximal normal subgroup $H$ of $G$. From this it follows that $G^{'} \subseteq \Psi(G)$. Since $\Psi(G)$ is solvable, $G^{'}$ is solvable, and hence, $G$ is solvable. □

**Theorem 14** *Let $G$ be a finitely generated group. Then $G$ is weakly nilpotent if and only if $G/\Phi(G)$ is nilpotent.*

***Proof.*** Suppose that $G/\Phi(G)$ is nilpotent. Let $H$ be a subgroup of $G$ such that $G = HG^{'}$. Then, inductively, we have $G = H\gamma_k(G)$ for all $k \geq 1$. Since $G/\Phi(G)$ is nilpotent, $\gamma_k(G/\Phi(G)) = \{1_{G/\Phi(G)}\}$ for some $k \geq 1$, and thus, $\gamma_k(G)\Phi(G)/\Phi(G) = \{1_{G/\Phi(G)}\}$. From this it follows that $\gamma_k(G) \subseteq \Phi(G)$. Therefore, we have $G = H\gamma_k(G) = H\Phi(G)$. Since $G$ is finitely generated, we may suppose that $H$ is maximal subgroup of $G$. Now $\Phi(G) \subseteq H$ implies that $G = H$. This shows that $G$ is weakly nilpotent.

Conversely, suppose that $G$ is weakly nilpotent. Let $H$ be a maximal subgroup of $G$. Then $HG^{'}$ is a subgroup of $G$ such that $H \subseteq HG^{'} \subseteq G$. If $G = HG^{'}$, then $H = G$, which is a contradiction. Hence, $H = HG^{'}$. But then $G^{'} \subseteq H$. Further, as $H$ is arbitrary, $G^{'} \subseteq \Phi(G)$. This shows that $G$ is nilpotent and hence, $G/\Phi(G)$ is nilpotent. □

For weakly solvable groups, we have the following analogue of Theorem 14.

**Theorem 15** *Let $G$ be a finite group. Then $G$ is weakly solvable if and only if $G/\Psi(G)$ is solvable.*

To this end, next we obtain a sufficient condition for a free group to be weakly solvable.

Let $F$ be a free group and $R$ be a subgroup of $F$ such that $F = RF^{'}$. If $R_n$, $F_n$ denote the $n$-th term of the lower central series of $R$ and $F$, respectively, then

$$\begin{aligned} F^{'} &= [RF^{'}, F] \\ &= [R, F][F^{'}, F] \\ &= [R, RF^{'}][F^{'}, F] \\ &= R^{'}[R, F^{'}][F^{'}, F] \\ &= R^{'}F_3. \end{aligned}$$

Hence, $F_2 = R_2F_3$. Suppose $F_{n-1} = R_{n-1}F_n$. Then

$$\begin{aligned} F_n &= [R_{n-1}F_n, F] \\ &= [R_{n-1}, F][F_n, F] \\ &= [R_{n-1}, RF^{'}][F_n, F] \\ &= R_n[R_{n-1}, F^{'}][F_n, F] \\ &\subseteq R_nF_{n+1} \\ &\subseteq F_n. \end{aligned}$$

This shows that $F_n = R_nF_{n+1}$ for all $n \geq 1$. Inductively, we have that

$$F_n = R_nF_{n+1} = R_nR_{n+1}F_{n+2} = R_nF_{n+2} = R_nF_{n+3} = \cdots .$$

From this it follows that $F_n = R_nF_k$ for all $k \geq n + 1$. Also,

$$F_n/F_{n+1} = R_nF_{n+1}/F_{n+1} \cong R_n/R_n \cap F_{n+1}.$$

Before stating another main result, we prove the following lemma.

**Lemma 3** *Let $F$ be a free group such that $F = RF^{'}$. If $R$ is a subgroup of $F$ such that rankF=rankR, then*

$$R_{n+1} = R \cap F_{n+1}.$$

***Proof.*** Since rank$F$ = rank$R$, we have $F \cong R$. Consequently, $F_n \cong R_n$ and $F_n/F_{n+1} \cong R_n/R_{n+1}$. Consider the following exact sequence of abelian groups:

$$1 \to \frac{R_n \cap F_{n+1}}{R_{n+1}} \to \frac{R_n}{R_{n+1}} \to \frac{R_n}{R_n \cap F_{n+1}} \to 1.$$

Since rank$(R_n/R_{n+1})$ = rank$(R_n/R_n \cap F_{n+1})$, rank$(R_n \cap F_{n+1})/R_{n+1} = 0$. But $(R_n \cap F_{n+1})/R_{n+1}$ is torsion free abelian and hence, it is a trivial group. This shows that $R_n \cap F_{n+1} = R_{n+1}$ for all $n \geq 1$. At this point, we claim that $R \cap F_{n+1} = R_{n+1}$. Since $R_n \cap F_{n+1} = R_{n+1}$, the result trivially holds for $n = 1$. Suppose result holds for $n$, i.e., $R \cap F_n = R_n$. Then

$$\begin{aligned} R \cap F_{n+1} &= R \cap F_{n+1} \cap F_n \\ &= R \cap F_n \cap F_{n+1} \\ &= R_n \cap F_{n+1} \\ &= R_{n+1}. \end{aligned}$$

This completes the proof. □

To this end, we state our final main result that provides the sufficient condition for a free group to be weakly solvable.

**Theorem 16** *Let $F$ be a free group of rank $n$. Then $F$ is weakly solvable if every normal subgroup $R$ of $F$ has rank $n$.*

***Proof.*** Suppose $F$ is a free group of rank $n$ with the assumption that every normal subgroup of $F$ has rank $n$. If $F = RF'$ for some normal subgroup $R$ of $F$, then

$$\frac{F}{F'} = \frac{RF'}{F'} \cong \frac{R}{R \cap F'}.$$

Hence, $\mathrm{rank}(R/R \cap F') = n$. Since the sequence

$$1 \to \frac{R \cap F'}{R'} \to \frac{R}{R'} \to \frac{R}{R \cap F'} \to 1$$

is exact, $\mathrm{rank}(R \cap F')/R' = 0$. Also $(R \cap F')/R'$ is a torsion free abelian group, therefore, $R \cap F' = R'$. Now $R' \subseteq [R, F] \subseteq R \cap F'$ implies that $R' = [R, F]$. Note that

$$[R/R', F/R'] = \frac{[R, F]}{R'} = 1.$$

This shows that $R/R' \subseteq Z(F/R')$. But $Z(F/R') \subseteq R/R'$. Hence, $Z(F/R') = R/R' \neq 1$. It is well-known that $Z(F/R')$ is non-trivial if and only if $F/R$ is a finite group [1]. Thus, $F/R$ is a finite group. Further, by Nielsen-Schrier's theorem ([8, Theorem 6.1.1]), it follows that

$$\mathrm{rank}R = 1 + |F/R|(\mathrm{rank}F - 1).$$

Since $\mathrm{rank}R = \mathrm{rank}F$, we must have $|F/R| = 1$. Hence, $F = R$. This shows that $F$ is weakly solvable. □

# 5 Discussion

We have introduced new classes of weakly nilpotent and weakly solvable groups and studied their properties. We have discussed several examples and counter-examples to support our arguments and characterized a finitely generated weakly nilpotent group. In addition to this, we have characterized the nilpotent and solvable groups in terms of weakly nilpotent and weakly solvable groups. Also, we have obtained a sufficient condition under which a free group turns out to be weakly solvable.

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