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On the invertibility of one integral operator

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Abstract. The present paper considers an integral operator defined on the entire real axis, which differs from the Hilbert transform with terms where kernels are constructed using integral exponential functions. The considered operator has similar properties with respect to the Hilbert transform. The form of the inverse operator is obtained.

Key Words: Integral operator, exponential integral function, $\mathcal{L}$-Wiener-Hopf operator
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Introduction

Suppose $w : \mathbb{R} \to [0; \infty)$ is a weight function from class $A_p(\mathbb{R})$, $1 < p < \infty$, i.e., a function satisfying the following condition:

$$\sup \left[\left(\frac{1}{|I|} \int\limits_I w^p(x)\, dx \right)^{1/p} \cdot \left(\frac{1}{|I|} \int\limits_I w^{-q}(x)\, dx \right)^{1/q} \right] < \infty$$

where I ranges over all bounded intervals of real axis $\mathbb{R}$, $|I|$ is the length of interval I, and $1/p + 1/q = 1$.

By $L_p(\mathbb{R}, w)$, $1 < p < \infty$, we denote Lebesgue space with the following norm:

$$\|f\|_{p.w} := \left(\int\limits_{\mathbb{R}} |f(x)|^p w^p(x)\, dx \right)^{1/p}.$$

Suppose $S : L_p(\mathbb{R}, w) \to L_p(\mathbb{R}, w)$ is a Hilbert transform, that is, a singular integral operator with the Cauchy kernel on the axis:

$$(Sy)(x) = \lim_{r \to 0} \frac{1}{\pi i} \int\limits_{|s-x|>r} \frac{1}{s-x}\, y(s)\, ds, \qquad x \in \mathbb{R}.$$

It is well-known [1, 2] that operator S is bounded in $L_p(\mathbb{R}, w)$ if and only if $w \in A_p(\mathbb{R})$.

Let m and μ are arbitrary positive numbers, $\xi = {}^1/_{2\mu} \ln({}^{m^2}/_{2\mu})$, $d \in \mathbb{C}$. We define function φ by the following expression:

$$\varphi(x) := \frac{m\, e^{\mu\xi}}{2\,\mathrm{ch}(\mu\,(x-\xi))} = \frac{2\,\mu\, m\, e^{\mu x}}{m^2 + 2\,\mu\, e^{2\mu x}}. \tag{1}$$

The main goal of the present work is to inverse the integral operator $T_d : L_p(\mathbb{R}, w) \to L_p(\mathbb{R}, w)$, $1 < p < \infty$, which acts as follows:

$$\begin{gathered}(T_d y)(x) := (Sy)(x) - \\ -\frac{1}{\pi i}\int\limits_{-\infty}^{\infty} \left(Ei(\mu(s-x)) - Ei(\mu(x-s))\right)\, \varphi(x)\,\varphi(s)\,y(s)\,ds + \\ + d\,\varphi(x)\int\limits_{-\infty}^{\infty} \varphi(s)\,y(s)\,ds,\end{gathered} \tag{2}$$

where

$$Ei(x) = \int\limits_{-\infty}^{x} \frac{e^t}{t}\,dt$$

is the exponential integral function. Note that for $x > 0$, we have Cauchy principal value integral. In (2), the integral in the second term is defined as

$$\int\limits_{-\infty}^{\infty} Ei(\pm\mu(x-s))\,\varphi(s)\,y(s)\,ds = \lim_{N\to\infty}\int\limits_{-N}^{N} Ei(\pm\mu(x-s))\,\varphi(s)\,y(s)\,ds$$

in the meaning of convergence in $L_p(\mathbb{R}, w)$.

The technique of the inversion of operator T_d is based on the theory of $\mathcal{L}$-convolution operators which was developed in [3]–[9]. In particular, in Theorem 1 below, it is proved that operator T_d is realized as an $\mathcal{L}$-convolution operator with symbol equal to $v(x) = -\mathrm{sgn}\,(x)$, and for a specific $\mathcal{L}$ it corresponds to some reflectionless potential of the Sturm-Liouville equation (see [10]-[15]). From this fact it follows that operator $T_d^+ : L_p(\mathbb{R}_+, w') \to L_p(\mathbb{R}_+, w')$ ($\mathbb{R}_+ = \{x > 0 : x \in \mathbb{R}\}$, $w' = w|_{\mathbb{R}_+}$) defined by the formula:

$$(T_d^+ y)(x) := \frac{1}{\pi i} \int_0^\infty \frac{1}{s-x}\, y(s)\, ds -$$
$$-\frac{1}{\pi i} \int_0^\infty \left(Ei(\mu(s-x)) - Ei(\mu(x-s))\right) \varphi(x)\, \varphi(s)\, y(s)\, ds +$$
$$+ d\, \varphi(x) \int_0^\infty \varphi(s)\, y(s)\, ds,$$

is the $\mathcal{L}$-Wiener-Hopf operator. The Fredholm theory of a class of operators involving operator T_d^+ is constructed in [8].

Wiener-Hopf integral equations with kernels containing the integral exponential function $Ei(\pm\mu(s-x))$ are often found in applications (see, for example, [16]–[18]). These equations are known in literature as Khvolson-Milne equations. They are usually found in study of physical processes in isotropic environments (see [19]).

1 $\mathcal{L}$-convolution operator

Let $\mathcal{L}$ be a selfadjoint operator generated by the following differential expression:

$$(\ell y)(x) = -y'' - \frac{2\mu^2}{\operatorname{ch}^2(\mu(x-\xi))}\, y(x)\,.$$

Operator $\mathcal{L}$ has only one eigenvalue $\lambda = (i\mu)^2$. This eigenvalue is simple. Function φ defined by (1) is a normal eigenfunction corresponding to this eigenvalue (see [8]).

Let us define functions:

$$u_-(x,\lambda) = e^{i\lambda x}\, \frac{\lambda + i\mu}{\lambda - i\mu} \left(1 - \frac{m\, e^{-\mu x}}{\mu - i\lambda}\, \varphi(x)\right),$$
$$u_+(x,\lambda) = e^{-i\lambda x}\, \frac{\lambda + i\mu}{\lambda - i\mu} \left(1 - \frac{2\mu}{m}\, \frac{e^{\mu x}}{\mu - i\lambda}\, \varphi(x)\right),$$

$x, \lambda \in \mathbb{R}$. It is obvious that potential $v(x) = {-2\mu^2}/{\operatorname{ch}^2(\mu(x-\xi))}$ satisfies the following condition:

$$\int_{-\infty}^\infty (1+|x|)\, v(x)\, dx < \infty$$

and is a reflectionless potential (see [10]–[15]).

Functions $u_{\mp}(x,\lambda)$ generate integrals

$$(U_{\mp}y)(\lambda) = \int_{-\infty}^{\infty} u_{\mp}(x,\lambda)\, y(x)\, dx, \qquad \lambda \in \mathbb{R}.$$

For $a \in L_\infty(\mathbb{R})$, by $m(a)$ we denote the operator of multiplication by a function a ($m(a)y := ay$) acting in functional spaces, and by $J : L_2(\mathbb{R}, w) \to L_2(\mathbb{R}, w)$ we denote the operator acting by the formula $(Jy)(x) = y(-x)$. Under the spectral transform of an operator $\mathcal{L}$, we consider the following operator:

$$U := m(\chi_+)U_- + m(\chi_-)JU_+ : L_2(\mathbb{R}, w) \to L_2(\mathbb{R}, w)$$

where χ_+ (χ_-) is the characteristic function of $\mathbb{R}_+$ ($\mathbb{R}_-$).

It is well-known (see [6, 3, 8]) that this operator is bounded and satisfies the equalities

$$U^*U = I - P, \quad UU^* = I, \tag{3}$$

where I is the identity operator and P is the orthogonal projector in $L_2(\mathbb{R})$ onto the span$\{\varphi\}$

$$(Py)(x) = \varphi(x) \int_{-\infty}^{\infty} \varphi(\tau)\, y(\tau)\, d\tau. \tag{4}$$

Operator defined by (4) is bounded in space $L_p(\mathbb{R}, w)$. Indeed, from Hölder inequality, we obtain

$$\left| \int_{-\infty}^{\infty} \varphi(\tau)\, y(\tau)\, d\tau \right| \le \|\varphi\|_{q,w^{-1}} \|y\|_{p,w}.$$

From here it follows that

$$\|Py\|_{p,w} \le \|\varphi\|_{p,w} \|\varphi\|_{q,w^{-1}} \|y\|_{p,w}.$$

Since $L_p(\mathbb{R}, w) \cap L_2(\mathbb{R})$ is a dense subset of $L_p(\mathbb{R}, w)$, we obtain $P^2 = P$ in $L_p(\mathbb{R}, w)$.

We call function $a \in L_p(\mathbb{R})$ a U-multiplicator and write $a \in \mathcal{M}_{p,w,\mathcal{L}}$ if the map $f \mapsto U^*m(a)Uf$ acts from $L_2(\mathbb{R}) \cap L_p(\mathbb{R}, w)$ to $L_2(\mathbb{R}) \cap L_p(\mathbb{R}, w)$ and there exists a constant $c > 0$ such that for all $f \in L_2(\mathbb{R}) \cap L_p(\mathbb{R}, w)$, we have

$$\|U^*m(a)Uf\|_{p,w} \le c\, \|f\|_{p,w}.$$

It means that we can expand $U^*m(a)U$ by continuity to the whole $L_p(\mathbb{R}, w)$. We denote this operator by $W^\circ_{\mathcal{L}}(a)$ and call it an $\mathcal{L}$-convolution operator on $L_p(\mathbb{R}, w)$ with symbol a (see [8]). If, in this definition, U is replaced by Fourier transform F:

$$(Fy)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} y(x)\, dx,$$

then we obtain multiplicators class $M_{p,w}$ and a classic convolution operator $W^\circ(a)$ (see [2]).

Lemma 1 *Let $w \in A_p(\mathbb{R})$. Then function $a(x) = -\operatorname{sgn}(x)$ belongs to $M_{p,w,\mathcal{L}}$.*

Proof. It is well-known (see Theorem 17.1 in [2]) that a piecewise-continuos function with bounded variation belongs to $\mathcal{M}_{p,w}$. According to Theorem 5.1 in [8], we obtain $\mathcal{M}_{p,w} \subset \mathcal{M}_{p,w,\mathcal{L}}$. □

2 Inversion formula

Theorem 1 *In case $d \neq 0$, operator T_d is bounded and invertible in $L_p(\mathbb{R}, w)$, $1 < p < \infty$. Moreover $T_d^{-1} = T_{d^{-1}}$. In case $d = 0$, operator T_0 is generalized invertible, i.e., $T_0 T_0 T_0 = T_0$. Equation*

$$T_0 = f, \qquad f \in L_p(\mathbb{R}, w)$$

has a solution if and only if

$$\int_{-\infty}^{\infty} f(x)\, \varphi(x)\, dx = 0.$$

If this condition is satisfied, the general solution has the form:

$$y = T_0 f + \alpha\varphi,$$

where α is an arbitrary complex number.

Proof. Let us show that

$$T_d = W^\circ_{\mathcal{L}}(-\operatorname{sgn}). \tag{5}$$

Note that by Lemma 1, operator $W^\circ_{\mathcal{L}}(-\operatorname{sgn})$ is bounded in $L_p(\mathbb{R}, w)$. It is known (see Theorem 3.1 in [5]) that in $L_2(\mathbb{R})$, it holds

$$W^\circ_{\mathcal{L}}(-\operatorname{sgn}) = S - m\,(\varphi\psi) S V_+ - m\,(\varphi\psi^{-1}) S V_-, \tag{6}$$

where operators V_+ and V_- are defined as folllows:

$$(V_+y)(x) = \psi^{-1}(x) \int\limits_x^{+\infty} \varphi(\tau)\, y(\tau)\, d\tau,$$

$$(V_-y)(x) = \psi(x) \int\limits_{-\infty}^{x} \varphi(\tau)\, y(\tau)\, d\tau,$$

and $\psi(x) = me^{-\mu x}$, $x \in \mathbb{R}$.

Let us define functions

$$f^{\pm}(\lambda) = \frac{1}{\mu \pm i\lambda}, \qquad \lambda \in \mathbb{R}.$$

These functions belong to $\mathcal{M}_{p,w}$ (see Proposition 5.1 in [8]), hence, operators $W^\circ(f^{\pm})$ are bounded in $L_p(\mathbb{R}, w)$, $1 < p < \infty$. On the other hand, it is not difficult to obtain from function convolution properties that the following equations:

$$\begin{aligned} W^\circ(f^+)\, m\, (\varphi\psi^{-1}) &= V_+\,, \\ W^\circ(f^-)\, m\, (\varphi\psi^{-1}) &= V_- \end{aligned}$$

are true in $L_2(\mathbb{R})$. From the above equations, it follows that these operators permit extention by continuity to the whole $L_p(\mathbb{R}, w)$.

Assuming the possibility of integrals order change, we obtain

$$(m\,(\varphi\psi) S V_+ y)(x) = \varphi(x) e^{-\mu x} \frac{1}{\pi i} \int\limits_{-\infty}^{\infty} \frac{1}{\tau - x} e^{\mu\tau} \int\limits_{\tau}^{+\infty} \varphi(s)\, y(s)\, ds\, d\tau =$$

$$= \varphi(x) e^{-\mu x} \frac{1}{\pi i} \int\limits_{-\infty}^{\infty} e^{\mu x} \int\limits_{-\infty}^{s} \frac{e^{\mu(\tau - x)}}{\mu(\tau - x)}\, \mu\, d\tau\, \varphi(s)\, y(s)\, ds =$$

$$= \varphi(x) \frac{1}{\pi i} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\mu(s-x)} \frac{e^t}{t}\, dt\, \varphi(s) y(s)\, ds = \varphi(x) \frac{1}{\pi i} \int\limits_{-\infty}^{\infty} Ei(\mu(s - x)) \varphi(s) y(s)\, ds.$$

Analogously,

$$(m\,(\varphi\psi^{-1}) S V_- y)(x) = -\varphi(x) \frac{1}{\pi i} \int\limits_{-\infty}^{\infty} Ei(\mu(x - s))\, \varphi(s)\, y(s)\, ds.$$

Hence, we conclude that equality (5) is true for the functions which have compact support, on the other hand, we have obtained that $W^\circ(f^{\pm})$ allows extention by continuity on the whole $L_p(\mathbb{R}, w)$. Therefore, we can do this with T_d, and equation (5) becomes obvious.

It remains verify that we can change the integrals order.

Let us define

$$I = \int\limits_{-\infty}^{\infty} \frac{1}{\tau - x} e^{\mu(\tau - x)} \int\limits_{\tau}^{+\infty} \varphi(s)\, y(s)\, ds\, d\tau$$

$$I' = \int\limits_{-\infty}^{\infty} Ei(\mu(s-x))\, \varphi(s)\, y(s)\, ds$$

and prove that $I = I'$. For this purpose, we use the approach suggested in § 7.1 in [20]. First we assume that $\operatorname{supp} y \subset [-N, N]$ for some sufficiently large N. Then we can write

$$\begin{aligned} I = &\int\limits_{R\setminus(x-\delta, x+\delta)} \frac{1}{\tau - x}\, e^{\mu(\tau - x)} \int\limits_{\tau}^{+\infty} \varphi(s)\, y(s)\, ds\, d\tau + \\ &+ \int\limits_{x-\delta}^{x+\delta} \frac{e^{\mu(\tau-x)}}{\tau - x} \int\limits_{\tau}^{+\infty} \varphi(s)\, y(s)\, ds\, d\tau = I_0 + I_\delta; \\ I' = &\int\limits_{-\infty}^{\infty} \varphi(s)\, y(s) \int\limits_{(-\infty, s)\setminus(x-\delta, x+\delta)} \frac{e^{\mu(\tau-x)}}{\tau - x}\, d\tau\, ds + \\ &+ \int\limits_{-\infty}^{\infty} \varphi(s)\, y(s) \int\limits_{x-\delta}^{x+\delta} \frac{e^{\mu(\tau-x)}}{\tau - x}\, d\tau\, ds = I_0' + I_\delta', \end{aligned}$$

because in I_0 and I_0' we have ordinary integrals so by Fubbini's theorem they equal to each other. Hence,

$$|I - I'| = |I_\delta - I_\delta'| \leq |I_\delta| + |I_\delta'|.$$

Let $\Gamma(\tau, x)$ satisfy Hölder condition due to both variables. Then

$$\int\limits_{x-\delta}^{x+\delta} \frac{\Gamma(\tau, x)}{\tau - x}\, d\tau \to 0, \qquad \text{when} \qquad \delta \to 0.$$

From here it follows that both I_δ and I_δ' converge to zero, i.e., $I = I'$.

Further, we have:

$$\begin{aligned} T_d = T_0 + P = W^\circ(f^\pm) + dP = U^* m(-\operatorname{sgn}) U + dP^2 = \\ = \begin{pmatrix} U^* & P \end{pmatrix} \begin{pmatrix} m(-\operatorname{sgn}) & 0 \\ 0 & m(d) \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix}. \end{aligned}$$

Similarly,

$$T_d^{-1} = \begin{pmatrix} U^* & P \end{pmatrix} \begin{pmatrix} m(-\mathrm{sgn}) & 0 \\ 0 & m(d^{-1}) \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix}.$$

Using (3), we conclude that:

$$T_d\,T_{d^{-1}} = T_{d^{-1}}\,T_d = I \quad \text{and} \quad T_0\,T_0\,T_0 = T_0.$$

Besides, $T_0^2 = U^*U = I - P$.

Due to the properties of generalized inversion equation, $T_0 y = f$ has a solution if and only if $T_0^2\, y = (I - P)\, y = y$, i.e., $P\, y = 0$.

If this condition is satisfied the general solution is given by

$$y = T_0\, f + u - T_0^2\, u = T_0\, f + P\, u,$$

where u is an arbitrary element of $L_p(\mathbb{R}, w)$. $\square$

Remark 1 *One can consider the case $m = 0$. Then $T_d = S$, and its inverse is also equals to S.*

Remark 2 *Operator T_1, like operator S, is involutive, i.e., $T_1^2 = I$.*

References

[1] R. Hunt, B. Muckenhoupt, and R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, Trans. Amer. Math. Soc., **176** (1973), pp. 227–251. https://doi.org/10.1090/s0002-9947-1973-0312139-8

[2] A. Böttcher, Yu. I. Karlovich, and I. M. Spitkovsky, Convolution operators and factorization of almost periodic matrix functions, Operator Theory: Advances and Applications, Birkhäuser, Basel, **131** (2002). https://doi.org/10.1007/978-3-0348-8152-4

[3] A. G. Kamalyan and I. M. Spitkovsky, On the Fredholm property of a class of convolution-type operators, Math. Notes, **104** (2018), pp. 404–416. https://doi.org/10.1134/s0001434618090080

[4] A. G. Kamalyan, M. I. Karakhanyan, and A. H. Hovhannisyan, On a class of $\mathcal{L}$-Wiener-Hopf operators, J. Contemp. Math. Anal., **53** (2018), pp. 134–138. https://doi.org/10.3103/s1068362318030032

[5] D. Hasanyan, A. Kamalyan, M. Karakhanyan, and I. M. Spitkovsky, Integral operators of the $\mathcal{L}$-convolution type in the case of a reflectionless potential, in Modern Methods in Operator Theory and Harmonic Analysis. OTHA 2018, ed. by A. Karapetyants, V. Kravchenko, and E. Liflyand, Springer Proceedings in Mathematics & Statistics, Springer, Cham, **291** (2019), pp. 175–197. https://doi.org/10.1007/978-3-030-26748-3_11

[6] H. A. Asatryan, A. G. Kamalyan, and M. I. Karakhanyan, On $\mathcal{L}$-convolution type operators with semialmost periodic symbols, Rep. Natl. Acad. Sci. Arm., **119** (2019), no. 1, pp. 22–28.

[7] H. A. Asatryan, A. G. Kamalyan, and M. I. Karakhanyan, On a class of integro-difference equations, Rep. Natl. Acad. Sci. Arm., **119** (2019), no. 2, pp. 103–109.

[8] A. G. Kamalyan and G. A. Kirakosyan, $\mathcal{L}$-Wiener-Hopf operators in weighted spaces in case of reflectionless potential, Journal of Contemporary Mathematical Analysis, **57** (2022), no. 2, pp. 112–121.

[9] H. S. Grigoryan, A. G. Kamalyan, and G. A. Kirakosyan, $\mathcal{L}$-Wiener-Hopf operators with piecewise continuous matrix-valued symbol on Lebesgue spaces with power weight, Reports of NAS RA, **121** (2021), no. 4, pp. 259–264 (in Russian).

[10] L. D. Faddeev, Inverse problem of the quantum scattering theory, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat., VINITI, Moscow, **3** (1974), pp. 93–180.

[11] V. A. Marchenko, Sturm-Liouville operators and their applications, Naukova Dumka, Kiev, (1977).

[12] V. Yurko, Introduction to the theory of inverse spectral problems, Fizmatlit, Moscow, (2007).

[13] P. L. Bhatnagar, Nonlinear waves in one-dimensional dispersive systems, Oxford Univ. Press, (1979).

[14] V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskii, Theory of solitons: method of inverse problem, Nauka, Moscow, (1980).

[15] F. Calogero and A. Degasperis, Spectral transform and solitons, North-Holland, Amsterdam, (1982).

[16] O. D. Khvolson, Grundzüge einer mathematischen theorie der inneren diffusion des lichtes, Proceedings of the Petersburg Academy of Science, **33** (1890), p. 221.

[17] E. A. Milne, Radiative equilibrium in the outer layers of a star, Monthly Notices of the RAS, **81** (1921), pp. 361–375. https://doi.org/10.1093/mnras/81.5.361

[18] V. V. Sobolev, Radiative transfer in the atmospheres of stars and planets, Moscow, Gostexizdat, (1956).

[19] M. G. Krein, Integral equations on a half-linement with a kernel depending upon the difference of the arguments, Amer. Math. Soc. Transl., **22** (1962), pp. 163–238.

[20] F. D. Gakhov, Boundary value problems, Dover Publ. Inc., New York, (1990).

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