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On interior regularity of solutions of a class of almost-hypoelliptic equations

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Abstract

In this paper it is proved that all distributional solutions of the non-degenerate, almost hypoelliptic (hypoelliptic by the one of variables) equation $P(D)u = P(D_1, D_2)u = 0$ are infinitely differentiable in the certain strip in E^2 under a priori assumption that they and its certain derivatives are square integrable with a certain exponential weight.

Key Words: hypoelliptic in respect to a group of variables operator, almost - hypoelliptic operator, weighted Sobolev spaces
Mathematics Subject Classification 2000: 35H10, 12E10

1 Definitions and preliminary facts

After in 1950's L.Hormander introduced the concept of a hypoelliptic differential equation $P(D)u = f$, all distributional solutions u of which with an infinitely differentiable right hand side f are infinitely differentiable (see. [1] - [2]), a problem arose of finding additional assumptions on solutions u of more general, non hypoelliptic equations ensuring that these solutions are infinitely differentiable.

This problem is closely related to the problem of finding suitable weight functions and corresponding weighted Sobolev -type function spaces, where one can study non-hypoelliptic differential equations. As for that kind of equations, note that such simple equations as $-\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^4 u}{\partial x_1^2\partial x_2^2}+\frac{\partial^4 u}{\partial x_2^4}=0$ or $\frac{\partial^2 u}{\partial x_1^2}-\frac{\partial^4 u}{\partial x_1^2\partial x_2^2}+\frac{\partial^2 u}{\partial x_2^2}=0$ are not hypoelliptic, despite the fact that corresponding characteristic polynomials (general symbols) $P_1(D)=D_1^2+D_1^2D_2^2+D_2^4$ and $P_2(D)=-(D_1^2-D_1^2D_2^2+D_2^2)$ of these equations are nondegenerate (regular) (for the corresponding definition see below).

On the way of the quest for new classes of equations which have more or less large set of infinitely differentiable solutions arose notions of partially hypoelliptic, almost - hypoelliptic, global hypoelliptic, hypoelliptic in respect to a group of variables and other classes of differential operators. Several monographs have already been devoted to this topic (see for example [2] - [4] and [15] - [16]). It was proved interior estimates for the solutions of some classes of elliptic, hypoelliptic and other equations as well as estimates near the boundary.

In [5] Ya.S.Bugrov constructed an example of a non-hypoelliptic equation, all solutions of which are infinitely differentiable provided they are square integrable in the half-space together with some of their derivatives.

In [6] and [17] V.I.Burenkov considered the equation $P(D)u=f$ in the cylinder $\Omega=\Omega_m\times E^{n-m}$ with $0\le m\le n$ where Ω_m is an open set in E_m (if $m=0$ then $\Omega=E^n$) and f and all its derivatives are $m-$ locally square integrable on Ω, i.e. square integrable on $Q_m\times E^{n-m}$ for all compacts $Q_m\subset\Omega_m$ (if $m=0$ square integrable on E^n). Necessary and sufficient conditions on P were found ensuring that all solutions u of this equation with any such f which are $m-$locally square integrable on Ω together with some of their derivatives, are of the same class as f (in particular are infinitely differentiable). The class of such operators is essentially wider than the class of hypoelliptic operators.

General objective laws which may be observed in the algebraic conditions of smoothness of the solutions of differential equations induce us to introduce in [7] - [8] the concept of a hypoelliptic number. It turned out that this numerical characteristic divides the set of differential operators into different classes. In this classification the hypoelliptic operators and the hyperbolic (by Petrovski or by Gording (see [9])) operators take up the extreme positions.

In the present article we study the interior regularity of the solutions of a class of equations with the given number of hypoellipticity.

We begin with some notations and definitions: N-is the set of natural numbers, $N_0=N\cup\{0\}$, $N_0^2=N_0\times N_0$-is the set of 2-dimensional multi - indexes, E^2 and R^2 are 2-dimensional Euclidean spaces. For $\xi\in R^2$, $x\in E^2$ and $\alpha\in N_0^2$ we put $|\xi|=\sqrt{\xi_1^2+\xi_2^2}$, $|\alpha|=\alpha_1+\alpha_2$, $\xi^\alpha=\xi_1^{\alpha_1}\cdot\xi_2^{\alpha_2}$, $D^\alpha=D_1^{\alpha_1}\cdot D_2^{\alpha_2}$, where $D_j=\partial/\partial\xi_j$ or $D_j=\frac{1}{i}\cdot\partial/\partial x_j$ $(i^2=-1, j=1,2)\cdot$

For a linear differential operator with constant coefficients $P(D)=\sum_\alpha\gamma_\alpha\cdot D^\alpha$ let $P(\xi)=\sum_\alpha\gamma_\alpha\cdot\xi^\alpha$ be its characteristic polynomial (complete symbol), where the sum extends over a finite collection of multiindexes $(P)=\{\alpha\in N_0^2,\gamma_\alpha\neq 0\}$.

The least convex polygon containing the set $(P)\cup\{0\}$ is called Newton or characteristic polygon of an operator $P(D)$ (polynomial $P(\xi)$, set of multiindexes $(P) \cup \{0\}$) (see [10]) and is denote by $\Re(P)$.

A polygon $\Re$ with vertices from N_0^2 is called complete, (see [10]), if $\Re$ has a vertex in the origin and other vertices on each coordinate axis of N_0^2. A complete polygon $\Re$ is called regular (totally regular), if the external normals of incoordinate sides of $\Re$ all have non-negative (positive) coordinates (see [11] - [12]).

An operator $P(D)$ (a polynomial $P(\xi)$) is called hypoelliptic (see [2]), if all solutions $u \in D'$ $(D' = D'(E^n)$ is the set of distributions) of the equation $P(D)u = f$ are infinitely differentiable, where f is infinitely differentiable, what is the same, all solutions $u \in D'$ of the equation $P(D)u = 0$ are infinitely differentiable.

L.Hormander proved (see [2], theorem 11.1.1 and theorem 11.1.3),that an operator $P(D)$ is hypoelliptic if and only if the following equivalent conditions are satisfied:

1) For every open set $\Omega \subset E^n$ and $u \in D'(\Omega)$ $SingSupp\ u = SingSupp\ P(D)u$

2) If $\Omega \subset E^n, u \in D'(\Omega)$ and $P(D)u = 0$, then $u \in C^\infty(\Omega)$

3) $P^{(\nu)}(\xi)/P(\xi) \equiv D^\nu P(\xi)/P(\xi) \to 0$ as $|\xi| \to \infty, 0 \neq \nu \in N_0^n$

4) If $d_P(\xi)$ - is the distance from $\xi \in R^n$ to $D(P) = \{\zeta; \zeta \in C^n, P(\zeta) = 0\}$, then $d_P(\xi) \to \infty$ when $|\ \xi| \to \infty$.

Definition 1.1 *(see [10]) An operator $P(D)$ (a polynomial $P(\xi)$) is called **non-degenerate** (regular) if there exists a constant $C > 0$ such that*

$$\sum_{\alpha \in \Re(P)} |\ \xi^\alpha| \leq C \cdot [|P(\xi)| + 1] \quad \forall \xi \in R^2\cdot \tag{1.1}$$

Definition 1.2 ([12]) *A polynomial $P(\xi)$) is called **almost - hypoelliptic** if there exist positive constants C and c such that*

$$\sum_\alpha |P^\alpha(\xi)| \equiv \sum_\alpha |D^\alpha P(\xi)| \leq C \cdot |P(\xi)| \quad \forall \xi \in R^2,\ |\xi| \geq c\cdot$$

In the work [13] for the polynomials increasing at infinity and in [14] in the general case it was shown that the polynomial $P(\xi)$ is almost hypoelliptic if and only if there exists a number $\delta > 0$ such that $N_\delta(P) \subset H_\delta^\infty$, where

$$N_\delta(P) = \{u \in D', u(x) \cdot e^{-\delta|x|} \in L_2(E^2), P(D)u = 0, x \in E^2\},$$

$$H_\delta^\infty = \{u \in D', D^\alpha u(x) \cdot e^{-\delta|x|} \in L_2(E^2) \quad \forall \alpha \in N_0^2\}\cdot$$

If this condition for the characteristic polynomial $P(\xi)$ of operator $P(D)$ is hold, then the operator $P(D)$ we call **almost - hypoelliptic**.

It was proved in [12], that the Newton polygon of almost hypoelliptic polynomial is regular.

Applying embedding theorems for weighted Sobolev spaces ([15]) we conclude that from the condition $N_\delta(P) \subset H_\delta^\infty$ it follows that $N_\delta(P) \subset C^\infty(E^2)\cdot$

Definition 1.3 *(see [7]) let* $k = 1,$ *or* $k = 2,$ *and* $m_k = ordP_{\xi_k}.$ *A polynomial* $P(\xi) = P(\xi_1, \xi_2)$ *is called hypoelliptic by variable* $\xi_k,$ *if*

$$D_k^j P(\xi)/P(\xi) \to 0 \;\; as \;\; |\xi| \to \infty \;\; (j = 1, 2, \cdots)\cdot$$

It was proved in [7] (see [7], theorem 0.1) that for the polynomials $P(\xi)$ increasing at infinity this condition is equivalent to condition

$$D_k P(\xi)/P(\xi) \to 0 \;\; as \;\; |\xi| \to \infty\cdot$$

We assume once and for all in this paper that $k = 1$ and for $j \in N$ by $P^{(j,\,0)}(D)$ we denote the differential operator,which is defined by replacing ξ_k by $D_k = -i.\partial/\partial x_k$ $k = 1, 2$ in the polynomial $D_1^j P(\xi) \equiv P^{(j,\,0)}(\xi)\cdot$

For $T > 0$ we denote by $\Omega_T = \{x \in E^2, |x_1| < T, x_2 \in E^1\}$ and put

$$N(P, \Omega_T) = \{u \in D^{'}(\Omega_T), P(D)u = 0, P^{(j,\,0)}(D)u \in L_2(\Omega_T),\; j = 0, 1, \cdots, m_1\}\cdot$$

It is proved in [8] that the polynomial $P(\xi)$ is hypoelliptic by ξ_1 if and only if $N(P, \Omega_T) \subset H^\infty(\Omega_T)$. In this case the operator $P(D)$ we call hypoelliptic by variable x_1.

Through the whole work we also assume that the even natural numbers m_1, m_2 and $\overline{m}_2$ are fixed such that $a = (m_2 - \overline{m}_2)/m_1 \geq 1,$ and we shall study non-degenerate (regular) operator $P(D) = P(D_1, D_2)$ with constant coefficients and with Newton polygon

$$\Re = \Re(m_1, m_2, a) = \{\alpha \in N_0^2,\; \alpha_1 \leq m_1, \;\; a \cdot \alpha_1 + \alpha_2 \leq m_2\}\cdot \tag{1.2}$$

It is obvious that $\Re$ is a regular polygon with the vertices $(0, 0), (m_1, 0),$ $(m_1, \overline{m}_2) = (m_1, m_2 - a \cdot m_1)$ and $(0, m_2)$ and operator $P(D)$ is almost hypoelliptic and hypoelliptic by variable x_1. In this connection, we note that since

$$D_2^{m_2} P(s, 0)/P(s, 0) = m_2! \cdot \gamma_{(m_1, \overline{m}_2)}/\gamma_{(m_1, 0)} \quad (s = 1, 2, \cdots),$$

then the polynomial $P(\xi)$ is not hypoelliptic by $\xi_2,$ while the following inequalities holds

Lemma 1.1 *Let* $P(\xi)$ *be regular polynomial with the Newton polygon* $\Re = \Re(m_1, m_2, a)$ *and* $a \geq 1.$ *There exists a number* $C_1 > 0$ *such that*

$$\sum_{j=0}^{m_1} (1 + |\,\xi|^j)|P^{(j,\,0)}(\xi)| \leq \sum_{j=0}^{m_1} (1 + |\,\xi_1| + |\,\xi_2|^a)^j \cdot |P^{(j,\,0)}(\xi)| \leq$$

$$\leq C_1 \cdot [\,1 + |P(\xi)|\,], \quad \forall \xi \in R^2\cdot \tag{1.3}$$

Proof. The first part of these inequalities is trivial. In order to prove the second part, it is sufficient to show that following monomials

$$\xi_1^{\alpha_1-j}\cdot\xi_2^{\alpha_2+j\cdot a}\quad (j=0,1,\cdots,\alpha_1)\ \forall\alpha\in\Re$$

are estimated by the right - hand side of (1.3).

V. P. Mikhajlov [10] proved that for polynomial $P(\xi)$ satisfying inequality (1.1) and for any point $\sigma\in E_+^2\cap\Re\equiv\{x\in E^2, x_j\geq 0\ \ (j=1,2)\}\cap\Re$ there exists a constant $C(\sigma)>0$ such that

$$|\,\xi^\sigma|\leq C(\sigma)\cdot[1+|P(\xi)|\,]\ \ \forall\xi\in R^2\cdot$$

On the other hand since $a\cdot\sigma_1^{k,j}(\alpha)+\sigma_2^{k,j}(\alpha)\equiv a\cdot[\alpha_1-(j-k)]+[\alpha_2+(j-k)]=a\cdot\alpha_1+\alpha_2$ for any multi - index $\alpha\in\Re$ and for each admittable pair (k,j) then $\sigma^{k,\,j}(\alpha)\in E_+^2\cap\Re\cdot$

Let $C_0=max\{C(\sigma^{k,\,j}(\alpha));\ \ \alpha\in\Re, k=0,1,\cdots,j;\ \ j=0,1,\cdots,m_1\}\cdot$ It remained only to denote by $C_1=M\cdot C_0$, where M is the number of points $\sigma^{k,\,j}(\alpha);\ \alpha\in\Re,\ \ 0\leq k\leq j,\ 0\leq j\leq m_1\cdot$

Lemma 1.1 is proved.

The following inequalities will be needed in Section 2 but besides seems to be of independent interest.

Lemma 1.2 *Let* $a_k, b_k\geq 0\ \ (k=0,1,\cdots,m)$ d *and* t *are positive numbers and*

$$C_1=2\cdot max\{[t(2d+1)]^m,[t(2d+1)]^{-m}\};\ \ C_2=2(d+1)^m\cdot$$

1) If

$$a_0=b_0;\ \ a_k\leq b_k+d\cdot\sum_{j=0}^{k-1}t^{k-j}\cdot a_j\ \ (k=1,2,\cdots,m),\tag{1.4}$$

then

$$\sum_{k=0}^m a_k\leq C_1\cdot\sum_{k=0}^m b_k\cdot\tag{1.5}$$

2) If

$$a_m=b_m;\ \ a_k\leq b_k+d\cdot\sum_{j=k+1}^m a_j\ \ (k=0,1,\cdots,m-1),\tag{1.4$'$}$$

then

$$\sum_{k=0}^m a_k\leq C_2\cdot\sum_{k=0}^m b_k\cdot\tag{1.5$'$}$$

Proof If $t=0$ inequality (1.5) is obvious. Let $t>0$ and $\theta=1/t(2d+1)\cdot$ Multiplying inequality (1.4_k) by θ^k $(k=1,\cdots,m)$ and summarizing by $k=0,1,\cdots,m$ we get

$$\sum_{k=1}^{m}\theta^k\cdot a_k\le\sum_{k=1}^{m}\theta^k\cdot b_k+d\cdot\sum_{k=1}^{m}\theta^k\cdot\sum_{j=0}^{k-1}t^{k-j}\cdot a_j\cdot \tag{1.6}$$

Changing the order of sums in the second item of right hand side in (1.6) we obtain

$$\sum_{k=1}^{m}\theta^k\cdot a_k\le\sum_{k=1}^{m}\theta^k\cdot b_k+d\cdot\sum_{j=0}^{m-1}a_j\cdot\theta^j\cdot\sum_{k=j+1}^{m}\theta^{k-j}\cdot t^{k-j}.$$

Since

$$d\cdot\sum_{k=j+1}^{m}(\theta\cdot t)^{k-j}=d\cdot\sum_{i=1}^{m-j}\frac{1}{(2d+1)^i}<\frac{1}{2},$$

then

$$\sum_{k=1}^{m}\theta^k\cdot a_k\le\sum_{k=1}^{m}\theta^k\cdot b_k+\frac{1}{2}\cdot\sum_{k=1}^{m}\theta^k\cdot a_k+\frac{1}{2}\cdot a_0\cdot$$

This implies

$$a_0+\sum_{k=1}^{m}\theta^k\cdot a_k\le 2\cdot(\sum_{k=1}^{m}\theta^k\cdot b_k+b_0)=2\cdot\sum_{k=0}^{m}\theta^k\cdot b_k a_0,$$

then

$$\sum_{k=0}^{m}\theta^k\cdot a_k\le 2\cdot\sum_{k=1}^{m}\theta^k\cdot b_k\cdot \tag{1.7}$$

If $\theta=1$ then (1.7) leads (1.5) with $C_1=2$. If $\theta\in(0,1)$ then

$$\theta^m\cdot\sum_{k=0}^{m}a_k<\sum_{k=0}^{m}\theta^k\cdot a_k\le 2\cdot\sum_{k=0}^{m}\theta^k\cdot b_k<2\cdot\sum_{k=0}^{m}b_k,$$

which leads (1.5) with $C_1=2.\theta^{-m}\cdot$

If $\theta>1$, then

$$\sum_{k=0}^{m}a_k<\sum_{k=0}^{m}\theta^k\cdot a_k\le 2\cdot\sum_{k=0}^{m}\theta^k\cdot b_k<2\cdot\theta^m\cdot\sum_{k=0}^{m}b_k,$$

which leads (1.5) with $C_1=2.\theta^m\cdot$

Thus, inequality (1.5) is proved.

To prove the inequality $(1.5')$ we denote by $\theta=(2d+1)^{-1}\cdot$ Multiplying inequality $(1.4'_k)$ by θ^{m-k} $(k=1,\cdots,m)$ and summarizing by $k=0,1,\cdots,m-1$ we get

$$\sum_{k=0}^{m-1} a_k \cdot \theta^{m-k} \le \sum_{k=0}^{m-1} b_k \cdot \theta^{m-k} + d \cdot \sum_{k=0}^{m-1} \theta^{m-k} \cdot \sum_{j=k+1}^{m} a_j \cdot \tag{1.8}$$

Repeating the previous argument and using $a_m = b_m$ leads to

$$\sum_{k=0}^{m-1} a_k \cdot \theta^{m-k} \le \sum_{k=0}^{m-1} b_k \cdot \theta^{m-k} + d \cdot \sum_{j=1}^{m} a_j \cdot \sum_{k=0}^{j-1} \theta^{m-k} =$$

$$= \sum_{k=0}^{m-1} b_k \cdot \theta^{m-k} + d \cdot \sum_{j=1}^{m} a_j \cdot \theta^{m-j} \cdot \sum_{k=0}^{j-1} \theta^{j-k} \le$$

$$\le \sum_{k=0}^{m-1} b_k \cdot \theta^{m-k} + \frac{1}{2} \cdot \sum_{k=1}^{m-1} a_k \cdot \theta^{m-k} + \frac{1}{2} \cdot b_m \cdot$$

As in the proof of (1.7) and using $a_m = b_m$ we get

$$a_0 \cdot \theta^m + \frac{1}{2} \cdot \sum_{k=1}^{m-1} a_k \cdot \theta^{m-k} + \frac{1}{2} \cdot a_m \le \sum_{k=0}^{m-1} b_k \cdot \theta^{m-k} + b_m = \sum_{k=0}^{m} b_k \cdot \theta^{m-k} \cdot$$

Since $\theta < 1$ we have from here

$$\frac{\theta^m}{2} \cdot \sum_{k=0}^{m} a_k \le \frac{1}{2} \cdot \sum_{k=0}^{m} a_k \cdot \theta^{m-k} \le \sum_{k=0}^{m} b_k \cdot \theta^{m-k} \le \sum_{k=0}^{m} b_k,$$

which proves the inequality $(1.5') \cdot$

Lemma 1.2 is proved.

2 Investigation of some weighted function spaces

At first we introduce some function spaces needed below. In future it would be more convenient to introduce the equivalent smooth weight function $g \in C^\infty$ instead of weight function $e^{-|x|}$.

Let $g \in C^\infty$ be fixed positive function of one variable $t \in E^1$, satisfying conditions:

1) there exists a constant $\kappa_0 > 1$ such that

$$\kappa_0^{-1}\, e^{-|t|} \le g(t) \le \kappa_0\, e^{-|t|} \quad t \in E^1,$$

2) for each $j \in N_0$ there exists a number $\kappa_j > 0$ such that

$$|g^{(j)}(t)| \equiv |D^j g(t)| \le \kappa_j\, g(t), \quad t \in E^1 \cdot$$

As a function g one can take a regularization of the function $G(t) = e^{-|t|}$ when $|t| > 1$ and $G(t) = e^{-1}$ when $|t| \le 1$ by the non-negative function $\varphi \in C_0^\infty$ for which $\int \varphi(t) dt = 1$ (see, for example [15]).

For $\delta > 0$ we put $g_\delta(t) = g(\delta\, t)$, then

$$\kappa_0^{-1}\, e^{-\delta|t|} \le g_\delta(t) \le \kappa_0\, e^{-\delta|t|}, \;\; t \in E^1, \tag{2.1}$$

$$|\, g_\delta^{(j)}(t)| \equiv |D^j g_\delta(t)| \le \kappa_j\, \delta^j\, g_\delta(t), \;\; t \in E^1, \;\; j = 1, \cdots, m_2. \tag{2.2}$$

Now we present some facts needed below:

Lemma 2.1. *Let* $b > 0$, $G \subset (-b, b)$, $\delta > 0$ and $\sigma_1 = \kappa_0^2\, e^{\delta\, b}$, $\sigma_2 = \kappa_1\, \delta\, b\, \sigma_1$. *Then the inequalities hold*

$$\sup_{\tau \in G} g_{\,\delta}(t + \tau) \;\le\; \sigma_1\, g_{\,\delta}\, (t) \quad \forall t \in E^{\,1}, \tag{2.3}$$

$$\sup_{\tau \in G} |\, g_{\,\delta}\, (t + \tau) - g_{\,\delta}\, (t)| \;\le\; \sigma_2 \;\, g_{\,\delta}\, (t) \quad \forall t \in E^{\,1}. \tag{2.4}$$

Proof. From the inequality $|t + \tau| \ge |t| - |\tau|$ and from (2.1) it follows that for any $t \in E^1$

$$\sup_{\tau \in G} |g_\delta(t+\tau)| \le \kappa_0 \sup_{\tau \in G} e^{-\delta|t+\tau|} \le \kappa_0\, e^{-\delta|t|} \sup_{\tau \in G} e^{\delta|\tau|} \le \kappa_0^2\, e^{\delta . b}\, g_\delta(t),$$

which proves (2.3).

For the proof of (2.4) we assume that the numbers t, τ and δ are fixed and denote by $f(z)$ the function $f(z) = g_{\,\delta}(t + z\,\tau)$. Since f is differentiable the following is obtained for a number $\theta = \theta(t, \tau) \in (0, 1)$

$$|g_{\,\delta}(t + \tau) - g_{\,\delta}(t)| = |f(1) - f(0)| = |f^{'}(\theta)| \le |g_\delta^{'}(t + \theta\, \tau)|\, |\tau|.$$

Since $\theta\, \tau \in (-b, b)$, hence from this and (2.2) - (2.3) it follows that

$$|g_\delta(t + \tau) - g_\delta(t)| \le \kappa_1\, \delta\, b\, g_\delta(t + \theta\tau) \le \kappa_1\, \delta\, b\, \sigma_1\, g_\delta(t).$$

Since the pair (x, y) is arbitrary it proves (2.4).

Lemma 2.1 is proved.

By now we are in a position to investigate some weighted Sobolev type spaces, where we will consider our problems.

For $\delta > 0$ we will denote by $L_{2,\,\delta} = L_{2,\,\delta}(E^2)$ the set of functions $\{u\}$ with finite norms

$$||u||_{L_{2,\,\delta}} = [\int_{E^2} |u(x)|^{\,2}\, e^{-2\;\delta\,|x_2|}\, dx]^{1/2} \tag{2.5}$$

For a domain $\Omega \subset E^2$, a regular polygon $\Re$, a number $\delta > 0$ and function g let us denote

$$H_\delta^{\Re}(\Omega) = \{u; D^\alpha u \in L_{2,\,\delta} \;\; \forall \alpha \in \Re\}.$$

Firstly notice that from the relation (2.1) it follows that in $L_{2,\delta}$ one can introduce the following norm, which is equivalent to norm (2.5)

$$||u||'_{L_{2,\,\delta}} = ||u\,g_\delta||_{L_2}. \tag{2.5'}$$

In addition to this we prove the following statement.

Lemma 2.2 *In* $H^{\Re}_\delta(\Omega)$ *one can introduce the following equivalent norms*

$$||u||' \equiv ||u||'_{H^{\Re}_\delta(\Omega)} = \sum_{|\alpha|\in\Re} ||(D^\alpha u)\,g_\delta||_{L_2(\Omega)}, \tag{2.7}$$

$$||u||'' \equiv ||u||''_{H^{\Re}_\delta(\Omega)} = \sum_{|\alpha|\in\Re} ||D^\alpha(u\,g_\delta)||_{L_2(\Omega)}. \tag{2.8}$$

Proof. We need to show that there exist positive numbers c and C such that for all $u \in H^{\Re}_\delta(\Omega)$

$$c\,||u||'' \le ||u||' \le C\,||u||'' \tag{2.9}$$

The left - hand side of inequality (2.9) follows immediately from the Leibnitz's formula, regularity of $\Re$ and property (2.2) of the function g. To prove the right - hand side of (2.9) it is enough for each $\alpha \in \Re$ to find the number $C_\alpha > 0$ such that

$$||(D^\alpha u)\,g_\delta||_{L_2(\Omega)} \le C_\alpha\,||u||''. \tag{2.10}$$

Let $\alpha \in \Re$ and $\Pi(\alpha) = \{\alpha^k \equiv (\alpha_1, k), k = 0, 1, \cdots, \alpha_2\}$. Since the polygon $\Re$ is regular then $\Pi(\alpha) \subset \Re$ and $D^{\alpha^k} u \in L_{2,\,\delta}$ for all $k = 0, 1, \cdots, \alpha_2$. In the other hand , since the function g only depend on variable x_2 hence $D^{\alpha^k}(u\,g_\delta) = D_2^k(D_1^{\alpha_1} u\,g_\delta)$ and $D^{\alpha^0}(u\,g_\delta) = (D_1^{\alpha_1} u)\,g_\delta$.

Therefore, by the Leibnitz formula we have for all $k = 0, 1, \cdots, \alpha_2$

$$D^{\alpha^k} u\,g_\delta = D^{\alpha^k}(u\,g_\delta) - \sum_{j=0}^{k-1} C_k^j\,D^{(\alpha_1,\,j)} u\,g_\delta^{(k-j)}.$$

Let $d_k = max\{C_k^j\,\kappa_j;\ 0 \le j \le k-1\}$. Applying the property (2.2) of function g we get

$$||D^{\alpha^k} u\,g_\delta||_{L_2(\Omega)} \le ||D^{\alpha^k}(u\,g_\delta)||_{L_2(\Omega)} + d_k \sum_{j=0}^{k-1} \delta^{k-j} ||D^{\alpha^j} u\,g_\delta||_{L_2(\Omega)}.$$

Denote by

$$a_k = ||D^{\alpha^k} u\,g_\delta||_{L_2(\Omega)},\ \ b_k = ||D^{\alpha^k}(u\,g_\delta)||_{L_2(\Omega)} \quad (k = 0, 1, \cdots, \alpha_2),$$

then $a_0 = b_0$ and

$$a_k \le b_k + d_k \sum_{j=0}^{k-1} \delta^{k-j} a_j \quad (k = 1, \cdots, \alpha_2).$$

Applying here Lemma 1.2 we get inequality (2.10). Lemma 2.2 is proved.

Lemma 2.3 *Let* $P(D) = P(D_1, D_2)$ *be an almost - hypoelliptic differential operator with constant coefficients and with (regular) Newton's polygon* $\Re = \Re(P),\ m_2 = ord_{x_2} P$, $T > 0$ *and as above* $\Omega_T = \{x \in E^2, |x_1| < T, x_2 \in E^1\}$. *Then there exist positive numbers* $\Delta_1 = \Delta_1(P)$ *and* $C = C(\Delta_1, P) = C(P)$ *such that for all* $\delta \in (0, \Delta_1),\ \psi \in C_0^\infty(-T, T)$ *and* $u \in H_\delta^{\Re}(\Omega_T)$

$$\sum_{j=0}^{m_2} ||\ P^{(0,\,j)}(D)(u\,\psi(x_1))\,g_\delta(x_2)\ ||_{L_2(E^2)} \le$$

$$\le C\,[\ ||\ P(D)(u\,\psi(x_1))\,g_\delta(x_2)\ ||_{L_2(E^2)} + ||\ u\,\psi\,g\ ||_{L_2(E^2)}\ ], \tag{2.11}$$

where as above we assume that the functions $u.\psi$ *are continued outside of* Ω_T *by zero.*

Proof. Since the polygon $\Re$ is regular then $u\,\psi \in H_\delta^{\Re}(E^2)$ for any $\psi \in C_0^\infty(-T, T)$ and $u \in H_\delta^{\Re}(\Omega_T)$. Applying the generalized Leibnitz formula (see [16], formula (1.1.10)) we deduce that for all $j = 0, 1, \cdots, m_2,\ \psi \in C_0^\infty(-T, T)$ and $u \in H_\delta^{\Re}(\Omega_T)$ (below $||.|| = ||.||_{L_2(E^2)}$)

$$||\ P^{(0,\,j)}(D)(u.\psi)\,g_\delta\ || \le ||\ P^{(0,\,j)}(D)(u\,\psi\,g_\delta)\ ||+$$

$$+\sum_{l=1}^{m_2-j} \frac{1}{l!}\,||\ P^{(0,\,j+l)}(D)(u\,\psi)\,g_\delta^{(l)}\ ||.$$

In view of property (1.2) of the function g we have from here

$$||\ P^{(0,\,j)}(D)(u\,\psi)\,g_\delta\ || \le ||\ P^{(0,\,j)}(D)(u\,\psi\,g_\delta)\ ||+$$

$$+\sum_{r=j+1}^{m_2} \frac{\kappa_{r-j}}{(r-j)!}\,\delta^{r-j}\,||\ P^{(0,\,r)}(D)(u\,\psi)\,g_\delta\ ||\quad (j = 0, 1, \cdots, m_2).$$

Summarizing these inequalities by j and changing the order of sums the following is obtained

$$\sum_{j=0}^{m_2} ||\ P^{(0,\,j)}(D)(u\,\psi)\,g_\delta\ || \le \sum_{j=0}^{m_2} ||\ P^{(0,\,j)}(D)(u\,\psi\,g_\delta)\ ||+$$

$$+\sum_{j=0}^{m_2} \sum_{r=j+1}^{m_2} \frac{\kappa_{r-j}}{(r-j)!}\,\delta^{r-j}\,||\ P^{(0,\,r)}(D)(u\,\psi)\,g_\delta\ || =$$

$$= \sum_{j=0}^{m_2} ||\ P^{(0,\,j)}(D)(u\,\psi\,g_\delta)\ ||+$$

$$+\sum_{r=1}^{m_2}\sum_{j=0}^{r-1}\frac{\kappa_{r-j}}{(r-j)!}\,\delta^{r-j}\,||\;P^{(0,\,r)}(D)(u.\psi)\,g_\delta\;|| =$$

$$=\sum_{j=0}^{m_2}||\;P^{(0,\,j)}(D)(u\,\psi\,g_\delta)\;||+$$

$$+\sum_{r=1}^{m_2}||\;P^{(0,\,r)}(D)(u\,\psi)\,g_\delta\;||\sum_{j=0}^{r-1}\frac{\kappa_{r-j}}{(r-j)!}\,\delta^{r-j} =$$

$$=\sum_{j=0}^{m_2}||\;P^{(0,\,j)}(D)(u\,\psi\,g_\delta)\;||+$$

$$+\sum_{r=1}^{m_2}||\;P^{(0,\,r)}(D)(u\,\psi)\,g_\delta\;||\sum_{l=1}^{r}\frac{\kappa_l\,\delta^l}{(l)!}\leq$$

$$\leq\sum_{j=0}^{m_2}||\;P^{(0,\,j)}(D)(u\,\psi\,g_\delta)\;||+\kappa\sum_{j=0}^{m_2}||\;P^{(0,\,j)}(D)(u\,\psi\,g_\delta)\;||,$$

where

$$\kappa=\kappa(\delta)=\sum_{l=1}^{m_2}\frac{\kappa_l\,\delta^l}{(l)!}.$$

Choosing number $\delta_1>0$ such that $1-\kappa(\delta_1)=1/2$, we get from here for all $\delta\in(0,\delta_1)$ and $u\in H_\delta^{\Re}(\Omega_T)$

$$\sum_{j=0}^{m_2}||\;P^{(0,\,j)}(D)(u\,\psi)\,g_\delta\;||\leq 2\sum_{j=0}^{m_2}||\;P^{(0,\,j)}(D)(u\,\psi\,g_\delta)\;||.$$

Since operator $P(D)$ is almost hypoelliptic hence using the Fourier transform and the Parseval's equality we can rewrite this inequality as

$$\sum_{j=0}^{m_2}||\;P^{(0,\,j)}(D)(u\,\psi)\,g_\delta)||\leq 2\sum_{j=0}^{m_2}||\;P^{(0,\,j)}(\xi)F(u\,\psi\,g_\delta)(\xi)\;||\leq$$

$$\leq C_1\,[||P(\xi)\,F(u\,\psi\,g_\delta)(\xi)||+||F(u\,\psi\,g_\delta)||]=$$

$$=C_1\,[||P(D)(u\,\psi\,g_\delta)||+||u\,\psi\,g_\delta||]\quad\forall u\in H_\delta^{\Re}(\Omega_T),$$

where $C_1=C_1(P)>0$.

By the Leibnitz's formula we obtain

$$\sum_{j=0}^{m_2}||\;P^{(0,\,j)}(D)(u\,\psi)\,g_\delta)||\leq C_1\,[||P(D)(u\,\psi)\,g_\delta||+$$

$$+\sum_{j=1}^{m_2}\frac{1}{j!}\,||\;P^{(0,\,j)}(D)(u\,\psi)\,g_\delta^{(j)})||+||u\,\psi\,g_\delta||]\quad\forall u\in H_\delta^{\Re}(\Omega_T).$$

Applying once more property (2.2) of the function g we conclude that

$$\sum_{j=0}^{m_2}||\;P^{(0,\,j)}(D)(u\,\psi)\,g_\delta)||\le C_1\,[||P(D)(u\,\psi)\,g_\delta||+$$

$$+\sum_{j=1}^{m_2}\frac{\kappa_j\,\delta^j}{j!}\,||\;P^{(0,\,j)}(D)(u\,\psi)\,g_\delta)||+||u\,\psi\,g_\delta||]\quad\forall u\in H_\delta^{\Re}(\Omega_T).$$

Choosing the number $\Delta_1\in(0,\,\delta_1)$ such that

$$max\{1-\frac{\kappa_j\,\Delta_1^j}{j!};\;1\le j\le m_2\}\ge\frac{1}{2}$$

we get (2.11) for all $\delta\in(0,\Delta_1)$. Lemma 2.3 is proved.

In this lemma we assume the operator $P(D)$ be almost hypoelliptic as well regular

Lemma 2.4 *Let $P(D)$ be regular almost hypoelliptic operator with Newton's polygon $\Re$, $m_j=ord_{x_j}P\;(j=1,2)$, $T>0$ and the number $\Delta_1=\Delta_1(P)$ is defined as in Lemma 1.3. Then, there exists a constant C ¿ 0 such that for all $\delta\in(0,\Delta_1)$ $\psi\in C_0^\infty(-T,T)$ and $u\in H_\delta^{\Re}(\Omega_T)$ we have*

$$\sum_{\alpha\in\Re}||D^\alpha(u.\psi).g_\delta||_{L_2(E^2)}\le C\,[\,||P(D)(u\,\psi)\,g_\delta\,||_{L_2(E^2)}\;+||\,u\,\psi\,g_\delta\,||_{L_2(E^2)}\,].$$

Proof. Since the polygon $\Re$ is regular then by Lemma 2.2 there exists a constant $C_1>0$ such that for all $\delta\in(0,\Delta_1)$ (below $||\cdot||=||\cdot||_{L_2(E^2)}$)

$$\sum_{\alpha\in\Re}||D^\alpha(u\,\psi)\,g_\delta||\le C_1\sum_{\alpha\in\Re}||D^\alpha(u\,\psi g_\delta)||\;\;\forall u\in H_\delta^{\Re}(\Omega_T)\cdot$$

Since the operator $P(D)$ is regular hence by Parseval's formula we get the following inequality with a constant $C_2=C_2(P,g)>0$

$$\sum_{\alpha\in\Re}||D^\alpha(u\,\psi)\,g_\delta||\le C_1\sum_{\alpha\in\Re}||\xi^\alpha F(u\,\psi\,g_\delta)||\le$$

$$\le C_2\,[||P(\xi)\,F(u\,\psi\,g_\delta)||+||F(u\,\psi\,g_\delta)||]=$$

$$=C_2\,[||P(D)(u\,\psi\,g_\delta)||+||(u\,\psi\,g_\delta)||]\;\;\forall u\in H_\delta^{\Re}(\Omega_T)\cdot$$

Applying Leibnitz' formula, the estimate (2.4) and the Lemma 2.3 we have with a positive constant C_3

$$\sum_{\alpha\in\Re}||D^\alpha(u\,\psi)\,g_\delta||\le C_2\,[||P(D)(u\,\psi)\,g_\delta||+$$

$$+\sum_{j=1}^{m_2}\frac{1}{j!}\,||P^{(0,\,j)}(u\,\psi)\,g_\delta^{(j)}||+||u\,\psi\,g_\delta||]\leq$$

$$\leq C_3\,[\sum_{j=1}^{m_2}||P^{(0,\,j)}(u\,\psi)\,g_\delta||+||u\,\psi\,g_\delta||]\ \ \forall u\in H_\delta^{\Re}(\Omega_T)\cdot$$

Lemma 2.4 is proved.

Let as above $T>0,\delta>0,\ \ \Omega_T=\{x\in E^2,\ |x_1|<T\}$ and $L_{2,\,\delta}(\Omega_T)=\{u;\ u.g_\delta(x_2)\in L_2(\Omega_T)\}$. For $\varepsilon>0, S_1=\{x\in E^2,|x|<1\}$ and the functions $u\in L_{2,\,\delta}(\Omega_T),\ \varphi\in C_0^\infty(S_1)$, $\varphi\geq 0,\int\varphi dx=1$ and $\psi\in C_0^\infty(-T,T)$ we put $\varphi_\varepsilon(x)=\varepsilon^{-2}\,\varphi(\frac{x}{\varepsilon}),\ \ u_\varepsilon=u\,\psi*\varphi_\varepsilon$.

Without changing notations as above we may assume that the function $u\,\psi$ is continued in E^2 such that $u(x)\,\psi(x_1)=0$ when $x\notin\Omega_T$, then it is clear that $u\,\psi\in L_{2,\,\delta}(E^2)$.

For any $k\in N_0$ and regular polygon $\Re$ we denote by $\Re_k$ the Newton polygon of set $\{\alpha\in N_0^2;\alpha=\beta+\gamma;\beta,\gamma\in N_0^2,\beta\in\Re,|\gamma|\leq k\}$ and put

$$H_\delta^{\Re\infty}(\Omega_T)=\bigcap_{k=0}^{\infty}H_\delta^{\Re\,k}(\Omega_T).$$

Notice that : a) the set $H_\delta^{\Re\infty}$ does not depend on $\Re$, i.e. $H_\delta^{\Re\infty}=H_\delta^{\Re'\infty}$ for any pair regular polygons $\Re$ and $\Re'$. Therefore, hereinafter the set $H_\delta^{\Re\infty}$ we denote by H_δ^∞, b) $L_{2,\,\delta}$ and $H_\delta^{\Re}$ are Banach spaces and H_δ^∞ is Frechet space for any $\delta>0$, weight function g and regular polygon $\Re$.

Lemma 2.5 *Let* $u\in L_{2,\,\delta}(\Omega_T)$, *then*

1) $u_\varepsilon\in H_\delta^\infty(E^2)$ *for any* $\varepsilon>0$

2)$||u_\varepsilon-u\,\psi||_{L_{2,\,\delta}(E^2)}\to 0\quad as\quad\varepsilon\to+o.$

Proof. By Lemma 2.2 and the Leibnitz formula we obtain for any regular polygon $\Re$

$$||\,u_\varepsilon\,||_{\,H_\delta^{\Re}(E^2)}=\sum_{\alpha\in\Re}||\,D^{\,\alpha}\,[\int(u\,\psi)(z)\,g_\delta(|x_2-z_2|)\,\varphi_\varepsilon(x-z)dz\,]\,||_{L_2(E^2)}\leq$$

$$\leq\sum_{\alpha\in\Re}\sum_{j=0}^{\alpha_2}C_{\alpha_2}^{j}\,\varepsilon^{-(|\alpha|-j)}\,||\int|(u\psi)(z)|\,|g_\delta^{(j)}(|x_2-z_2|)|\,|(D^{(\alpha_1,\,\alpha_2-j)}\varphi)_\varepsilon(x-z)|dz||_{L_2(E^2)}.$$

Applying properties (2.2) and (2.3) of function g and the Young's inequality we obtain from this

$$||\,u_\varepsilon\,||_{\,H_\delta^{\Re}(E^2)}\leq$$

$$\leq\sum_{\alpha\in\Re}\sum_{j=0}^{\alpha_2}C_{\alpha_2}^{j}\,\kappa_j\,\delta^j\,\varepsilon^{-(|\alpha|-j)}\,\sigma_1(S_\varepsilon)\,||(u\,\psi\,g_\delta)\,(D^{(\alpha_1,\,\alpha_2-j)}\varphi)_\varepsilon||_{L_2(E^2)}\leq$$

$$\leq C_1 \,||u\,\psi\,g_\delta||_{L_2(E^2)}\;||(D^{(\alpha_1,\,\alpha_2-j)}\varphi)_\varepsilon||_{L_1(E^2)} \leq C_2\,||u\,\psi\,g_\delta||_{L_2(\Omega_T)},$$

where C_1 and C_2 are positive constants.

Since polygon $\Re$ is supposed to be an arbitrary regular polygon, it follows that $u_\varepsilon \in H^\infty_\delta(E^2)$. The proof of the first part of lemma is complete.

The proof of the second part concludes from (2.4). We deduce that

$$||u_\varepsilon - u\,\psi||_{L_{2,\,\delta}(E^2)} = ||u_\varepsilon\, g_\delta(x_2) - u\,\psi(x_1)\,g_\delta(x_2)||_{L_2(E^2)} \leq$$

$$\leq ||(u\,\psi\,g_\delta)_\varepsilon - (u\,\psi\,g_\delta)||_{L_2(E^2)} + ||(u\,\psi\,g_\delta)_\varepsilon - u_\varepsilon\,g_\delta||_{L_2(E^2)} =$$

$$= ||(u\,\psi\,g_\delta)_\varepsilon - (u\,\psi\,g_\delta)||_{L_2(E^2)} + ||\int u(y)\,\psi(x_1-y_1)\,g_\delta(x_2-y_2)\,\varphi_\varepsilon(y)dy -$$

$$-\int u(x-y)\,\psi(x_1-y_1)\,\varphi_\varepsilon(y) g_\delta(x_2)||_{L_2(E^2)} = ||(u\,\psi\,g_\delta)_\varepsilon - (u\,\psi\,g_\delta)||_{L_2(E^2)} +$$

$$+||\int\; u(x-y)\,\psi(x_1-y_1)\,[g_\delta(x_2-y_2) - g_\delta(x_2)]\,]\,\varphi_\varepsilon(y)dy||_{L_2(E^2)} \leq$$

$$\leq ||(u\,\psi\,g_\delta)_\varepsilon - (u\,\psi\,g_\delta)||_{L_2(E^2)} +$$

$$+||\int |u(y)\,\psi(x_1-y_1)|\,|g_\delta(x_2-y_2) - g_\delta(x_2)|\,\varphi_\varepsilon(y)dy||_{L_2(E^2)} \leq$$

$$\leq ||(u\,\psi\,g_\delta)_\varepsilon - (u\,\psi\,g_\delta)||_{L_2(E^2)} + \sigma_2(\varepsilon)\,||\int |(u\,\psi\,g_\delta)(x-y)|\,\varphi_\varepsilon(y)dy||_{L_2(E^2)} =$$

$$= ||(u\,\psi\,g_\delta)_\varepsilon - (u\,\psi\,g_\delta)||_{L_2(E^2)} + \sigma_2(\varepsilon)\,||\,|u\,\psi\,g_\delta| * \varphi_\varepsilon||_{L_2(E^2)}.$$

Applying here Young's inequality the following is obtained

$$||\,u_\varepsilon - u\,\psi\,||_{L_{2,\,\delta}(E^2)} \leq [\sup_{y\in S_\varepsilon} ||\,(u\,\psi\,g_\delta)(.-y) - (u\,\psi\,g_\delta)(.)\,||_{L_2(E^2)} +$$

$$+\sigma_2(\varepsilon)\,||\,u\,\psi\,g_\delta\,||_{L_2(E^2)}\,]\,||\,\varphi_\varepsilon\,||_{L_1(E^2)}.$$

The second part of lemma follows since $||\,\varphi_\varepsilon\,||_{L_1(E^2)} = 1,\; u + g_\delta \in L_2$ for any $u \in L_{2,\,\delta}(\Omega_T)$ and functions from L_2 are continuous in mean.

Lemma 2.5 is proved.

Let $P(D)$ be nonnegative operator with the regular Newton polygon $\Re = \Re(m_1, m_2, a)$ (see section 0^0), the domain Ω_T and the functions φ, ψ be the same as above and in addition that $\psi(x_1) = 1$ when $|x_1| < \frac{1}{2}.(T+T_1)$ for a number $T_1 \in (0,T)\cdot$ It is assumed that the functions $u, \psi.u$ and $P(D)u$ are continued outside of Ω_T by zero and

$$N(P,\delta,\Omega_T)=\{P(D)u=0,\ P^{(j,\,0)}(D)u\in L_{2,\,\delta}(\Omega_T)\ \ (j=0,1,\dots,m_1)\,\}.$$

For future reference we note the following useful properties of sets $H^{\Re}_{\delta}$ and $N(P,\delta,\Omega_T)$:

Lemma 2.6

$1) u\,\psi\in H^{\Re}_{\delta}(E^2)\ \ if\ \ u\in H^{\Re}_{\delta}(\Omega_T)$

$2) u_\varepsilon\in H^{\Re}_{\delta}(E^2)\ \ if\ \ u\in H^{\Re}_{\delta}(\Omega_T)$

$3) u_\varepsilon\in N(P,\delta,\Omega_{T_1})\ \ if\ \ u\in N(P,\delta,\Omega_T)\ \ and\ \ \varepsilon\in(0,\frac{1}{2}\,(T-T_1))\cdot$

Proof The property 1) follows from the Leibnitz formula and the regularity of polygon $\Re$.

Since $D^\alpha u_\varepsilon(x)=[\,D^\alpha(u.\psi)*\varphi_\varepsilon\,](x)$ for $x\in E^2$ (see for example [15], 6.3. (2)) and $H^{\Re}_{\delta}(\Omega_T)\subset L_{2,\,\delta}(\Omega_T)$ then the property 2) immediately follows from Lemma 1.5 .

Since $\psi(x_1-y_1)=1$ if $|x_1|<T_1$ by definition of ψ and $|y_1|<\varepsilon$ then for any $x\in\Omega_T$, $|x_1|<T_1$

$$u_\varepsilon(x)=\int\limits_{E^2}u(x-y)\,\psi(x_1-y_1)\,\varphi_\varepsilon(y)dy=(u*\varphi_\varepsilon)(x)\cdot$$

This implies that

$$P(D)u_\varepsilon(x)=[\,(P(D)u)*\varphi_\varepsilon)\,](x)=0$$

for $|x_1|<T_1,$ which proves point 3). Lemma 2.6 is proved.

Lemma 2.7 *Let* φ,ψ *be the same functions and operator* $P(D)$ *be the same as above and* $T_1\in(0,T)\cdot$ *Then there exists a number* $C>0$ *such that for any* $\delta\in(0,\Delta_1(P))$ *and* $0<\varepsilon<\frac{1}{2}\,(T-T_1)$

$$1)||\,u_\varepsilon\,||_{H^{\Re}_{\delta}(\Omega_T)}\le C\,||\,u\,||_{H^{\Re}_{\delta}(\Omega_T)}\ \ \forall u\in H^{\Re}_{\delta}(\Omega_T)$$

$$2)\sum_{j=0}^{m_1}||\,P^{(j,\,0)}(D)u_\varepsilon\,||_{L_{2,\,\delta}(\Omega_{T_1})}+||\,u_\varepsilon\,||_{L_{2,\,\delta}(\Omega_{T_1})}\le$$

$$\le C\,\sum_{j=0}^{m_1}||\,P^{(j,\,0)}(D)u\,||_{L_{2,\,\delta}(\Omega_T)}+||\,u\,||_{L_{2,\,\delta}(\Omega_T)}\ \ \forall u\in N(P,\delta,\Omega_T)$$

$$3)||\,u_\varepsilon-u\,||_{H^{\Re}_{\delta}(\Omega_{T_1})}\to 0\ \ as\ \ \varepsilon\to+0\ \ \forall u\in H^{\Re}_{\delta}(\Omega_T)$$

$$4)\sum_{j=0}^{m_1}||\,P^{(j,\,0)}(D)(u_\varepsilon-u)\,||_{L_{2,\,\delta}(\Omega_{T_1})}+$$

$$+||\,u_\varepsilon-u\,||_{L_{2,\,\delta}(\Omega_{T_1})}\to 0\ \ as\ \ \varepsilon\to+0\ \ \forall u\in N(P,\delta,\Omega_T)\cdot$$

Proof By the first point of Lemma 1.6 and according to the property (1.4) of function g the following inequality is obtained

$$|| u_\varepsilon ||_{H^{\Re}_\delta(\Omega_T)} = \sum_{\alpha\in\Re} ||(D^\alpha(u\,\psi) * \varphi_\varepsilon)\, g_\delta||_{L_2(\Omega_T)} \leq$$

$$\leq \sum_{\alpha\in\Re} ||(D^\alpha(u\,\psi).g_\delta) * \varphi_\varepsilon||_{L_2(E^2)} + \sum_{\alpha\in\Re} ||(D^\alpha(u\,\psi)\, g_\delta) * \varphi_\varepsilon -$$

$$-(D^\alpha(u\,\psi) * \varphi_\varepsilon)\, g_\delta||_{L_2(E^2)} \leq \sum_{\alpha\in\Re} ||(D^\alpha(u\,\psi)\, g_\delta) * \varphi_\varepsilon||_{L_2(E^2)} +$$

$$+\sigma_2\,(S_\varepsilon) \sum_{\alpha\in\Re} ||(D^\alpha(u\,\psi)\, g_\delta) * \varphi_\varepsilon||_{L_2(E^2)}.$$

By the definition of function ψ and applying the Young's inequality the following inequality is obtained for a constant $C_1 > 0$

$$|| u_\varepsilon ||_{H^{\Re}_\delta(\Omega_T)} \leq \sum_{\alpha\in\Re} ||D^\alpha(u\,\psi)\, g_\delta||_{L_2(E^2)}\, ||\varphi_\varepsilon||_{L_1(E^2)} +$$

$$+\sigma_2(S_\varepsilon) \sum_{\alpha\in\Re} ||D^\alpha(u\,\psi)\, g_\delta||_{L_2(E^2)}\, ||\varphi_\varepsilon||_{L_1(E^2)} \leq$$

$$\leq C_1 \sum_{\alpha\in\Re} ||D^\alpha(u\,\psi)\, g_\delta||_{L_2(E^2)} = C_1 \sum_{\alpha\in\Re} ||D^\alpha(u\,\psi)\, g_\delta||_{L_2(\Omega_T)}.$$

Because of $\psi \in C_0^\infty(-T, T)$ and the regularity of polygon $\Re$ we get the inequality 1) with a constant $C = C(\sigma_2, \psi) > 0\cdot$

Since $P^{(j,\,0)}(D)u_\varepsilon(x) = [\,(P^{(j,\,0)}(D)u)\,\psi * \varphi_\varepsilon\,](x)$ if $|x_1| < T_1$ and $\varepsilon \in (0, \frac{1}{2}\,(T - T_1))$ for all $j \in N_0$ (see proof of point 3) of Lemma 2.6) then the proof of inequality 2) can be done in a similar way as it was done in the proof of inequality 1).

Let us prove the relation 3). For given $u \in H^{\Re}_\delta(\Omega_T)$ and $\varepsilon \in (0, \frac{1}{2}\,(T - T_1))$, applying point 1) of Lemma 1.6 we have

$$|| u_\varepsilon - u ||_{H^{\Re}_\delta(\Omega_{T_1})} = || u_\varepsilon - u\,\psi ||_{H^{\Re}_\delta(\Omega_{T_1})} \leq || u_\varepsilon - u\,\psi ||_{H^{\Re}_\delta(E^2)} =$$

$$= \sum_{\alpha\in\Re} ||[D^\alpha(u\,\psi) * \varphi_\varepsilon) - D^\alpha(u\,\psi)]\, g_\delta||_{L_2(E^2)}.$$

From here applying the estimate (2.4) and Young's inequality we obtain

$$|| u_\varepsilon - u ||_{H^{\Re}_\delta(\Omega_{T_1})} \leq \sum_{\alpha\in\Re} ||(D^\alpha(u\,\psi)\, g_\delta)_\varepsilon - D^\alpha(u\,\psi)\, g_\delta||_{L_2(E^2)} +$$

$$+ \sum_{\alpha\in\Re} ||(D^\alpha(u\,\psi)\, g_\delta)_\varepsilon - (D^\alpha(u\,\psi))_\varepsilon\, g_\delta||_{L_2(E^2)} \leq$$

$$\leq \sum_{\alpha\in\Re} ||(D^\alpha(u\,\psi)\, g_\delta)_\varepsilon - D^\alpha(u\,\psi)\, g_\delta||_{L_2(E^2)} +$$

$$+\sigma_2(S_\varepsilon)\sum_{\alpha\in\Re}||\,|D^\alpha(u\,\psi)\,g_\delta|*\varphi_\varepsilon||_{L_2(E^2)}||\le$$

$$\le\sum_{\alpha\in\Re}\sup_{y\in S_\varepsilon}||(D^\alpha(u\,\psi)\,g_\delta)(\cdot-y)-D^\alpha(u\,\psi)\,g_\delta||_{L_2(E^2)}\,||\varphi_\varepsilon||_{L_1(E^2)}+$$

$$+\sigma_2(S_\varepsilon)\sum_{\alpha\in\Re}||(D^\alpha(u\,\psi)\,g_\delta)||_{L_2(E^2)}\,||\varphi_\varepsilon||_{L_1(E^2)}\cdot$$

Since by Lemma 2.6 $D^\alpha(u\,\psi)\,g_\delta\in L_2(E^2)$ for all $\alpha\in\Re$, then from here and by the definition of number $\sigma_2(S_\varepsilon)$ we obtain the relation 3).

Because of $P(D)u_\varepsilon(x)=[\,(P(D)u)\psi*\varphi_\varepsilon\,](x)$ if $|x_1|<T_1,\quad \varepsilon\in(0,\frac{1}{2}\,(T-T_1))$, by the definition of function ψ and by the condition $u\in N(P,\delta,\Omega_T)$, then one can proof the relation 4) in the same way as relation 3). Lemma 2.7 is proved.

Lemma 2.8 *Let* $P(D)$ *be a differential operator with constant coefficients and with regular Newton* polygon $\Re$ *and let* $m_2=ordP_{x_2},\ \ \delta\in(0,1)\cdot$ *Then there exists a constant* $c>0$ *such that for all* $u\in H^{\Re}_\delta(\Omega_T)$

$$\sum_{k=0}^{m_2}||(P^{(0,\,k)}(D)u)\,g_\delta(x_2)||_{L_2(\Omega_T)}\le c\sum_{k=0}^{m_2}||P^{(0,\,k)}(D)(u\,g_\delta(x_2))||_{L_2(\Omega_T)}\cdot \qquad (2.12)$$

Proof First of all note that $(P^{(0,\,k)}(D)u)\,g_\delta$ and $P^{(0,\,k)}(D)(u\,g_\delta)$ belong to $L_2(\Omega_T)$ and for each $k=0,1,\cdots,m_2$ denote by

$$a_k=||(P^{(0,\,k)}(D)u)\,g_\delta(x_2)||_{L_2(\Omega_T)},\ \ b_k=||P^{(0,\,k)}(D)(u\,g_\delta(x_2))||_{L_2(\Omega_T)}\ \ \cdot$$

We will prove that the numbers $\{a_k\}$ and $\{b_k\}$ satisfy the conditions of second part of Lemma 1.2. Indeed, since $P^{(0,\,m_2)}=const$, then $a_{m_2}=b_{m_2}$ and by the Leibnitz formula for differential operators (see, for example [16], formula 1.1.10) we get

$$a_k=||\,P^{(0,\,k)}(D)(u\,g_\delta)-\sum_{j=1}^{m-k}\frac{1}{j!}\,[P^{(0,\,k+j)}(D)u]\,g_\delta^{(j)}||_{L_2(\Omega_T)}\le$$

$$\le b_k+\sum_{j=1}^{m-k}\frac{1}{j!}\,||[P^{(0,\,k+j)}(D)u]\,g_\delta^{(j)}||_{L_2(\Omega_T)}\cdot$$

From here applying the property (2.2) of function g we get

$$a_k\le b_k+d\sum_{j=k+1}^{m_2}\delta^j\,||[P^{(0,\,k+j)}(D)u]\,g_\delta||_{L_2(\Omega_T)}\le b_k+d\sum_{j=k+1}^{m_2}a_j,$$

where $d=max\{\kappa_j\ \ 0\le j\le m_2\}\cdot$ The latest one and Lemma 1.2 are proved Lemma 2.8.

For any regular polygon $\Re$, any weight function g, satisfying conditions (2.1) - (2.2) and domain Ω_T we introduce the Sobolev- type space $H^{\Re}_\delta(\Omega_T)$ as above and local space $H^{\Re}_{\delta,\,loc}(\Omega_T)$, corresponding to space $H^{\Re}_\delta(\Omega_T)$ as follows

$$H^{\Re}_{\delta,\,loc}(\Omega_T) = \{u \in H^{\Re}_{\delta}(\Omega_{T_1}) \quad \forall\, T_1 \in (0,\, T)\}\cdot$$

Lemma 2.9 *Let $P(D)$ be almost hypoelliptic operator with (regular) Newton polygon $\Re$ and $m_j = ordP_{x_j}$ $(j = 1, 2)\cdot$ There are exist the numbers $\Delta_2 = \Delta_2(P) > 0$ and $C = C(\Delta_2) > 0$ such that for any $\delta \in (0,\, \Delta_2)$ $\psi \in C_0^\infty(-T, T)$ and for all $u \in H^{\Re}_{\delta,\,loc}(\Omega_T)$*

$$\sum_{k=0}^{m_2} ||\, P^{(0,\,k)}(D)(u.\psi(x_1).g_\delta(x_2))\, ||_{L_2(\Omega_T)} \le$$

$$\le C.[\, ||\, P(D)(u.\psi).g_\delta\, ||_{L_2(\Omega_T)} + ||\, u.\psi.g_\delta\, ||_{L_2(\Omega_T)}\,]\cdot \tag{2.13}$$

Proof First we show that $v \equiv \psi\, u \in H^{\Re}_{\delta}(\Omega_T)$ for any $\psi \in C_0^\infty(-T, T)$ and $u \in H^{\Re}_{\delta,\,loc}(\Omega_T)\cdot$ Let $supp\, \psi \subset [\, -T_2, T_2\,]$ $(T_2 < T),$ and

$$A \equiv \sup_{t \in\, [-T_2, T_2\,]} \{D_1^j \psi(t),\ (j = 0, 1, \cdots, m_1,)\},$$

then by the Leibnitz formula we get

$$\sum_{\alpha\in\Re} ||\, D^\alpha v\, g_\delta\, ||_{L_2(\Omega_T)} = \sum_{\alpha\in\Re} ||\, D^\alpha v\, g_\delta\, ||_{L_2(\Omega_{T_2})} \le$$

$$\le \sum_{\alpha\in\Re} \sum_{j=0}^{\alpha_1} C^j_{\alpha_1} ||\, (D^{(\alpha_1 - j,\, \alpha_2)} u)\, \psi_1^{(j)}\, g_\delta\, ||_{L_2(\Omega_{T_2})} \le$$

$$\le A \sum_{\alpha\in\Re} \sum_{j=0}^{\alpha_1} C^j_{\alpha_1} ||\, (D^{(\alpha_1 - j,\, \alpha_2)} u)\, g_\delta\, ||_{L_2(\Omega_{T_2})} \le A_1\, ||u||_{H^{\Re}_{\delta}(\Omega_T)}$$

with a constant $A_1 > 0$, which proves that $v \in H^{\Re}_{\delta}(\Omega_T)\cdot$

Let now $\omega(x) = v(x)$ if $x \in \Omega_T$ and $\omega(x) = 0$ if $E^2 \setminus \Omega_T\cdot$ It is then clear that $\omega \in H^{\Re}_{\delta}(E^2)\cdot$ Applying the Parseval equality we get

$$\sum_{k=0}^{m_2} ||\, P^{(0,\,k)}(D)(v\, g_\delta)\, ||_{L_2(\Omega_T)} = \sum_{k=0}^{m_2} ||\, P^{(0,\,k)}(D)(\omega\, g_\delta)\, ||_{L_2(E^2)} =$$

$$= \sum_{k=0}^{m_2} ||\, P^{(0,\,k)}(\xi), F(\omega\, g_\delta)\, ||_{L_2(R^2)}, \tag{2.14}$$

where F is the Fourier transform.

Since the operator $P(D)$ is almost hypoelliptic then

$$\sum_{k=0}^{m_2} |\, P^{(0,\,k)}(\xi)\, | \le \rho\, [\, |\, P(\xi)\, | + 1\,] \quad \forall \xi \in R^2$$

with a constant $\rho = \rho(P) > 0\cdot$ Then taking into account (2.14) the following is obtained

$$\sum_{k=0}^{m_2} ||P^{(0,\,k)}(D)(v\,g_\delta)||_{L_2(\Omega_T)} \le \rho\,[\,||P(\xi)F(\omega\,g_\delta)(\xi)||_{L_2(R^2)} + ||\,F(\omega\,g_\delta)||_{L_2(R^2)}\,]\cdot$$

From here applying once more the Parseval equality and the Leibnitz formula we get

$$\sum_{k=0}^{m_2} ||\,P^{(0,\,k)}(D)(v\,g_\delta)\,||_{L_2(\Omega_T)} \le \rho\,[\,||\,P(D)(\omega\,g_\delta)\,||_{L_2(R^2)} + ||\,(\omega\,g_\delta)\,||_{L_2(R^2)}\,] =$$

$$= \rho\,[\,||\,P(D)(v\,g_\delta)\,||_{L_2(\Omega_T)} + ||\,v\,g_\delta\,||_{L_2(\Omega_T)}\,] =$$

$$= \rho\,[\,||\,(P(D)v)\,g_\delta\,||_{L_2(\Omega_T)} + \sum_{k=1}^{m_2} \frac{1}{k!}\,||(P^{(0,\,k)}(D)v).g_\delta^{(k)}||_{L_2(\Omega_T)} +$$

$$+||\,v\,g_\delta\,||_{L_2(\Omega_T)}\,]\cdot$$

Let the number d is the same as in Lemma 2.8 . Then, this inequality together with property (2.2) of function g leads for any $\delta \in (0,1)$

$$\sum_{k=0}^{m_2} ||\,P^{(0,\,k)}(D)(v\,g_\delta)\,||_{L_2(\Omega_T)} \le \rho\,[\,||\,(P(D)v)\,g_\delta)\,||_{L_2(\Omega_T)} +$$

$$+||\,v\,g_\delta\,||_{L_2(\Omega_T)}\,] + \rho\,d \sum_{k=1}^{m_2} \delta^k\,||(P^{(0,\,k)}(D)v)\,g_\delta||_{L_2(\Omega_T)} \le$$

$$\le \rho\,[\,||\,(P(D)v)\,g_\delta)\,||_{L_2(\Omega_T)} + ||\,v\,g_\delta\,||_{L_2(\Omega_T)}\,] +$$

$$+\rho\,d\,\delta \sum_{k=1}^{m_2} ||(P^{(0,\,k)}(D)v)\,g_\delta||_{L_2(\Omega_T)}\cdot \qquad (2.15)$$

Since $v = \psi\,u \in H_\delta^{\Re}(\Omega_T)$ then we can apply the Lemma 2.8 to estimate the third item of the right - hand side of (2.15). It leads

$$\sum_{k=0}^{m_2} ||P^{(0,\,k)}(D)(v\,g_\delta)||_{L_2(\Omega_T)} \le \rho\,[\,||\,(P(D)v)\,g_\delta)\,||_{L_2(\Omega_T)} +$$

$$+||\,v\,g_\delta\,||_{L_2(\Omega_T)}\,] + \rho\,d\,c\,\delta \sum_{k=0}^{m_2} ||P^{(0,\,k)}(D)(v\,g_\delta)||_{L_2(\Omega_T)}\cdot$$

If we choose the number Δ_2 such that $1 - \rho\,d\,c\,\Delta_2 = 1/2,$ i.e. $\Delta_2 = 1/(2\rho\,d\,c)$ then (2.15) leads (2.13) for all $\delta \in (0,\,\Delta_2)\cdot$

Lemma 2.9 is proved.

In the remainder of this paper we assume that $\Delta = min\{\Delta_1, \Delta_2, 1\}$, where the number Δ_1 is taken from Lemma 2.3 and the number Δ_2 from Lemma 2.9.

3 The main result

In this section our main results are stated and proved. We shall study non-degenerate differential operators $P(D) = P(D_1, D_2)$ with constant coefficients and with given regular Newton polygon $\Re = \Re(m_1, m_2, a)$ such that $P(D)$ is almost hypoelliptic and hypoelliptic by variable x_1 . We establish the interior regularity (smoothness) of solutions of equation $P(D)u = 0$ in the strip $\Omega_T \subset E^2$.

First of all we show that the contraction of functions from $N(P, D, \Omega_T)$ on Ω_{T_1} for any $T_1 < T$ belong to $H^{\Re}_{\delta, loc}(\Omega_T) = \{u;\ u \in H^{\Re}_{\delta}(\Omega_{\overline{T}}) \quad \forall \overline{T} < T\}\cdot$ Then we show that $N(P, \delta, \Omega_T) \subset H^{\infty}_{\delta, loc}(\Omega_T)$ and finally we prove that $N_{loc}(P, \delta, \Omega_T) \subset H^{\infty}_{\delta, loc}(\Omega_T)\cdot$

Applying embedding theorems for Sobolev-type spaces (see, for example [15], Theorem 10.4) we get our main result: all solutions $u \in H^{\Re}_{\delta}(\Omega_T)$ of non-degenerate hypoelliptic by variable x_1 equation $P(D)u = 0$ are infinitely differentiable in the strip $\Omega_T \subset E^2$.

Theorem 3.1. Let $P(D)$ *be a mentioned above non-degenerate operator with constant coefficients and with regular Newton polygon* $\Re = \Re(m_1, m_2, a)$ *and let* $\delta \in (0, \Delta(P))\cdot$ *Then for any* $T_1 < T$ *the contraction of function* $u \in N(P, D, \Omega_T)$ on Ω_{T_1} *belongs to* $H^{\Re}_{\delta}(\Omega_{T_1})$, i.e. $N(P, \delta, \Omega_T) \subset H^{\Re}_{\delta, loc}(\Omega_T)\cdot$

Proof. Let fix a function $u \in N(P, D, \Omega_T)$, numbers $T_1, T_2 : \quad 0 < T_1 < T_2 < T$ and the functions

$$\psi_0 \in C_0^{\infty}(-\frac{T + T_2}{2}, \frac{T + T_2}{2}), \quad \psi_1 \in C_0^{\infty}(-\frac{T_1 + T_2}{2}, \frac{T_1 + T_2}{2}),$$

such that $\psi_0(t) = 1$ when $|t| < T_2$, $\psi_1(t) = 1$ when $|t| < T$. Denote that $\psi_1(t) = 1$ on the $supp\psi_1\cdot$

It is assumed that the functions $u, D^{\alpha}u \quad \alpha \in \Re$ and $v = u\,\psi_0$ are continued outside of Ω_T and put $u_\varepsilon = v * \varphi_\varepsilon$, where the function φ and for a given $\varepsilon > o$ the function φ_ε is defined as above .Then by Lemma 2.5 $u_\varepsilon \in H^{\Re}_{\delta}(E^2)$ for any $\varepsilon > 0\cdot$ Since $\psi_1(x_1) = 1$ if $|x_1| < T_1$ and by Lemma 2.2 the two norms in $H^{\Re}_{\delta}(\Omega_T)$ are equivalent, then we have

$$||\, u_\varepsilon\, ||''_{H^{\Re}_{\delta}(\Omega_{T_1})} = \sum_{\alpha \in \Re} ||\, D^{\alpha}(u_\varepsilon\, g_\delta)\, ||_{L_2(\Omega_{T_1})} =$$

$$= \sum_{\alpha \in \Re} ||\, D^{\alpha}(u_\varepsilon\, \psi_1\, g_\delta)\, ||_{L_2(\Omega_{T_1})} = \sum_{\alpha \in \Re} ||\, D^{\alpha}(u_\varepsilon\, \psi_1\, g_\delta)\, ||_{L_2(E^2)}\cdot$$

Since by regularity of the polygon $\Re$ $D^{\alpha}(u_\varepsilon\, \psi_1\, g_\delta) \in L_2$ for all $\alpha \in \Re$ and since operator $P(D)$ is regular, then by the Parseval equality it follows from here with a constant $C_1 = C_1(\Re, P) > 0$ (here $F(w)$ is Fourier transform of function w)

$$||\, u_\varepsilon\, ||''_{H^{\Re}_{\delta}(\Omega_{T_1})} \le \sum_{\alpha \in \Re} ||\, \xi^{\alpha}\, F(u_\varepsilon\, \psi_1\, g_\delta)\, ||_{L_2(R^2)} \le$$

$$\le C_1\, [\, ||\, P(\xi)\, F(u_\varepsilon\, \psi_1\, g_\delta)\, ||_{L_2(R^2)} + ||\, F(u_\varepsilon\, \psi_1\, g_\delta)\, ||_{L_2(R^2)}\,] =$$

$$= C_1\,[\,||\,P(D)\,(u_\varepsilon\,\psi_1\,g_\delta)\,||_{L_2(E^2)} + ||\,(u_\varepsilon\,\psi_1\,g_\delta)\,||_{L_2(E^2)}\,]\cdot$$

Since $\delta \in (0,\,\Delta(P))$ then by the Lemma 2.3 and by the generalized Leibnitz formula (see [16], formula (1.1.10)) we obtain from here with the positive constants C_2, C_3, C_4

$$||\,u_\varepsilon\,||_{H_\delta^{\mathfrak{R}}(\Omega_{T_1})} \le C_2\,[\,||\,(P(D)(u_\varepsilon\,\psi_1))\,g_\delta\,||_{L_2(E^2)} + ||\,(u_\varepsilon\,\psi_1\,g_\delta)\,||_{L_2(E^2)}\,] \le$$

$$\le C_3\left[\sum_{j=0}^{m_1} ||\,(P^{(j,\,0)}(D)u_\varepsilon)\,\psi_1^{(j)}\,g_\delta\,||_{L_2(E^2)} + ||\,(u_\varepsilon\,\psi_1\,g_\delta)\,||_{L_2(E^2)}\right] \le$$

$$\le C_4\left[\sum_{j=0}^{m_1} ||\,(P^{(j,\,0)}(D)u_\varepsilon)\,g_\delta\,||_{L_2(\Omega_{(T_1+T_2)/2})} + ||\,(u_\varepsilon\,g_\delta)\,||_{L_2(\Omega_{(T_1+T_2)/2})}\right]\cdot$$

Let $|\,x_1\,| < (T_1+T_2)/2,\ |\,y_1\,| < \varepsilon$ and $\varepsilon \in (0,\,(T_2-T_1)/2),$ then $\psi_0(x_1-y_1) = 1\cdot$ On the other hand since $u \in N(P,\delta,\Omega_T)$ and $\rho(x,\partial\Omega_T) > \varepsilon$ if $|\,x_1\,| < (T_1+T_2)/2$ and $\varepsilon \in (0,\,(T-T_2)/2)$ (see for example [15], 6.3. (2)) then $P(D)u_\varepsilon(x) = (P(D)u)_\varepsilon(x) = 0$ for $|\,x_1\,| < (T_1+T_2)/2,$ and $\varepsilon \in (0,\,T_3) \equiv (0,\,min\{(T_2-T_1)/2,\,(T-T_2)/2\})\cdot$

In view of above mentioned we now obtain from last inequality

$$||\,u_\varepsilon\,||_{H_\delta^{\mathfrak{R}}(\Omega_{T_1})} \le C_4\,[\sum_{j=1}^{m_1} ||\,(P^{(j,\,0)}(D)u_\varepsilon)\,g_\delta\,||_{L_2(\Omega_{(T_1+T_2)/2})} +$$

$$+||\,(u_\varepsilon\,g_\delta)\,||_{L_2(\Omega_{(T_1+T_2)/2})}\,]\cdot \tag{3.1}$$

Since by definition of set $N(P,\delta,\Omega_T)$ $P^{(j,\,0)}(D)u \in L_{2,\,\delta}(\Omega_T)$ and $P^{(j,\,0)}(D)u_\varepsilon(x) = (P^{(j,\,0)}(D)(u\,\psi_0))_\varepsilon(x)$ $(j = 1,\cdots,m_1)$ for $|\,x_1\,| < (T_1+T_2)/2,$ $\varepsilon \in (0,\,(T-T_2)/2),$ then we get from (3.1)

$$||\,u_\varepsilon\,||_{H_\delta^{\mathfrak{R}}(\Omega_{T_1})} \le C_4\,[\sum_{j=1}^{m_1} ||\,(P^{(j,\,0)}(D)(u\,\psi_0))_\varepsilon\,g_\delta\,||_{L_2(\Omega_{(T_1+T_2)/2})} +$$

$$+||\,(u.\psi_0)_\varepsilon\,g_\delta)\,||_{L_2(\Omega_{(T_1+T_2)/2})}\,] \le$$

$$\le C_4\,[\sum_{j=1}^{m_1} ||\,((P^{(j,\,0)}(D)(u\,\psi_0))\,g_\delta)_\varepsilon\,||_{L_2(\Omega_{(T_1+T_2)/2})} +$$

$$+\sum_{j=1}^{m_1} ||\,((P^{(j,\,0)}(D)(u\,\psi_0)\,g_\delta)_\varepsilon - (P^{(j,\,0)}(D)(u\,\psi_0))_\varepsilon)\,g_\delta\,||_{L_2(\Omega_{(T_1+T_2)/2})} +$$

$$+||(u\,\psi_0\,g_\delta)_\varepsilon - u\,\psi_0\,g_\delta\,||_{L_2(\Omega_{(T_1+T_2)/2})} + ||(u\,\psi_0\,g_\delta)_\varepsilon\,||_{L_2(\Omega_{(T_1+T_2)/2})}\,]\cdot$$

Applying here estimate (2.4) we get

$$|| u_\varepsilon ||_{H_\delta^{\Re}(\Omega_{T_1})} \leq C_4 \, [\, \sigma_2(S_\varepsilon) \sum_{j=1}^{m_1} || \, |(P^{(j,\,0)}(D)(u.\psi_0))\, g_\delta| * \varphi_\varepsilon \, ||_{L_2(\Omega_{(T_1+T_2)/2)})} +$$

$$+ \sum_{j=1}^{m_1} || \, |(P^{(j,\,0)}(D)(u\,\psi_0))\, g_\delta| * \varphi_\varepsilon \, ||_{L_2(\Omega_{(T_1+T_2)/2)})} +$$

$$+ \sigma_2(S_\varepsilon) \, || \, (\, (u\,\psi_0)\, g_\delta) * \varphi_\varepsilon \, ||_{L_2(\Omega_{(T_1+T_2)/2)})} +$$

$$+ || \, |\, (u\,\psi_0))\, g_\delta| * \varphi_\varepsilon \, ||_{L_2(\Omega_{(T_1+T_2)/2)})} \,]\cdot$$

Applying here Young's inequality and then the generalized Leibnitz formula (see [16], formula (1.1.10)) we obtain with positive constants C_5, C_6

$$|| u_\varepsilon ||_{H_\delta^{\Re}(\Omega_{T_1})} \leq C_5 \, [\sum_{j=1}^{m_1} || \, |P^{(j,\,0)}(D)(u\,\psi_0))\, g_\delta| \, ||_{L_2(\Omega_{(T_1+T_2)/2)})} +$$

$$+ || \, u\,\psi_0\, g_\delta \, ||_{L_2(\Omega_{(T_1+T_2)/2)})}] \, || \, \varphi_\varepsilon \, ||_{L_1(E^2)} \leq$$

$$\leq C_5 \, [\sum_{j=1}^{m_1} \sum_{k \geq 0} \frac{1}{k!} \, || \, (P^{(j+k,\,0)}(D) u) \, \psi_0^{(k)} \, g_\delta \, ||_{L_2(\Omega_{(T_1+T_2)/2)})} +$$

$$+ || \, u\,\psi_0\, g_\delta \, ||_{L_2(\Omega_{(T_1+T_2)/2)})} \,] \leq$$

$$\leq C_6 \, [\sum_{j=1}^{m_1} || \, P^{(j,\,0)}(D) u \, g_\delta \, ||_{L_2(\Omega_T)} + || \, u \, g_\delta \, ||_{L_2(\Omega_T)} \,]\cdot \tag{3.2}$$

This shows that the set $\{u_\varepsilon : u \in N(P, \delta, \Omega_T)\}$ is uniformly bounded by ε in $H_\delta^{\Re}(\Omega_{T_1})$ for any $\varepsilon \in (0,\, T_3)\cdot$

Let $\varepsilon, \theta \in (0,\, T_3)\cdot$ In the same way, which we applied to get the inequality (3.1), we can see that

$$||u_\varepsilon - u_\theta \, ||_{H_\delta^{\Re}(\Omega_{T_1})} \leq C_4 \, [\sum_{j=1}^{m_1} || \, P^{(j,\,0)}(D)(u_\varepsilon - u_\theta)\, g_\delta \, ||_{L_2(\Omega_{(T_1+T_2)/2})} +$$

$$+ || \, (u_\varepsilon - u_\theta)\, g_\delta \, ||_{L_2(\Omega_{(T_1+T_2)/2})}\cdot \tag{3.3}$$

Taking into account the fact that

$$P^{(j,\,0)}(D) u_\varepsilon(x) = (P^{(j,\,0)}(D)(u\,\psi_0))_\varepsilon(x)$$

if $|\, x_1 \,| < (T_1 + T_2)/2$ and $\varepsilon \in (0,\, T_3)$ the following is obtained from (3.3)

$$||u_\varepsilon - u_\theta \, ||_{H_\delta^{\Re}(\Omega_{T_1})} \leq C_4 \, [\sum_{j=1}^{m_1} || \, [\, (P^{(j,\,0)}(D)(u\,\psi_0))_\varepsilon -$$

$$-(P^{(j,\,0)}(D)(u\,\psi_0))_\theta\,]\,g_\delta\,||_{L_2(\Omega_{(T_1+T_2)/2})}+$$

$$+||\,[\,(u\,\psi_0)_\varepsilon-(u\,\psi_0)_\theta\,]\,g_\delta\,||_{L_2(\Omega_{(T_1+T_2)/2})}\,]\le$$

$$\le C_4\,[\sum_{j=1}^{m_1}[\,||\,[\,(P^{(j,\,0)}(D)(u\,\psi_0))_\varepsilon)-P^{(j,\,0)}(D)(u\,\psi_0)\,]\,g_\delta\,||_{L_2(\Omega_{(T_1+T_2)/2})}+$$

$$+\sum_{j=1}^{m_1}||\,[\,(P^{(j,\,0)}(D)(u\,\psi_0))_\theta-P^{(j,\,0)}(D)(u\,\psi_0)\,]\,g_\delta\,||_{L_2(\Omega_{(T_1+T_2)/2})}+$$

$$+||\,[\,(u\,\psi_0)_\varepsilon-u\,\psi_0\,]\,g_\delta\,||_{L_2(\Omega_{(T_1+T_2)/2})}+||\,[\,(u\,\psi_0)_\theta-u\,\psi_0\,]\,g_\delta\,||_{L_2(\Omega_{(T_1+T_2)/2})}\,]\cdot$$

Proceeding as in the proof of the second point of Lemma 2.5 and according to the fact that $u\,\psi_0,\ P^{(j,\,0)}(D)(u\,\psi_0)\in L_{2,\,\delta}(E^2)\quad (j=1,\cdots,m_1)$ we get

$$||u_\varepsilon-u_\theta\,||_{H_\delta^{\Re}(\Omega_{T_1})}\to 0\quad as\quad \varepsilon\to+0,\quad \theta\to+0, \tag{3.4}$$

i.e. the set $\{u_\varepsilon;\ \varepsilon\in(0,T_3)\}$ is precompact in $H_\delta^{\Re}(\Omega_{T_1})$ for any $u\in N(P,\delta,\Omega_T)\cdot$

Because of the space $H_\delta^{\Re}(\Omega_{T_1})$ is complete, the operator of generalized differentiation is closed (see for example [15]) and

$$||u_\varepsilon-u\,||_{L_{2,\,\delta}(\Omega_{T_1})}\to 0\quad as\quad \varepsilon\to+0,$$

then we get from here that for any $u\in N(P,\delta,\Omega_T)$ the sequence $\{u_\varepsilon\}$ converges to the contraction of function u on Ω_{T_1} as $\varepsilon\to+0$ by the norm of $H_\delta^{\Re}(\Omega_{T_1})\cdot$ This completes the proof.

Theorem 3.1 is proved.

Remark 3.1 *Note that we have actually proved more (see estimate (2.1)). Namely there exists a constant* $C_0>0$ *such that for all* $\delta\in(0,\,\Delta),\ \ u\in N(P,\delta,\Omega_T)$ and $T_1\in(0,\,T_1)$

$$||\,u\,||_{H_\delta^{\Re}(\Omega_{T_1})}\le C_6.[\sum_{j=1}^{m_1}||\,P^{(j,\,0)}(D)u\,||_{L_{2,\,\delta}}(\Omega_T)+||\,u\,||_{L_{2,\,\delta}}(\Omega_T)\,]\cdot \tag{3.5}$$

This means that for any $T_1<T$ the space $N(P,\delta,\Omega_T)$ is topologically included in $H_\delta^{\Re}(\Omega_{T_1})\cdot$

Corollary 3.1 *Let*

$$N_{loc}(P,\delta,\Omega_T)=\{u;\ u\in N(P,\delta,\Omega_{T_1})\quad\forall\,T_1\in(0,T)\}\cdot$$

It follows from Theorem 2.1 that $N_{loc}(P,\delta,\Omega_T)\subset H^{\Re}_{\delta,\,loc}(\Omega_T)\cdot$

This theorem is a sharpened version of Theorem 3.1

Theorem 3.2 *Let the hypotheses of Theorem 3.1 are fulfilled and* $T>0\cdot$ *Then*

$$N(P,\delta,\Omega_T)\subset H^{\infty}_{\delta,\,loc}(\Omega_T)\cdot$$

Proof It is clear that we only need to show that

$$N(P,\delta,\Omega_T)\subset H^{\Re_m}_{\delta}(\Omega_{T_1}) \tag{3.6}$$

for all $m\in N_0$ and $T_1\in(0,\,T)\cdot$

The proof of (3.6) carried out by induction on $m\cdot$ For $m=0$ the inclusion (3.6) follows from Theorem 3.1. Assume that (3.6) holds for $m\le k;$ we will prove it for $m=k+1\cdot$

Let the numbers $T_1,T_2,T\quad(0<T_1<T_2<T)$ and functions ψ_0,ψ_1,φ are fixed as in the beginning of proof of the Theorem 3.1, $u\in N(P,\delta,\Omega_T)$ and $u_\varepsilon=(u\,\psi_0)*\varphi_\varepsilon$ $(\varepsilon\in(0,min\{(T-T_2)/2,\,(T_2-T_1)/2\})\equiv(0,T_3))\cdot$

Under this assumption first we prove that the set $\{u_\varepsilon\}$ is uniformly bounded in $H^{\Re_{k+1}}_{\delta}(\Omega_{T_1})$ for the $\varepsilon\in(0,T_3)\cdot$ Then we prove that this set is precompact in $H^{\Re_{k+1}}_{\delta}(\Omega_{T_1}),$ which means (without loss of generality) that $\{u_\varepsilon\}$ converges to a function v as $\varepsilon\to+0\cdot$

Since the space $H^{\Re_{k+1}}_{\delta}(\Omega_{T_1})$ is complete and the operator of generalized differentiation is closed then $v\in H^{\Re_{k+1}}_{\delta}(\Omega_T)\cdot$ On the other hand by Lemma 2.7 $\{u_\varepsilon\}$ converges to u as $\varepsilon\to+0,$ i.e. $u=v\in H^{\Re_{k+1}}_{\delta}(\Omega_{T_1})\cdot$

Thus let us prove that the set $\{u_\varepsilon\}$ $\varepsilon\in(0,\,T_3)$ is uniformly bounded in $H^{\Re_{k+1}}_{\delta}(\Omega_{T_1})\cdot$

Let $u\in N(P,\delta,T)$ $(\subset L_{2,\,\delta}(\Omega_{T_1}))\cdot$ By the Lemma 2.5 $u_\varepsilon\in H^{\infty}_{\delta}(\Omega_{T_1})$ for any $\varepsilon>0\cdot$ Moreover, by the inductive assumption $u\in H^{\Re_m}_{\delta}(\Omega_{T_1})$ for any $T_1\in(0,\,T)$ and $m\le k\cdot$ Therefore we get

$$A_{k+1}\equiv||u_\varepsilon||_{H^{\Re_{k+1}}_{\delta}(\Omega_{T_1})}=\sum_{\alpha\in\Re_{k+1}}||D^\alpha u_\varepsilon||_{L_{2,\,\delta}(\Omega_{T_1})}=$$

$$=\sum_{\alpha\in\Re_{k+1}\setminus\Re_k}||D^\alpha u_\varepsilon||_{L_{2,\,\delta}(\Omega_{T_1})}+\sum_{\alpha\in\Re_k}||D^\alpha u_\varepsilon||_{L_{2,\,\delta}(\Omega_{T_1})}\cdot$$

Since by the inductive assumption $D^\alpha u\in L_{2,\,\delta}(\Omega_{T_0})$ for all $T_0\in(0,T)$ and $\alpha\in\Re_k$ then $D^\alpha u_\varepsilon(x)=D^\alpha[\,(u.\psi_0)*\varphi_\varepsilon\,](x)=[D^\alpha(u\psi_0)*\varphi_\varepsilon](x)$ for $x\in\Omega_{T_1}$ and $\varepsilon\in(0,\,T_3)$ (see [15]] , 6.3.(2)). Therefore

$$A_{k+1}=\sum_{\alpha\in\Re_{k+1}\setminus\Re_k}||D^\alpha u_\varepsilon\,g_\delta||_{L_2(\Omega_{T_1})}+\sum_{\alpha\in\Re_k}||(D^\alpha(u\,\psi_0)*\varphi_\varepsilon).g_\delta||_{L_2(\Omega_{T_1})}\cdot \tag{3.7}$$

In order to estimate the second term of right - hand side of (3.7), we apply the estimate (2.4) and Young's inequality, which gives

$$\sum_{\alpha\in\Re_k}||(D^\alpha(u\,\psi_0)*\varphi_\varepsilon)\,g_\delta||_{L_2(\Omega_{T_1})}\le$$

$$\le\sum_{\alpha\in\Re_k}||(D^\alpha(u\,\psi_0)\,g_\delta)*\varphi_\varepsilon-(D^\alpha(u\,\psi_0)*\varphi_\varepsilon)\,g_\delta||_{L_2(E^2)}+$$

$$+\sum_{\alpha\in\Re_k}||(D^{\alpha}(u\,\psi_0),g_\delta)*\varphi_\varepsilon||_{L_2(E^2)}\leq$$

$$\leq\sigma_2(\varepsilon).\sum_{\alpha\in\Re_k}||\,|D^{\alpha}(u.\psi_0).g_\delta|*\varphi_\varepsilon||_{L_2(E^2)}+$$

$$+\sum_{\alpha\in\Re_k}||(D^{\alpha}(u\,\psi_0)\,g_\delta)*\varphi_\varepsilon||_{L_2(E^2)}\leq$$

$$\leq(\sigma_2(\varepsilon)+1)\sum_{\alpha\in\Re_k}||(D^{\alpha}(u\,\psi_0)\,g_\delta)||_{L_2(E^2)}\;||\varphi_\varepsilon||_{L_1(E^2)}\cdot$$

Let $\sigma=max\{\sigma_2(\varepsilon)+1)\,||\,\varphi_\varepsilon\,||_{L_1(E^2)}\};\;\;\varepsilon\in(0,T_3)\cdot$ By the definition of function ψ_0 and by the inductive assumption we get from here that with a constant $C_1>0$

$$\sum_{\alpha\in\Re_k}||(D^{\alpha}(u\,\psi_0)*\varphi_\varepsilon)\,g_\delta||_{L_2(\Omega_{T_1})}\leq\sigma\sum_{\alpha\in\Re_k}||D^{\alpha}(u\,\psi_0)\,g_\delta||_{L_2(\Omega_{(T+T_1)/2})}\leq$$

$$\leq\sigma\sum_{\alpha\in\Re_k}\sum_{\beta_1\leq\alpha_1}C^{\beta_1}_{\alpha_1}\,||(D^{(\alpha_1-\beta_1,\,\alpha_2)}u)\,\psi_0^{(\beta_1)}\,g_\delta\,||_{L_2(\Omega_{(T+T_1)/2})}\leq$$

$$\leq C_1\sum_{\alpha\in\Re_k}||(D^{\alpha}u)\,g_\delta||_{L_2(\Omega_{(T+T_1)/2})}\quad\forall\varepsilon\in(0,T_3)\cdot\tag{3.8}$$

To estimate the first term of right - hand side of (3.7), we apply the definitions of $\Re_m$ and ψ_1 and Lemma2.2 about two norms equivalence, which yields with a constant $C_2>0$

$$\sum_{\alpha\in\Re_{k+1}\setminus\Re_k}||D^{\alpha}u_\varepsilon\,g_\delta\,||_{L_2(\Omega_{T_1})}\leq\sum_{\beta\in\Re}\sum_{|\gamma|=k+1}||\,D^{\beta}(D^{\gamma}u_\varepsilon)\,g_\delta\,||_{L_2(\Omega_{T_1})}=$$

$$=\sum_{\beta\in\Re}\sum_{|\gamma|=k+1}||\,D^{\beta}[\,(D^{\gamma}u_\varepsilon)\,\psi_1\,]\,g_\delta\,||_{L_2(\Omega_{T_1})}\leq$$

$$\leq C_2\sum_{\beta\in\Re}\sum_{|\gamma|=k+1}||\,D^{\beta}[\,(D^{\gamma}u_\varepsilon)\,\psi_1\,g_\delta]\,||_{L_2(E^2)}\cdot$$

By the Lemma 2.5 $(D^{\gamma}u_\varepsilon)\,\psi_1\,g_\delta\in L_2(E^2)$ for any $\gamma\in N_0^2\cdot$ On the other hand since the operator $P(D)$ is regular then applying the Parseval equality get from here that with a constant $C_3>0$

$$\sum_{\alpha\in\Re_{k+1}\setminus\Re_k}||D^{\alpha}u_\varepsilon\,g_\delta\,||_{L_2(\Omega_{T_1})}\leq C_3\sum_{|\gamma|=k+1}\sum_{\beta\in\Re}||\,\xi^{\beta}F[\,(D^{\gamma}u_\varepsilon)\,\psi_1\,g_\delta\,]\,||_{L_2(R^2)}\leq$$

$$\leq C_3\sum_{|\gamma|=k+1}[\,||\,P(\xi)\,F[\,(D^{\gamma}u_\varepsilon)\,\psi_1\,g_\delta\,]\,||_{L_2(R^2)}+||\,F[\,(D^{\gamma}u_\varepsilon)\,\psi_1\,g_\delta\,]\,||_{L_2(R^2)}\,]=$$

$$= C_3 \sum_{|\gamma|=k+1} [\, ||\, P(D)[\,(D^\gamma u_\varepsilon)\, \psi_1\, g_\delta\,]\, ||_{L_2(E^2)} + ||(D^\gamma u_\varepsilon)\, \psi_1\, g_\delta\, ||_{L_2(E^2)} \cdot$$

Applying Lemma 2.9 and definition of function ψ_1 we get from here that with a constant $C_4 > 0$

$$\sum_{\alpha \in \Re_{k+1} \setminus \Re_k} ||D^\alpha u_\varepsilon\, g_\delta\, ||_{L_2(\Omega_{T_1})} \leq$$

$$\leq C_4 \sum_{|\gamma|=k+1} [\, ||\, P(D)[\,(D^\gamma u_\varepsilon)\, \psi_1]\, g_\delta\, ||_{L_2(\Omega_{(T_1+T_2)/2})} +$$

$$+ ||(D^\gamma u_\varepsilon)\, \psi_1\, g_\delta\, ||_{L_2(\Omega_{(T_1+T_2)/2})}] \cdot \qquad (3.9)$$

Since $\gamma \in \Re_k$ when $\gamma = k+1$ then reasoning analogously as in the estimate of the second term of right - hand side of (3.7) we get for the second term of (3. 9) with a constant $C_5 > 0$

$$\sum_{|\gamma|=k+1} ||\,(D^\gamma u_\varepsilon)\, \psi_1\, g_\delta||_{L_2(\Omega_{T_1+T_2)/2})} \leq \sum_{|\gamma|=k+1} ||\,(D^\gamma u_\varepsilon)\, g_\delta||_{L_2(\Omega_{(T_1+T_2)/2})} \leq$$

$$\leq C_5 \sum_{|\gamma| \leq k+1} ||\,[\,(D^\gamma u)\, g_\delta\, ||_{L_2(\Omega_{(T_1+T_2)/2})} \quad \forall \varepsilon \in (0, T_3) \cdot \qquad (3.10)$$

Using the Leibnitz formula we have for the first term of right - hand side of (3.9) with a constant $C_6 > 0$

$$\sum_{|\gamma|=k+1} ||\, P(D)[\,(D^\gamma u_\varepsilon)\, \psi_1]\, g_\delta\, ||_{L_2(\Omega_{(T_1+T_2)/2})} \leq$$

$$\leq \sum_{|\gamma|=k+1} \sum_{j=0}^{m_1} \frac{1}{j!} ||\, P^{(j,\,0)}(D)(D^\gamma u_\varepsilon)\, \psi_1^{(j)}\, g_\delta\, ||_{L_2(\Omega_{(T_1+T_2)/2})} \leq$$

$$\leq C_6 \sum_{|\gamma|=k+1} \sum_{j=0}^{m_1} ||\, P^{(j,\,0)}(D)(D^\gamma u_\varepsilon)\, g_\delta\, ||_{L_2(\Omega_{(T_1+T_2)/2})} \quad \forall \varepsilon \in (0, T_3) \cdot \qquad (3.11)$$

Since $\psi_0(x_1 - y_1) = 1$ when $\varepsilon \in (0, T_3)$, $x \in \Omega_{(T_1+T_2)/2})$ and $|y_2| < \varepsilon$ then $P(D)[\,(D^\gamma u_\varepsilon)](x) = (D^\gamma P(D))(u.\psi_0 * \varphi_\varepsilon)(x) = D^\gamma[(P(D)(u\,\psi_0 * \varphi_\varepsilon)](x) = D^\gamma[P(D)(u\,\psi_0) * \varphi_\varepsilon](x) = 0 \cdot$ Therefore, the following inequality with a constant $C_7 > 0$ is obtained from (3.11)

$$\sum_{|\gamma|=k+1} ||\, P(D)[\,(D^\gamma u_\varepsilon).\psi_1.g_\delta\,]\, ||_{L_2(E^2)} \leq$$

$$\leq C_6 \sum_{|\gamma|=k+1} \sum_{j=1}^{m_1} \frac{1}{j!} ||\,(P^{(j,\,0)}(D) D^\gamma u_\varepsilon)\, \psi_1^{(j)}\, g_\delta\, ||_{L_2(\Omega_{(T_1+T_2)/2})} \leq$$

$$\leq C_7 \sum_{|\gamma|=k+1} \sum_{j=1}^{m_1} ||\, (P^{(j,\,0)}(D) D^\gamma u_\varepsilon)\, g_\delta\, ||_{L_2(\Omega_{(T_1+T_2)/2})}.$$

By Lemma 1.1 $\alpha - (j,\,0) + \gamma \in \Re_k$ for any $\alpha \in \Re$ $(\alpha_1 \neq 1)$, $\gamma \in N_0^2 : |\,\gamma\,| = k+1$ and $j = 1, 2, \cdots$ Therefore in view of inductive assumption and proceeding as in the proof of estimate of second term of right - hand side of (3.7) we get from here with a constant $C_8 > 0$ for any $\varepsilon \in (0, T_3)$

$$\sum_{|\gamma|=k+1} ||\, P(D)[\,(D^\gamma u_\varepsilon)\, \psi_1\, g_\delta\,]\, ||_{L_2(E^2)} \leq$$

$$\leq C_8 \sum_{|\gamma|=k+1} \sum_{j=1}^{m_1} ||\, (P^{(j,\,0)}(D) D^\gamma u_\varepsilon)\, g_\delta\, ||_{L_2(\Omega_{(T_1+T_2)/2})}. \tag{3.12}$$

Applying estimates (3.7) - (3.12) and we deduce that with a number $C_9 > 0$ (independent of ε)

$$\sum_{\alpha \in \Re_{k+1}} ||(D^\alpha u_\varepsilon)\, g_\delta||_{L_2(\Omega_{T_1})} \leq C_9 \quad \forall \varepsilon \in (0, T_3),$$

i.e. the set $\{u_\varepsilon\}$ is uniformly bounded in $H_\delta^{\Re_{k+1}}(\Omega_{T_1})$ for any $T_1 \in (0, T)\cdot$

Proceeding as in the proof of the relation (3.4) and estimate (3. 12) and applying the estimate (2.4), we can prove that

$$||u_\varepsilon - u_\theta||_{H_\delta^{\Re_{k+1}}(\Omega_T)} \to 0 \;\; as \;\; \varepsilon, \theta \to +0,$$

which means that the set $\{u_\varepsilon\}$, $\varepsilon \in (0, T_3)$ is precompact in $H_\delta^{\Re_{k+1}}(\Omega_T)\cdot$ Then without loss of generality one can assume that $u_\varepsilon \to v$ as $\varepsilon \to +0$ by the norm of $H_\delta^{\Re_{k+1}}(\Omega_T)\cdot$ Since this space is complete, the operator of generalized differentiation is closed (see for instance [15], Lemma 2.6.2) and in view of point 3) of Lemma 2.7

$$||u_\varepsilon - u||_{L_{2,\,\delta}(\Omega_{T_1})} \to 0 \;\; as \;\; \varepsilon, \theta \to +0,$$

we get that $v = u \in H_\delta^{\Re_{k+1}}(\Omega_{T_1})\cdot$

Under inductive assumption this means that

$$u \in \bigcap_{k=0}^{\infty} H_\delta^{\Re_{k+1}}(\Omega_{T_1}) \equiv H_\delta^\infty(\Omega_{T_1}) \;\; \forall\; T_1 \in (0, T)\cdot$$

Since $u \in N(P, \delta, \Omega_T)$ is arbitrary then this means that $N(P, \delta, \Omega_T) \subset H_\delta^\infty(\Omega_{T_1})$ $\forall\; T_1 \in (0, T)$, which completes the proof.

Theorem 3.2 is proved.

Corollary 3.2 *Under the hypotheses of Theorem 3.1*

$$N_{loc}(P,\delta,\Omega_T) \subset H^{\infty}_{\delta,\,loc}(\Omega_T)\cdot$$

Summarizing these results together with embedding theorems for weighted Sobolev spaces (see for instance [15]) one can now formulate our main result

Theorem 3.3 $N_{loc}(P,\delta,\Omega_T) \subset C^{\infty}(\Omega_T)\cdot$

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