ARMENIAN JOURNAL OF MATHEMATICS
Volume 3, Number 2, 2010, 61–91

Forced Flow by Powers of the m^{th} Mean Curvature [1]

Chuanxi Wu*, Daping Tian** and Guanghan Li***

School of Mathematics and Computer Science,
Key Laboratory of Applied Mathematics of Hubei Province,
Hubei University, Wuhan, 430062, P. R. China

* cxwu@hubu.edu.cn

** tdp761128@163.com

*** liguanghan@163.com

Received by the editors September 1, 2009; accepted for publication June 7, 2010.

Abstract

In this paper, we consider the m^{th} mean curvature flow of convex hypersurfaces in Euclidean spaces with a general forcing term. Under the assumption that the initial hypersurface is suitably pinched, we show that the flow may shrink to a point in finite time if the forcing term is small, or exist for all time and expand to infinity if the forcing term is large enough. The flow can also converge to a round sphere for some special forcing term and initial hypersurface. Furthermore, the normalization of the flow is carried out so that long time existence and convergence of the rescaled flow are studied. Our work extends Schulze's flow by powers of the mean curvature and Cabezas-Rivas and Sinestrari's volume-preserving flow by powers of the m^{th} mean curvature.

Key Words: m^{th} mean curvature, parabolic equation, maximum principle, forcing term, normalization.
Mathematics Subject Classification 2000: 53C44, 53A05, 35K55

[1]Research partially supported by NSFC (No.10971055) and Project of Hubei Provincial Department of Education (No.T200901).

1 Introduction

Let M^n be a compact oriented manifold of dimension $n \geq 2$ without boundary, and $X_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth hypersurface immersion of M^n which is strictly convex. We consider a smooth family of maps $X_t = X(\cdot, t)$ evolving according to

$$\begin{cases} \frac{\partial}{\partial t} X(x,t) &= \{h(t) - F(x,t)\}\mathbf{v}(x,t), \quad x \in M^n, \\ X(\cdot, 0) &= X_0, \end{cases} \tag{1.1}$$

where F is a symmetric function of the principal curvatures of $M_t = X_t(M^n)$, $\mathbf{v}$ the outer unit normal vector field, and $h(t)$ a nonnegative continuous function.

In order to specify the class of speeds F we are going to consider, let us introduce some notation. We denote by M_t both the immersion $X_t : M^n \to \mathbb{R}^{n+1}$ and the image $X_t(M^n)$. We call $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ the principal curvatures of M_t. We use the letters H and K for the mean curvature and Gauss curvature respectively, i.e. $H = \lambda_1 + \cdots + \lambda_n$ and $K = \lambda_1 \cdots \lambda_n$. In addition, for any integer $m \in \{1, \cdots, n\}$, we denote by H_m the m^{th} mean curvature, defined as

$$H_m = \frac{m!(n-m)!}{n!} \sum_{1 \leq i_1 < \cdots < i_m \leq n} \lambda_{i_1} \cdots \lambda_{i_m}.$$

We remark that $H = nH_1$ and $K = H_n$. In addition, H_2 coincides, up to a constant factor, with the scalar curvature. Thus, the m^{th} mean curvatures can be regarded as generalizations of these quantities.

In this paper we consider the flow (1.1) with the speed F given by a power of the m^{th} mean curvature, namely

$$F(\lambda_1, \cdots, \lambda_n) = H_m^\gamma(\lambda_1, \cdots, \lambda_n) \tag{1.2}$$

for some $\gamma > \frac{1}{m}$. In this way F is a homogenous function of the principal curvatures with a degree $m\gamma > 1$. Our analysis is focused on the behavior of convex hypersurfaces. The curvature flow (1.1) is a strictly parabolic equation and the short time existence easily follows from [8]. Therefore we suppose that the evolution equation (1.1) has a smooth solution on a maximal time interval $[0, T_{\max})$ for some $T_{\max} > 0$. Often different forcing term will lead to different maximal time interval. We always assume that $h(t)$ is continuous in $[0, T_{\max})$.

If $h(t) = 0$ and $m = 1$, (1.1) is just the flow by powers of the mean curvature [14, 15]. In this case, (1.1) is contracting and $T_{\max}$ is finite. If $h(t)$ is the average of powers of the m^{th} mean curvature on M_t, i.e. $h(t) = \int_{M_t} F d\mu_t / \int_{M_t} d\mu_t$, where $d\mu_t$ is the area element of M_t, (1.1) is then the volume-preserving flow by powers of the m^{th} mean curvature [2], which exists on all time $[0, \infty)$, and the solution converges to a round sphere. If $m = \gamma = 1$, (1.1) is just the forced mean curvature flow in Euclidean spaces studied by the third author and Salavessa [9]. Some special cases of certain volume preserving mean curvature flows are studied in [7, 11, 12]. When F is a smooth and symmetric function of homogenous degree one and satisfies suitable properties, the mixed volume preserving curvature flow is studied in [13], and the curvature flow with a general forcing term is studied in [10].

One of the main reasons for authors in [2] to consider the curvature functions (1.2) is to obtain the higher order derivative estimates of curvatures. However, to the authors' knowledge, there is no regular paper on the contractive case, i.e. $h(t)=0$. In this paper, as in [9, 10] we study the curvature flow (1.1) with a general forcing term $h(t)$ such that the limit $\lim_{t\to T_{\max}} h(t)$ exists, which includes the contracting flow. We want to show that if the initial hypersurface is convex and compact, the shape of M_t approaches the shape of a round sphere as $t\to T_{\max}$. In order to describe the shape of the limiting hypersurface, we carry out a normalization as in [6, 9]. For any time t, where the solution $X(\cdot,t)$ of (1.1) exists, let $\psi(t)$ be a positive factor such that the hypersurface $\tilde{M}_t$ given by

$$\tilde{X}(x,t)=\psi(t)X(x,t)$$

has total area equal to $|M_0|$, the area of M_0

$$\int_{\tilde{M}_t} d\tilde{\mu}_t = |M_0|, \qquad \text{for all } t\in[0,T_{\max}).$$

After choosing the new time variable $\tilde{t}(t)=\int_0^t \psi(\tau)^{m\gamma+1}d\tau$, we will see that $\tilde{X}$ satisfies the following evolution equation

$$\begin{cases} \frac{\partial}{\partial \tilde{t}}\tilde{X} = \{\tilde{h}-\tilde{F}\}\tilde{\mathbf{v}}+\frac{1}{n}\tilde{\theta}\tilde{X}, \\ \tilde{X}(\cdot,0)=X_0, \end{cases} \tag{1.3}$$

where $\tilde{h}=\psi^{-m\gamma}h$, $\tilde{\theta}=\psi^{-m\gamma-1}\theta$ and θ is given by

$$\theta=\frac{\int_M (F-h)H\,d\mu}{\int_M d\mu}.$$

In Section 4, we have a time sequence $\{T_i\}$ such that $T_i\to T_{\max}$ as $i\to\infty$, and a limit

$$\lim_{T_i\to T_{\max}} \psi(T_i)=\Lambda.$$

We now state our main theorem:

Theorem 1.1. *Let $n\geq 2$ and M_0 be an n-dimensional smooth, compact and strictly convex hypersurface immersed in $\mathbb{R}^{n+1}$. Given $m\in\{1,\cdots,n\}$ and $\gamma>\frac{1}{m}$ there exists a constant $0<C_0(n,m,\gamma)<\frac{1}{n^n}$ such that, if the initial hypersurface X_0 is pinched in the sense that*

$$\frac{K(x)}{H^n(x)} \geq C_0(n,m,\gamma) \qquad \textit{for all } x\in M^n, \tag{1.4}$$

then for any nonnegative continuous function $h(t)$*, inequality* (1.4) *holds everywhere on* M_t *for all* $t > 0$ *as long as the flow exists, and there exists a unique, smooth solution to the evolution equation* (1.1) *on a maximal time interval* $[0, T_{\max})$ *for some* $T_{\max} > 0$*. If additionally the following limit exists and satisfies*

$$\lim_{t \to T_{\max}} h(t) = \bar{h} < +\infty, \tag{1.5}$$

then we have:

(I) *If* $\Lambda = \infty$*, then* $T_{\max} < \infty$ *and the curvature flow* (1.1) *converges uniformly to a point as* $t \to T_{\max}$*. Moreover the normalized equation* (1.3) *has a solution* $\tilde{X}(x, \tilde{t})$ *for all time* $0 \leq \tilde{t} < \infty$*, and the hypersurfaces* $\tilde{M}(x, \tilde{t})$ *converge, exponentially in the* $C^\infty-$*topology, to a round sphere of area* $|M_0|$*, as* $\tilde{t} \to \infty$*.*

(II) *If* $0 < \Lambda < \infty$*, then* $T_{\max} = \infty$*, and the solutions to* (1.1) *converge, exponentially in the* $C^\infty-$*topology, to a round sphere as* $t \to \infty$*.*

(III) *If* $\Lambda = 0$*, then* $T_{\max} = \infty$*. Moreover if* $\bar{h} \neq 0$*, the solutions to* (1.1) *expand uniformly to* ∞ *as* $t \to \infty$ *and if the rescaled solutions to* (1.3) *converge to a smooth hypersurface, then the limit must be a round sphere of total area* $|M_0|$*.*

Remark 1.2. (1) *If* $h = 0$*, Theorem* 1.1 *is just the curvature flow by powers of the* m^{th} *mean curvature, which partially generalizes Schulze's flow by powers of the mean curvature* [*14*, *15*]*. Theorem* 1.1 *also includes Cabezas-Rivas and Sinestrari's volume-preserving flow by powers of the* m^{th} *mean curvature* [*2*]*.*

(2) *The assumption* (1.5) *seems not natural since often the maximal existing time* $T_{\max}$ *of* (1.1) *depends on* $h(t)$*. In fact we can use a stronger assumption that* $h(t)$ *is a nonnegative continuous function on* $[0, \infty)$ *and satisfies* $\lim_{t \to \infty} h(t) < +\infty$*. Our result still includes all cases in* (1)*.*

The extreme cases of Theorem 1.1 can also be considered.

Remark 1.3. (1) *For case* (I)*, when* $\bar{h} = \infty$*,* $T_{\max}$ *may not be finite, even though* M_t *is contracting* (*see Remark* 5.2 (3))*. A sphere:* $r(t) = \frac{1}{t+1}$*,* $h(t) = (t+1)^{m\gamma} - \frac{1}{(t+1)^2}$*, is such an example, whose maximal existing time* $T_{\max} = \infty$*.*

(2) *For case* (III)*, if* $\bar{h} = 0$*,* $T_{\max}$ *is also infinite* (*see Section* 7)*. We do not know whether the solutions to* (1.1) *expand uniformly to* ∞ *as* $t \to \infty$*, but we can find the special solution satisfying that condition. In fact, a sphere:* $r(t) = \sqrt{t+1}$*,* $h(t) = \frac{(\sqrt{t+1})^{m\gamma-1}+2}{2(\sqrt{t+1})^{m\gamma}}$*, is such a particular example, for which* M_t *expands to infinity. If* $\bar{h} = \infty$*, by similar discussion as in Section* 7*, we can show that* M_t *expands to infinity, but* $T_{\max}$ *may not be* ∞*. For example, the sphere* $r(t) = \frac{1}{1-t}$*,* $h(t) = (1-t)^{m\gamma} + \frac{1}{(1-t)^2}$ *is a solution to* (1.1)*, for which* $T_{\max} = 1$*, and* $r \to \infty$*, as* $t \to 1$*.*

This paper is organized as follows: Section 2 introduces basic properties of the m^{th} mean curvature, some known facts of convex hyperfaces and an interior Hölder estimate, which will be used later. In Section 3, we compute the evolution equations for several geometric quantities of the flow (1.1), and prove the preservation of the initial pinching condition. In Section 4, we carry out the normalization of (1.1), compute the evolution equations for several rescaled geometric quantities, and estimate the inner and outer radii of the rescaled convex hypersurfaces. In terms of the limiting shape of the scaling factor $\psi(t)$ as $t \to T_{\max}$, long time existence and convergence of solutions to (1.1) or (1.3) are proved in Section 5, 6 and 7, separately, and therefore the proof of Theorem 1.1 is completed.

2 Preliminaries

Let M^n be a smooth hypersurface immersion in $\mathbb{R}^{n+1}$. We will use the same notation as in [2, 9, 15]. In particular, for a local coordinate system $\{x^1, \cdots, x^n\}$ of M^n, $g = g_{ij} = (\cdot,\cdot)$ and $A = h_{ij}$ denote respectively the metric and second fundamental form of M^n. The eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ of A are called the principal curvatures. We say that M_t is convex if $\lambda_1 \geq 0$ everywhere and that it is uniformly convex if $\lambda_1 > 0$ everywhere. Then further important quantities are the mean curvature $H = g^{ij}h_{ij} = \sum_i \lambda_i$, the normed square of the second fundamental form $|A|^2 = g^{ij}g^{kl}h_{ik}h_{jl} = \sum_i \lambda_i^2$ and the Gauss curvature $K = (\det h_{ij})/(\det g_{ij}) = \prod_i \lambda_i$, where g^{ij} is the (i,j)-entry of the inverse of the matrix (g_{ij}). More generally, we call the m^{th} mean curvature of a hypersurface the function H_m, that is, the m^{th} elementary symmetric polynomial of the principal curvatures, up to a normalizing factor. Since H_m is homogeneous of degree m, the speed F is a homogeneous function of degree $m\gamma$ in the curvatures λ_i. Throughout this paper we sum over repeated indices from 1 to n unless otherwise indicated.

We shall often use λ to denote the vector $(\lambda_1, \cdots, \lambda_n)$ whose entries are the principal curvatures or, depending on the context, a generic element of $\mathbb{R}^n$. We denote by $\Gamma_+ \subset \mathbb{R}^n$ the positive cone, i.e.

$$\Gamma_+ = \{\lambda = (\lambda_1, \cdots, \lambda_n) : \lambda_i > 0 \text{ for all } i\}.$$

Observe that H, K, H_m, F may be regarded as functions of λ, or as functions of A, or as functions of h_{ij} and g_{ij}, or also as functions of space and time variables on M_t. For the sake of simplicity, we denote these functions by the same letters in all cases, since the meaning should be clear from the text.

We use the notation

$$F^i = \frac{\partial F}{\partial \lambda_i}, \quad \operatorname{tr}(F^i) = \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} = \frac{\partial F}{\partial h_{ij}} g^{ij}.$$

In addition, we denote by (F^{ij}) the matrix of the first partial derivatives of F with respect to the components of its arguments:

$$\frac{\partial}{\partial s} F(A + sB)|_{s=0} = F^{ij}(A)B_{ij},$$

where A and B are any symmetric matrices. Similarly for the second partial derivatives of F, we write

$$\frac{\partial^2}{\partial s^2}F(A+sB)|_{s=0} = F^{ij,kl}(A)B_{ij}B_{kl}.$$

For any differentiable function φ on M_t, we define the operator $\mathcal{L}_\eta$ according to a matrix η^{ij}:

$$\mathcal{L}_\eta\varphi = \eta^{ij}\nabla_i\nabla_j\varphi.$$

In addition, for any other differentiable function ψ on M_t, we write

$$|\nabla\varphi|^2_\eta = \eta^{ij}\nabla_i\varphi\nabla_j\varphi, \quad (\nabla\varphi,\nabla\psi)_\eta = \eta^{ij}\nabla_i\varphi\nabla_j\psi.$$

In particular, $|\nabla\varphi|^2 = g^{ij}\nabla_i\varphi\nabla_j\varphi$. The gradient and Beltrami-Laplace operator on M_t are denoted by ∇ and $\triangle$ respectively.

First we need the following lemma (see [2]) which will be used repeatedly throughout the paper.

Lemma 2.1. *For any $m\in\{1,\cdots,n\}$, the function H_m satisfies:*

(1) $\frac{\partial H_m}{\partial\lambda_i}(\lambda)>0$ *for all $i\in\{1,\cdots,n\}$ and $\lambda\in\Gamma_+$.*

(2) $H_m^{1/m}$ *is concave in Γ_+.*

(3) $\mathrm{tr}(F^i)\ge m\gamma F^{1-\frac{1}{m\gamma}}$.

(4) $H_m^{1/m}\le\frac{H}{n}$, *or equivalently, $F\le\left(\frac{H}{n}\right)^{m\gamma}$.*

(5) H_m*, as a function of h_{ij}, is also a homogeneous polynomial of degree m; in addition, as a function on M, it satisfies $\nabla_j\left(\frac{\partial H_m}{\partial h_{ij}}\right)=0$ for any $i\in\{1,\cdots,n\}$, where ∇ is again the covariant derivative on M.*

Let $|M|$ be the area of M, and $|V|$ the volume of the region V contained inside M. We denote by r_- the inner radius of $M=\partial V$ and by r_+ the outer radius, which are respectively the radii of the biggest ball enclosed by M and of the smallest ball enclosing M. We have the following relations between $|V|$ and $|M|$ by Aleksandrov-Fenchel inequality and divergence theorem (see Theorem 2.3 in [12], or also see [9, 10]).

Lemma 2.2. *Let M be a compact and convex hypersurface embedded into $\mathbb{R}^{n+1}$ satisfying $H>0$ and $h_{ij}\ge\varepsilon Hg_{ij}$, for some $\varepsilon\in(0,\frac{1}{n}]$. Then there exists a constant C_1 depending on n and ε such that*

$$C_1^{-1}|M|^{\frac{n+1}{n}}\le|V|\le C_1|M|^{\frac{n+1}{n}}.$$

The following result with regard to inner and outer radii observed by Andrews in [1], shows that a pinching inequality on the curvatures implies a bound on the ratio between outer radius and inner radius.

Lemma 2.3. *Let M be a smooth, compact and convex hypersurface in $\mathbb{R}^{n+1}$. Suppose that there exists a positive constant C_2 such that M satisfies the pointwise pinching estimate $\lambda_n(x) \le C_2\lambda_1(x)$, for every $x \in M$. Then there exists a constant C_3 depending on n and C_2 such that*

$$r_+ \le C_3 r_-.$$

Next we recall an algebraic property proved by Schulze in [15] (or also see [2]).

Lemma 2.4. *For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ with the following property: if we have $h_{ij} \ge \varepsilon H g_{ij} > 0$ at some point of an n-dimensional hypersurface, then at the same point we have*

$$\frac{n|A|^2 - H^2}{H^2} \ge \delta\Big(\frac{1}{n^n} - \frac{K}{H^n}\Big). \tag{2.1}$$

Finally, we recall an interior Hölder estimate, due to Di Benedetto and Friedman (see [4], or also see [2]). Given $r > 0$, we denote by B_r the ball of radius $r > 0$ in $\mathbb{R}^n$ centered at the origin. For degenerate parabolic equation

$$\frac{\partial}{\partial t}\nu - D_i\big(a^{ij}(x,t,D\nu)D_j\nu^d\big) = f(x,t,\nu,D\nu), \tag{2.2}$$

being $d > 1$, we assume that $a^{ij} = a^{ji}$ and that a^{ij} is uniformly elliptic, that is, there exist two constants ω, $\Omega > 0$ such that

$$\omega|v|^2 \le a^{ij}(x,t)v_i v_j \le \Omega|v|^2$$

for $v \in \mathbb{R}^n$ and $(x,t) \in B_r \times [0,T]$. Then the following estimate holds:

Lemma 2.5. *Let $\nu \in C^2\big(B_r \times [0,T]\big)$ be a nonnegative solution to (2.2). Let B_1, B_2, $N > 0$ be such that*

$$|f(x,t,\nu,D\nu)| \le B_1|D\nu^d| + B_2,$$

and

$$\sup_{0<t<T} \|\nu(\cdot,t)\|^2_{L^2(B_r)} + \|D\nu^d\|^2_{L^2(B_r\times[0,T])} \le N.$$

Then for any $0 < \delta < T$ and $0 < r' < r$, we have

$$\|\nu\|_{C^\alpha(B_{r'}\times[\delta,T])} \le \mathcal{C},$$

for suitable $\mathcal{C} > 0$, $\alpha \in (0,1)$ depending only on n, N, ω, Ω, δ, B_1, B_2, r and r'.

3 Evolution equations and the pinching estimate

As it is pointed out in [7], the forcing term $h(t)$ does not influence the parabolicity of the equation (1.1). Then the short time existence easily follows (see [2, 13]). That is, there exists a unique smooth solution $X(\cdot,t)$ of equation (1.1), for some time interval $[0,T_{\max})$, with $T_{\max}>0$.

For convenience we write $b^{ij}=\big(h_{ij}\big)^{-1}$. As in [2, 14, 15], we have the following evolution equations for various geometric quantities under the flow (1.1).

Lemma 3.1. *The following evolution equations hold for any solution to equation* (1.1).

(1) $\frac{\partial}{\partial t}g_{ij}=2(h-F)h_{ij}$.

(2) $\frac{\partial}{\partial t}\mathbf{v}=\nabla F$.

(3) $\frac{\partial}{\partial t}d\mu_t=(h-F)Hd\mu_t$.

(4) $\frac{\partial}{\partial t}F=\mathcal{L}_F F-(h-F)F^{pq}h_{pk}h^k_q$.

(5) $\frac{\partial}{\partial t}h_{ij}=\mathcal{L}_F h_{ij}+F^{pq,rs}\nabla_i h_{pq}\nabla_j h_{rs}+F^{pq}h_{pk}h^k_q h_{ij}+\big(h-(m\gamma+1)F\big)h_{ik}h^k_j$.

(6) $\frac{\partial}{\partial t}H=\mathcal{L}_F H+F^{pq,rs}\nabla^i h_{pq}\nabla_i h_{rs}-\big(h+(m\gamma-1)F\big)|A|^2+F^{pq}h_{pk}h^k_q H$.

(7) $\frac{\partial}{\partial t}K=\mathcal{L}_F K-\frac{|\nabla K|^2_F}{K}+K\big(b^{ij}F^{pq,rs}\nabla_i h_{pq}\nabla_j h_{rs}-F^{pq}\nabla_p b^{ij}\nabla_q h_{ij}\big)$
$\quad-\big(h+(m\gamma-1)F\big)HK+nKF^{pq}h_{pk}h^k_q$.

(8) *For the position vector field X on M_t:*

$$\frac{\partial}{\partial t}(X,\mathbf{v})=\mathcal{L}_F(X,\mathbf{v})+F^{pq}h_{pk}h^k_q(X,\mathbf{v})+\big(h-(m\gamma+1)F\big).$$

In order to write down the evolution equation for H_m, we set

$$c^{ij}=\frac{\partial H_m}{\partial h_{ij}}.$$

Since the symmetric tensor c is divergence-free (cf. part (5) of Lemma 2.1 or see [2]), we have:

Lemma 3.2. *If M_t is a hypersurface in $\mathbb{R}^{n+1}$ evolving under* (1.1)*, the $m^{\rm th}$ mean curvature H_m and its $\gamma^{\rm th}$ power F satisfy the following evolution equations:*

$$\frac{\partial}{\partial t}H_m=\gamma H_m^{\gamma-1}\Big(\mathcal{L}_c H_m+(\gamma-1)\frac{|\nabla H_m|^2_c}{H_m}\Big)+(F-h)c^{ij}h_{ik}h^k_j,$$

and

$$\frac{\partial}{\partial t}F=\gamma H_m^{\gamma-1}\mathcal{L}_c F+\gamma\frac{F-h}{H_m}Fc^{ij}h_{ik}h^k_j. \tag{3.1}$$

By direct calculation, we have the following evolution equation (see [2, 15]).

Lemma 3.3. *The quantity $Q=K/H^n$ evolves under flow* (1.1) *satisfying*

$$\begin{aligned}\frac{\partial}{\partial t}Q \;=\;& \mathcal{L}_F Q+\frac{n+1}{nH^n}\big(\nabla H^n,\nabla Q\big)_F-\frac{n-1}{nK}\big(\nabla K,\nabla Q\big)_F-\frac{H^n}{nK}\big|\nabla Q\big|^2_F\\ &+\frac{Q}{H^2}\big|H\nabla_i h_{pq}-h_{pq}\nabla_i H\big|^2_{F,b}+Q\Big(b^{ij}-\frac{n}{H}g^{ij}\Big)F^{pq,rs}\nabla_i h_{pq}\nabla_j h_{rs}\\ &+\big(h+(m\gamma-1)F\big)\frac{Q}{H}\big(n|A|^2-H^2\big),\end{aligned} \tag{3.2}$$

where

$$\big|H\nabla_i h_{pq} - h_{pq}\nabla_i H\big|^2_{F,b} = F^{ij}b^{pr}b^{qs}\big(H\nabla_i h_{pq} - h_{pq}\nabla_i H\big)\big(H\nabla_j h_{rs} - h_{rs}\nabla_j H\big).$$

We want to show a monotonicity property for the quotient Q. Such a quotient, which was also considered in [2, 3, 15], plays an important role in studying flow (1.1).

In order to apply the maximum principle to equation (3.2), we need some preliminary inequalities (see [2, 15]).

Lemma 3.4. *If for some $\varepsilon > 0$ the inequality $h_{ij} \geq \varepsilon H g_{ij} > 0$ holds at a point of a hypersurface immersed in $\mathbb{R}^{n+1}$, then $\varepsilon \leq \frac{1}{n}$ and at a same point we also have*

$$|H\nabla_i h_{pq} - h_{pq}\nabla_i H|^2 \geq \frac{n-1}{2}\varepsilon^2 H^2 |\nabla A|^2. \tag{3.3}$$

Lemma 3.5. *Given any $\varepsilon \in (0, \frac{1}{n}]$, let $C_0(n,m,\gamma)$ be the minimal constant such that $0 \leq C_0 < \frac{1}{n^n}$, for any $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ with $\lambda_i \geq 0$ for all $i = 1, \cdots, n$. Then we have*

$$\frac{K(\lambda)}{H^n(\lambda)} \geq C_0 \Rightarrow \min_{1\leq i\leq n} \lambda_i \geq \varepsilon H(\lambda).$$

By these two lemmas, a pinching estimate for flow (1.1) immediately follows (see [2, 15]).

Corollary 3.6. *There exists a constant $C_0(n,m,\gamma) \in (0, \frac{1}{n^n})$ with the following property: if $X_t : M \times (0, T_{\max}) \to \mathbb{R}^{n+1}$, is a smooth solution to equation (1.1), such that the initial immersion X_0 satisfies (1.4), and the solution M_t satisfies $H > 0$ at $t = 0$, then the minimum of the quotient Q on M_t is nondecreasing in time.*

Proof. Let us set $\mathcal{Q}(t) = \min_M Q(\cdot, t)$. Thanks to (3.2) we get

$$\begin{aligned}\frac{d}{dt}\mathcal{Q} &\geq \mathcal{Q}\Big(\frac{1}{H^2}\big|H\nabla_i h_{pq} - h_{pq}\nabla_i H\big|^2_{F,b} + \big(b^{ij} - \frac{n}{H}g^{ij}\big)F^{pq,rs}\nabla_i h_{pq}\nabla_j h_{rs}\Big)\\ &\geq \mathcal{Q}\Big(\frac{1}{H^2}\big|H\nabla_i h_{pq} - h_{pq}\nabla_i H\big|^2_{F,b} - \Big|b^{ij} - \frac{n}{H}g^{ij}\Big||F^{pq,rs}(\nabla A, \nabla A)|\Big),\end{aligned} \tag{3.4}$$

where we have used that the last term in (3.2) is nonnegative, since the elementary inequality $|A|^2 \geq H^2/n$.

We can show that the above expression is nonnegative provided the second fundamental form is suitably pinched. To this purpose we need to bound from below the positive term. Doing computations at a point where we choose an orthonormal basis which diagonalizes h_{ij}, we first deduce

$$\begin{aligned}\big|H\nabla_i h_{pq} - h_{pq}\nabla_i H\big|^2_{F,b} &= \sum_{i,p,q} F^i \frac{1}{\lambda_p}\frac{1}{\lambda_q}(H\nabla_i h_{pq} - h_{pq}\nabla_i H)^2\\ &\geq \frac{1}{|A|^2}\sum_{i,p,q} F^i (H\nabla_i h_{pq} - h_{pq}\nabla_i H)^2,\end{aligned} \tag{3.5}$$

since $0 < \lambda_p < |A|$ for all p.

Note that each F^i is positive in the interior of the positive cone Γ_+. More precisely, let us set, for any $\varepsilon \in (0, \frac{1}{n}]$

$$\mathcal{K}_\varepsilon = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n : \min_{1\le i\le n} \lambda_i \ge \varepsilon H > 0\},$$

$$N_1(\varepsilon) = \min\{F^i(\lambda) : 1 \le i \le n, \lambda \in \mathcal{K}_\varepsilon, |\lambda| = 1\}.$$

Observe that $F^i > 0$ on $\mathcal{K}_\varepsilon$ for all i, by Lemma 2.1 (1). Therefore $N_1(\varepsilon) > 0$, being the minimum of a finite family of positive smooth functions on a compact set. In addition, since the cone $\mathcal{K}_\varepsilon$ becomes smaller as ε increases, $N_1(\varepsilon)$ is an increasing function of ε. By homogeneity, we conclude

$$F^i(\varepsilon) \ge N_1(\varepsilon)|\lambda|^{m\gamma-1}, \quad \lambda \in \mathcal{K}_\varepsilon.$$

Substituting this into (3.5) and using Lemma 3.4, on a hypersurface satisfying $h_{ij} \ge \varepsilon H g_{ij}$, we obtain

$$\left|H\nabla_i h_{pq} - h_{pq}\nabla_i H\right|^2_{F,b} \ge \frac{n-1}{2} N_1(\varepsilon)\varepsilon^2 |A|^{m\gamma-3} H^2 |\nabla A|^2. \tag{3.6}$$

Now we can estimate the term $|F^{pq,rs}(\nabla A, \nabla A)|$ from above. Observe that the quantity $F^{pq,rs}(\nabla A, \nabla A)$ is homogeneous of degree $m\gamma - 2$ in the curvatures and quadratic in ∇A. It is smooth as long as the curvatures are all positive, while it may be in general not defined when one or more curvatures vanish. With an argument similar to the previous one, we see that, for some $\varepsilon \in (0, \frac{1}{n}]$, there exists a constant $N_2(\varepsilon)$ such that, at any point where $h_{ij} \ge \varepsilon H g_{ij}$,

$$|F^{pq,rs}(\nabla A, \nabla A)| \le N_2(\varepsilon)|A|^{m\gamma-2}|\nabla A|^2. \tag{3.7}$$

The constant $N_2(\varepsilon)$ is decreasing in ε, since it gives a bound from above.

To conclude, we show that $\left|b^{ij} - \frac{n}{H}g^{ij}\right|$ is small if the second fundamental form is pinched enough. Clearly, we have

$$\left|b^{ij} - \frac{n}{H}g^{ij}\right| \le \max\Big\{\sqrt{n}\Big(\frac{1}{\lambda_1} - \frac{n}{H}\Big), \sqrt{n}\Big(\frac{n}{H} - \frac{1}{\lambda_n}\Big)\Big\}.$$

If $h_{ij} \ge \varepsilon H g_{ij}$ for some $\varepsilon \in (0, \frac{1}{n}]$, then $\lambda_1 \ge \varepsilon H$ and $\lambda_n \le \big(1 - (n-1)\varepsilon\big)H$. It follows that

$$\frac{1}{\lambda_1} - \frac{n}{H} \le \frac{1 - n\varepsilon}{\varepsilon H},$$

and

$$\frac{n}{H} - \frac{1}{\lambda_n} \le \frac{n(n-1)(1-n\varepsilon)}{H}.$$

Since $\varepsilon \le \frac{1}{n}$, we deduce that

$$\left|b^{ij} - \frac{n}{H}g^{ij}\right| \le (1 - n\varepsilon)\frac{n^{\frac{3}{2}}(n-1)}{H}. \tag{3.8}$$

Plugging (3.6), (3.7) and (3.8) into (3.4), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{Q} &\geq \frac{n-1}{2}\mathcal{Q}|A|^{m\gamma-3}|\nabla A|^2\Big(N_1(\varepsilon)\varepsilon^2 - 2n^{\frac{3}{2}}(1-n\varepsilon)\frac{|A|}{H}N_2(\varepsilon)\Big) \\ &\geq \frac{n-1}{2}\mathcal{Q}|A|^{m\gamma-3}|\nabla A|^2\Big(N_1(\varepsilon)\varepsilon^2 - 2n^{\frac{3}{2}}(1-n\varepsilon)N_2(\varepsilon)\Big). \end{aligned} \tag{3.9}$$

To apply the maximum principle, we need that $N_1(\varepsilon)\varepsilon^2 - 2n^{\frac{3}{2}}(1-n\varepsilon)N_2(\varepsilon) \geq 0$ on our hypersurface. Since $N_1(\varepsilon)$ is increasing and $N_2(\varepsilon)$ is decreasing, such a quantity is a strictly increasing function of ε. In addition, it is negative for ε close to zero and positive for ε close to $\frac{1}{n}$. The optimal condition is obtained if we fix $\varepsilon \in (0, \frac{1}{n})$ to be the unique value such that

$$N_1(\varepsilon)\varepsilon^2 - 2n^{\frac{3}{2}}(1-n\varepsilon)N_2(\varepsilon) = 0. \tag{3.10}$$

By Lemma 3.5 there exists a constant $C_0 \in [0, \frac{1}{n^n})$ such that $K/H^n \geq C_0$ implies $h_{ij} \geq \varepsilon H g_{ij}$, with ε given by (3.10). Then, if $K/H^n \geq C_0$ everywhere on our hypersurface, we have $\frac{d}{dt}\mathcal{Q} \geq 0$ by (3.9). By the maximum principle, this proves that, for any $C_0 \in [0, \frac{1}{n^n})$, the property $K/H^n \geq C_0$ is invariant under the flow. □

Corollary 3.6 states that inequality $K/H^n \geq C_0$ holds for all $t \in [0, T_{\max})$. In addition, by the definition of C_0, we have that

$$\lambda_i \geq \varepsilon H \quad \text{on } M \times [0, T_{\max}) \qquad \text{for each } i \tag{3.11}$$

with $\varepsilon \in (0, \frac{1}{n})$. In particular, the solution is convex for all t and therefore satisfies

$$\lambda_j \leq H \quad \text{on } M \times [0, T_{\max}) \qquad \text{for each } j. \tag{3.12}$$

4 The normalized equation

The solution of the curvature flow (1.1) may shrink to a point if h is small enough (e.g. $h = 0$ [14]), or expand to infinity if h is large enough (e.g. h is a constant and $h > \sup_{x\in M^n} F(x, 0)$). The solution can also converge to a smooth hypersurface, for some special initial hypersurface and h (e.g. volume-preserving flow by powers of the m^{th} mean curvature [2]). In order to see this, we normalize the equation (1.1) by keeping some geometrical quantity fixed, for example as in [6, 9] the total area of the hypersurfaces M_t. As that mentioned in Section 1, multiplying the solution X of (1.1) at each time $0 \leq t < T_{\max}$ with a positive constant $\psi(t)$ such that the total area of the hypersurfaces $\tilde{M}_t$ given by

$$\tilde{X}(x,t) = \psi(t)X(x,t)$$

has total area equal to $|M_0|$, the area of M_0

$$\int_{\tilde{M}_t} d\tilde{\mu}_t = |M_0|, \qquad 0 \leq t < T_{\max}. \tag{4.1}$$

Then we introduce a new time variable $\tilde{t}(t)=\int_0^t \psi(\tau)^{m\gamma+1}d\tau$, such that $\frac{\partial \tilde{t}}{\partial t}=\psi^{m\gamma+1}$.

As in [6, 9], for a geometric quantity P on M_t, we denote by $\tilde{P}$ the corresponding quantity on the rescaled hypersurface $\tilde{M}_{\tilde{t}}$. By direct calculation we have

$$\begin{aligned} \tilde{g}_{ij} &= \psi^2 g_{ij}, & \tilde{h}_{ij} &= \psi h_{ij},\\ \tilde{H} &= \psi^{-1}H, & |\tilde{A}|^2 &= \psi^{-2}|A|^2,\\ \tilde{K} &= \psi^{-n}K, & \tilde{H}_m &= \psi^{-m}H_m,\\ \tilde{F} &= \psi^{-m\gamma}F, & d\tilde{\mu} &= \psi^n d\mu, \end{aligned}$$

and so on. If we differentiate (4.1) for time t, we obtain

$$\psi^{-1}\frac{\partial\psi}{\partial t}=\frac{1}{n}\frac{\int_M (F-h)Hd\mu}{\int_M d\mu}=\frac{1}{n}\theta.$$

Now by differentiating $\tilde{X}$ with respect to $\tilde{t}$, we derive the normalized evolution equation for a different maximal time interval $0\le\tilde{t}<\tilde{T}_{\max}$

$$\begin{cases} \frac{\partial}{\partial\tilde{t}}\tilde{X}(x,\tilde{t})=\{\tilde{h}(\tilde{t})-\tilde{F}(x,\tilde{t})\}\tilde{\mathbf{v}}(x,\tilde{t})+\frac{1}{n}\tilde{\theta}(\tilde{t})\tilde{X}(x,\tilde{t}),\\ \tilde{X}(\cdot,0)=X_0, \end{cases}\tag{4.2}$$

where $\tilde{h}=\psi^{-m\gamma}h$, $\tilde{\theta}=\psi^{-m\gamma-1}\theta$ and θ is given by

$$\theta=\frac{\int_M (F-h)Hd\mu}{\int_M d\mu}.\tag{4.3}$$

As in Lemma 3.1, we have the following evolution equations for various geometric quantities under the rescaled flow (1.3).

Lemma 4.1. *The following evolution equations hold for any solution to equation* (1.3).

(1) $\frac{\partial}{\partial\tilde{t}}\tilde{g}_{ij}=2(\tilde{h}-\tilde{F})\tilde{h}_{ij}+\frac{2}{n}\tilde{\theta}\tilde{g}_{ij}$.

(2) $\frac{\partial}{\partial\tilde{t}}d\tilde{\mu}_{\tilde{t}}=(\tilde{h}-\tilde{F})\tilde{H}d\tilde{\mu}_{\tilde{t}}+\tilde{\theta}d\tilde{\mu}_{\tilde{t}}$.

(3) $\frac{\partial}{\partial\tilde{t}}\tilde{F}=\mathcal{L}_{\tilde{F}}\tilde{F}-(\tilde{h}-\tilde{F})\tilde{F}^{pq}\tilde{h}_{pk}\tilde{h}^k_q-\frac{m\gamma}{n}\tilde{\theta}\tilde{F}$.

(4) $\frac{\partial}{\partial\tilde{t}}\tilde{h}_{ij}=\mathcal{L}_{\tilde{F}}\tilde{h}_{ij}+\tilde{F}^{pq,rs}\nabla_i\tilde{h}_{pq}\nabla_j\tilde{h}_{rs}+\tilde{F}^{pq}\tilde{h}_{pk}\tilde{h}^k_q\tilde{h}_{ij}+\big(\tilde{h}-(m\gamma+1)\tilde{F}\big)\tilde{h}_{ik}\tilde{h}^k_j+\frac{1}{n}\tilde{\theta}\tilde{h}_{ij}$.

(5) $\frac{\partial}{\partial\tilde{t}}\tilde{H}=\mathcal{L}_{\tilde{F}}\tilde{H}+\tilde{F}^{pq,rs}\nabla^i\tilde{h}_{pq}\nabla_i\tilde{h}_{rs}-\big(\tilde{h}+(m\gamma-1)\tilde{F}\big)|\tilde{A}|^2+\tilde{F}^{pq}\tilde{h}_{pk}\tilde{h}^k_q\tilde{H}-\frac{1}{n}\tilde{\theta}\tilde{H}$.

(6) $\frac{\partial}{\partial\tilde{t}}\tilde{K}=\mathcal{L}_{\tilde{F}}\tilde{K}-\frac{|\nabla\tilde{K}|^2_{\tilde{F}}}{\tilde{K}}+\tilde{K}\big(\tilde{b}^{ij}\tilde{F}^{pq,rs}\nabla_i\tilde{h}_{pq}\nabla_j\tilde{h}_{rs}-\tilde{F}^{pq}\nabla_p\tilde{b}^{ij}\nabla_q\tilde{h}_{ij}\big)$
$\quad-\big(\tilde{h}+(m\gamma-1)\tilde{F}\big)\tilde{H}\tilde{K}+n\tilde{K}\tilde{F}^{pq}\tilde{h}_{pk}\tilde{h}^k_q-\tilde{\theta}\tilde{K}$.

Lemma 4.2. *If $\tilde{M}_{\tilde{t}}$ is a hypersurface in $\mathbb{R}^{n+1}$ evolving under* (1.3)*, the m^{th} mean curvature $\tilde{H}_m$ and its γ^{th} power $\tilde{F}$ satisfy the following evolution equations:*

$$\frac{\partial}{\partial\tilde{t}}\tilde{H}_m=\gamma\tilde{H}_m^{\gamma-1}\Big(\mathcal{L}_{\tilde{c}}\tilde{H}_m+(\gamma-1)\frac{|\nabla\tilde{H}_m|^2_{\tilde{c}}}{\tilde{H}_m}\Big)+(\tilde{F}-\tilde{h})\tilde{c}^{ij}\tilde{h}_{ik}\tilde{h}^k_j-\frac{m}{n}\tilde{\theta}\tilde{H}_m$$

and

$$\frac{\partial}{\partial \tilde{t}}\tilde{F} = \gamma \tilde{H}_m^{\gamma-1}\mathcal{L}_{\tilde{c}}\tilde{F} + \gamma\frac{\tilde{F}-\tilde{h}}{\tilde{H}_m}\tilde{F}\tilde{c}^{ij}\tilde{h}_{ik}\tilde{h}_j^k - \frac{m\gamma}{n}\tilde{\theta}\tilde{F}. \tag{4.4}$$

In the remainder of this section, we will estimate the inner and outer radii of the normalized hypersurfaces $\tilde{M}$. First we see that since at each time the whole configuration of $\tilde{M}$ is only dilated by a constant factor ψ, the solutions to (4.2) are compact and convex hypersurfaces, then Lemma 2.3 still holds for $\tilde{r}_+$ and $\tilde{r}_-$ with the same constant C_3. It is easy to see that Lemma 3.5 and Corollary 3.6 still hold for the rescaled flow (1.3) because of the scaling invariance of the quotient $\tilde{Q} = \tilde{K}/\tilde{H}^n$. Combination of (3.11) and (3.12) yields

$$\varepsilon\tilde{H} \le \tilde{\lambda}_i \le \tilde{H} \quad \text{on } \tilde{M}\times[0,\tilde{T}_{\max}) \quad \text{for each } i, \tag{4.5}$$

for some $\varepsilon \in (0, \frac{1}{n}]$. The hypersurface $\tilde{M}$ encloses a region $\tilde{V}$ of volume $|\tilde{V}|$. Then by Lemma 2.2

$$C_1^{-1}|\tilde{M}|^{\frac{n+1}{n}} \le |\tilde{V}| \le C_1|\tilde{M}|^{\frac{n+1}{n}}. \tag{4.6}$$

Since $|\tilde{V}|$ is controlled by the volume of its inner and outer sphere

$$C_4\tilde{r}_-^{n+1} \le |\tilde{V}| \le C_4\tilde{r}_+^{n+1},$$

for a constant C_4, we obtain the following estimate by the fixed total area of $\tilde{M}$ by (4.6)

$$\tilde{r}_+ \ge C_5 \quad \text{and} \quad \tilde{r}_- \le C_6, \tag{4.7}$$

for some two positive constants C_5 and C_6.

By Lemma 2.3 and (4.7) we have

Proposition 4.3. *The lower bound of the inner radius and the upper bound of the outer radius of $\tilde{M}_{\tilde{t}}$ are all uniformly bounded, i.e.*

$$C_7^{-1} \le \tilde{r}_- \le \tilde{r}_+ \le C_7$$

for some constant C_7 and all $\tilde{t} \in [0, \tilde{T}_{\max})$.

Now for any given time sequence $\{T_i\}$, $T_i \in [0, T_{\max})$, such that $T_i \to T_{\max}$ as $i \to \infty$, there corresponds to a sequence $\{\psi_i = \psi(T_i)\}$. By limiting theory, there exists at least one accumulation of this sequence. Denote by Λ_i the minimal accumulation of the sequence $\{\psi_i = \psi(T_i)\}$. We define Λ to be the infimum of Λ_i for all possible sequences $\{\psi_i = \psi(T_i)\}$, i.e.

$$\Lambda = \inf\{\Lambda_i | \Lambda_i \text{ is the minimal accumulation of a sequence } \{\psi_i = \psi(T_i)\}, \\ \text{where } \{T_i\} \text{ is any sequence in } [0, T_{\max}) \text{ such that } T_i \to T_{\max} \text{ as } i \to \infty\}.$$

Therefore by the method of extracting diagonal subsequences we have a subsequence, still denoted by $\{\psi_i = \psi(T_i)\}$, which converges to Λ as $T_i \to T_{\max}$ (or $i \to \infty$), that is to say we have the following limit

$$\lim_{i\to\infty} \psi_i = \Lambda. \tag{4.8}$$

There are three cases in terms of the limit Λ: $\Lambda = \infty$, $0 < \Lambda < \infty$ and $\Lambda = 0$. We will consider the three cases separately in the sequel.

5 Case (I) $\Lambda = \infty$

In this section we consider the case $\Lambda = \infty$, and prove Theorem 1.1 (I). Since $\tilde{r}_+ = \psi r_+$, we have by Proposition 4.3

$$\frac{C_7^{-1}}{\psi} \le r_+ \le \frac{C_7}{\psi},$$

which implies that for the sequence $\{T_i\}$ in last section (see (4.8)), we have a limit

$$\lim_{T_i\to T_{\max}} r_+(T_i) = 0. \tag{5.1}$$

By limiting theory, there exists a time $T^* < T_{\max}$ such that for any $T_i \ge T^*$, $r_+(T_i)$ is less than any given positive number r^*. By the assumption (1.5), $h(t)$ has a uniformly upper bound h^+ on $[0, T_{\max})$ (We can always assume $h^+ > 0$ even in the case of $h(t) = 0$). We now choose r^* is less than $(1/h^+)^{1/m\gamma}$.

We follow an idea in [1, 9, 18] to prove the following lemma which implies that when t is very near $T_{\max}$, M_t is in fact contracting.

Lemma 5.1. *When $t \ge T^*$, the regions enclosed by the hypersurfaces M_t are decreasing. Furthermore $T_{\max} < \infty$, and the solutions to* (1.1) *converge uniformly to a point in $\mathbb{R}^{n+1}$ as $t \to T_{\max}$.*

Proof. Let $\partial B_{r^*}(O)$ be a sphere in $\mathbb{R}^{n+1}$ centered at the origin O, with radius r^*. Since the outer radius of M_{T^*} is less than r^*, without loss of generality, we can assume that the hypersurface M_{T^*} is enclosed by $\partial B_{r^*}(O)$. Now we evolve the sphere $\partial B_{r^*}(O)$ in terms of (1.1), the radius $r_B(t)$ satisfies

$$\begin{cases} \frac{dr_B(t)}{dt} = h - \frac{1}{r_B(t)^{m\gamma}} \le h^+ - \frac{1}{r_B(t)^{m\gamma}}, \quad t \ge T^*, \\ r_B(T^*) = r^*, \end{cases} \tag{5.2}$$

which yields that $r_B(t)$ is decreasing because $r^* < (1/h^+)^{1/m\gamma}$. Then by containment principle, we see that the enclosed regions of M_t are decreasing for $t \ge T^*$.

Furthermore it can be checked that the solution to the differential inequality (5.2) is given by

$$r_B(t) + \frac{1}{h^+}\int_{r^*}^{r_B(t)} \frac{1}{r_B(t)^{m\gamma} - \frac{1}{h^+}} dr_B(t) \geq h^+(t - T^*) + r^*, \tag{5.3}$$

which yields the finiteness of $T_{\max}$ since the left hand side of (5.3) is uniformly bounded for $t \geq T^*$.

By convexity in Lemma 3.5, the pinching estimate in Lemma 2.3 will imply the uniformly convergence of solutions to (1.1) to a point if we can show that the enclosed area of M_t tends to 0 as $t \to T_{\max}$. We claim that $\limsup_{t\to T_{\max}} \max_{M_t} |A|^2 = +\infty$. Suppose not, we can show that under the assumption $\limsup_{t\to T_{\max}} \max_{M_t} |A|^2 < +\infty$, the surface M_t converges to a smooth limiting surface $M_{T_{\max}}$ (see Proposition 3.5 in [14]). By the short time existence we get a contradiction to the maximality of $T_{\max}$. Therefore $X(\cdot, t)$ must converge to a point as $t \to T_{\max}$. This completes the proof of the lemma. $\square$

Remark 5.2. (1) *By the strict parabolic maximum principle, as for the mean curvature flow, we have the containment principle. If two closed initial hypersurfaces* M_1 *lies in the domain enclosed by* M_2*, then they remain so under the flow for* $0 \leq t < T_{\max}$*. To prove this, we assume that* M_1 *intersects* M_2 *at some point* $x_0 \in M^n$ *for the first time* t_0*. It can be seen that* $h^{(1)}_{ij}(x_0, t_0) \geq h^{(2)}_{ij}(x_0, t_0)$*, so by the monotonicity condition of* F *(see part* (1) *of Lemma* 2.1*),* $F\big(h^{(1)}_{ij}(x_0, t_0)\big) \geq F\big(h^{(2)}_{ij}(x_0, t_0)\big)$.

(2) *From the proof of Lemma* 5.1*, we see that the containment principle implies that* r_+ *tends to zero, as* $t \to T_{\max}$*. Therefore by Proposition* 4.3 *again, the function* $\psi(t)$ *must tend to infinity as* $t \to T_{\max}$*, i.e.*

$$\lim_{t\to T_{\max}} \psi(t) = \infty.$$

(3) *We can see that for* $\bar{h} = \infty$*,* (1.1) *is still contracting to a point. In fact from the limit of* $\psi(t)$ *in Section* 4*, we see that* Λ *is the smallest limit of* ψ*. That is to say if* $\Lambda = \infty$*, then for any sequence* $\{T_i\} \subset [0, T_{\max})$ *satisfying* $T_j \to T_{\max}$ *as* $j \to \infty$*,* $\lim_{j\to\infty} \psi(T_j) = \infty$*. Therefore similarly by Proposition* 4.3*, the inner and outer radii of the evolving hypersurfaces all tend to zero as* $t \to T_{\max}$*. Then the containment principle implies that the solutions to* (1.1) *converge to a point as* $t \to T_{\max}$ *for all possible limits of* $h(t)$*.*

To understand the solution $X(\cdot, t)$ near the maximal time $T_{\max}$, we consider the solution of the rescaled equation (1.3). We want to bound the m^{th} mean curvature $\tilde{H}_m$ and the mean curvature $\tilde{H}$ of $\tilde{M}_{\tilde{t}}$. For this purpose, we set $\mathcal{S} = (X, \mathbf{v})$ and use a trick of Chow (Tso) [16] (see also [1, 2, 9, 12, 18]) to consider the function

$$\Phi = \frac{F}{\mathcal{S} - \alpha}, \tag{5.4}$$

for a constant α to be chosen later. First we compute the evolution equation of Φ.

Lemma 5.3. *For $t \in [0, T_{\max})$, for any constant α we have*

$$\begin{aligned}\frac{\partial}{\partial t}\Phi &= \mathcal{L}_F\Phi + \frac{2}{\mathcal{S}-\alpha}(\nabla\Phi, \nabla\mathcal{S})_F \\ &\quad + \frac{1}{(\mathcal{S}-\alpha)^2}\big\{(m\gamma+1)F^2 - hF - \alpha F F^{pq}h_{pk}h_q^k - h(\mathcal{S}-\alpha)F^{pq}h_{pk}h_q^k\big\}.\end{aligned}$$

Proof. The proof is similar as in [9]. Because we shall consider the evolution equations of similar functions in Section 6 and 7, we outline its proof here. We first have

$$\nabla_i\Phi = \frac{\nabla_i F}{\mathcal{S}-\alpha} - \frac{F\nabla_i\mathcal{S}}{(\mathcal{S}-\alpha)^2},$$

and

$$\nabla_i\nabla_j\Phi = \frac{\nabla_i\nabla_j F}{\mathcal{S}-\alpha} - \frac{\nabla_i F\nabla_j\mathcal{S} + \nabla_i\mathcal{S}\nabla_j F}{(\mathcal{S}-\alpha)^2} - \frac{F\nabla_i\nabla_j\mathcal{S}}{(\mathcal{S}-\alpha)^2} + \frac{2F\nabla_i\mathcal{S}\nabla_j\mathcal{S}}{(\mathcal{S}-\alpha)^3},$$

which yields

$$\mathcal{L}_F\Phi = \frac{1}{\mathcal{S}-\alpha}\mathcal{L}_F F - \frac{F}{(\mathcal{S}-\alpha)^2}\mathcal{L}_F\mathcal{S} - \frac{2}{\mathcal{S}-\alpha}(\nabla\Phi, \nabla\mathcal{S})_F. \tag{5.5}$$

By Lemma 3.1 (4) and (8), we have the time derivative of Φ

$$\begin{aligned}\frac{\partial}{\partial t}\Phi &= \frac{1}{\mathcal{S}-\alpha}\mathcal{L}_F F - \frac{1}{\mathcal{S}-\alpha}(h-F)F^{pq}h_{pk}h_q^k \\ &\quad - \frac{F}{(\mathcal{S}-\alpha)^2}\mathcal{L}_F\mathcal{S} - \frac{F}{(\mathcal{S}-\alpha)^2}\mathcal{S}F^{pq}h_{pk}h_q^k - \frac{F}{(\mathcal{S}-\alpha)^2}\big(h-(m\gamma+1)F\big).\end{aligned} \tag{5.6}$$

Therefore by combining (5.5) and (5.6), we obtain the expression

$$\begin{aligned}\frac{\partial}{\partial t}\Phi &= \mathcal{L}_F\Phi + \frac{2}{\mathcal{S}-\alpha}(\nabla\Phi, \nabla\mathcal{S})_F - \frac{1}{\mathcal{S}-\alpha}(h-F)F^{pq}h_{pk}h_q^k \\ &\quad - \frac{1}{(\mathcal{S}-\alpha)^2}\mathcal{S}FF^{pq}h_{pk}h_q^k - \frac{1}{(\mathcal{S}-\alpha)^2}hF + \frac{1}{(\mathcal{S}-\alpha)^2}(m\gamma+1)F^2 \\ &= \mathcal{L}_F\Phi + \frac{2}{\mathcal{S}-\alpha}(\nabla\Phi, \nabla\mathcal{S})_F \\ &\quad + \frac{1}{(\mathcal{S}-\alpha)^2}\big\{(m\gamma+1)F^2 - hF - \alpha FF^{pq}h_{pk}h_q^k - h(\mathcal{S}-\alpha)F^{pq}h_{pk}h_q^k\big\},\end{aligned}$$

which establishes the lemma. □

For $t \in [0, T^*]$, M_t is smooth, compact and convex, and therefore the m^{th} mean curvature H_m and the mean curvature H are uniformly bounded in this time interval. Similarly, the m^{th} mean curvature $\tilde{H}_m$ and the mean curvature $\tilde{H}$ of $\tilde{M}$ are also bounded in the corresponding time interval. Moreover we can prove the following

Lemma 5.4. *There exist positive constants C_8 and C_9 such that for any $\tilde{t} \in [0, \tilde{T}_{\max})$,*

$$\tilde{H}_m(x, \tilde{t}) \le C_8 \quad \textit{and} \quad \tilde{H}(x, \tilde{t}) \le C_9, \quad \forall x \in M^n.$$

Proof. Let $\tilde{T}^* = \int_0^{T^*} \psi(t)^{m\gamma+1}dt$. For any $\tilde{t} \in [0, \tilde{T}^*]$, $\tilde{M}_{\tilde{t}}$ is a smooth, compact and convex hypersurface, the velocity $\tilde{F}$ of the flow is therefore uniformly bounded in $[0, \tilde{T}^*]$.

Consider any time $t_0 \in [T^*, T_{\max})$, and choose the origin of $\mathbb{R}^{n+1}$ to be the center of the sphere of radius $r_-(t_0)$, which is enclosed by $X(\cdot, t_0)$. By Lemma 5.1, on the time interval $[T^*, t_0]$, the function satisfies

$$\mathcal{S} = (X, \mathbf{v}) \geq r_-(t_0).$$

Let $\alpha = \frac{1}{2}r_-(t_0)$, we consider the function $\Phi(x,t)$ defined in (5.4) for any $(x,t) \in M^n \times [T^*, t_0]$. Let $(x_1, t_1) \in M^n \times [T^*, t_0]$ be such that Φ achieves the maximum $\sup\{\Phi(x,t) | (x,t) \in M^n \times [T^*, t_0]\}$. If $t_1 = T^*$, we are done, since in this case, $F(x, t_0) \leq constant$. Thus we may assume $t_1 > T^*$, then by Lemma 5.3, at (x_1, t_1)

$$(m\gamma+1)F^2 - hF - \alpha F F^{pq} h_{pk} h_q^k - h(\mathcal{S} - \alpha) F^{pq} h_{pk} h_q^k \geq 0.$$

We use

$$F^{pq} h_{pk} h_q^k = F^i \lambda_i^2 \geq \varepsilon H F^i \lambda_i = \varepsilon m\gamma F H$$

to obtain

$$(m\gamma+1)F(F+h) \geq (m\gamma+1)F^2 - hF \geq \alpha\varepsilon m\gamma H F(F+h),$$

then

$$H(x_1, t_1) \leq \frac{m\gamma+1}{\alpha\varepsilon m\gamma}.$$

By Lemma 2.1 (4) we have

$$F(x_1, t_1) \leq \Big(\frac{H(x_1,t_1)}{n}\Big)^{m\gamma} \leq \Big(\frac{m\gamma+1}{\alpha\varepsilon m n\gamma}\Big)^{m\gamma}.$$

Therefore for any $x \in M^n$,

$$\Phi(x, t_0) = \frac{F(x,t_0)}{\mathcal{S}(x,t_0) - \alpha} \leq \Phi(x_1, t_1),$$

which implies

$$F(x, t_0) \leq \Big(\frac{2(m\gamma+1)}{\varepsilon m n\gamma}\Big)^{m\gamma} \frac{1}{r_-(t_0)^{m\gamma}},$$

where we have used Lemma 2.3. By combining with Proposition 4.3, we have

$$\tilde{F}(x, \tilde{t}_0) \leq \Big(\frac{2C_7(m\gamma+1)}{\varepsilon m n\gamma}\Big)^{m\gamma},$$

for all $x \in M^n$. Here $\tilde{t}_0 = \int_0^{t_0} \psi(t)^{m\gamma+1}dt$.

Since $t_0 \in [T^*, T_{\max})$ is arbitrary, $\tilde{t}_0 \in [\tilde{T}^*, \tilde{T}_{\max})$ is also arbitrary, we thus have the uniform bound on $\tilde{F}$ in $[\tilde{T}^*, \tilde{T}_{\max})$. Combination with the bound in $[0, \tilde{T}^*]$, we at last arrive at the inequality $\tilde{F}(x, \tilde{t}) \leq C_{10}$, for a constant C_{10}.

Then

$$\tilde{H}_m(x,\tilde{t}) = \tilde{F}(x,\tilde{t})^{1/\gamma} \leq C_{10}^{1/\gamma} = C_8, \quad \forall x \in M^n.$$

Moreover, by homogeneity and the inequality (3) in Lemma 2.1, we have

$$m\gamma F = F^i\lambda_i \geq \varepsilon H \mathrm{tr}(F^i) \geq \varepsilon H m\gamma F^{1-\frac{1}{m\gamma}}.$$

We therefore at last get

$$\tilde{H}(x,\tilde{t}) \leq \varepsilon^{-1}\tilde{F}(x,\tilde{t})^{1/m\gamma} = \varepsilon^{-1}\tilde{H}_m^{1/m} \leq \varepsilon^{-1}C_8^{1/m} = C_9, \quad \forall x \in M^n. \tag{5.7}$$

□

The above result allows us to obtain a uniform upper bound on the quantity $\tilde{\theta}$.

Lemma 5.5. *There exists a positive constant* C_{11} *such that*

$$\tilde{\theta}(\tilde{t}) \leq C_{11} \tag{5.8}$$

holds for any $\tilde{t} \in [0, \tilde{T}_{\max})$.

Proof. Let $\tilde{T}^* = \int_0^{T^*} \psi(t)^{m\gamma+1}dt$. For any $\tilde{t} \in [0, \tilde{T}^*]$, $\tilde{h}(\tilde{t})$ is a nonnegative continuous function, $\tilde{h}$ is therefore uniformly bounded in $[0, \tilde{T}^*]$.

By the assumption (1.5), $h(t)$ has a uniformly upper bound h^+ on $[0, T_{\max})$. Since $\Lambda = \infty$, we can assume that for any $t \geq T^*$, $\psi(t)$ is greater than any given positive number N. Then, we have

$$\tilde{h}(\tilde{t}) = \psi^{-m\gamma}h \leq N^{-m\gamma}h^+, \quad \forall \tilde{t} \geq \tilde{T}^*,$$

here $\tilde{t} = \int_0^t \psi(\tau)^{m\gamma+1}d\tau$.

Therefore, combination with the bound in $[0, \tilde{T}^*]$, we at last arrive at the uniform bound on $\tilde{h}$ in $[0, \tilde{T}_{\max}]$, that is, there exists a constant C_{12} such that for any $\tilde{t} \in [0, \tilde{T}_{\max})$,

$$\tilde{h}(\tilde{t}) \leq C_{12}. \tag{5.9}$$

Note that

$$\tilde{\theta} = \psi^{-m\gamma-1}\frac{\int_M (F-h)H d\mu}{\int_M d\mu} = \frac{\int_M(\tilde{F}-\tilde{h})\tilde{H}d\mu}{\int_M d\mu} \leq \frac{\int_M(\tilde{F}+\tilde{h})\tilde{H}d\mu}{\int_M d\mu} \leq (\tilde{F}+\tilde{h})\tilde{H},$$

combination with (5.9), the lemma follows. □

Since our hypersurfaces are convex, the bound on $\tilde{H}$ we have just obtained implies a bound on all principal curvatures. As a consequence, one can prove that $\tilde{M}_{\tilde{t}}$ can be locally written as a graph with uniformly bounded C^2 norm (see [2, 15, 17]).

Corollary 5.6. *There exists* $\tilde r, \eta > 0$ (*depending only on* $\max \tilde H$) *with the following property. Given any* $(\bar p, \bar t) \in \tilde M \times (0, \tilde T_{\max})$, *there is a neighborhood* $\tilde{\mathcal{U}}$ *of the point* $\bar x = \tilde X(\bar p, \bar t)$ *such that* $\tilde M_{\tilde t} \cap \tilde{\mathcal{U}}$ *coincides with the graph of a smooth function*

$$\tilde u : B_{\tilde r} \times \mathcal{I} \to \mathbb{R}, \quad \textit{for all } \tilde t \in \mathcal{I}.$$

Here $B_{\tilde r} \subset T_{\bar p} \tilde M_{\bar t}$ *is the ball of radius* $\tilde r$ *centered at* $\bar x$ *in the hyperplane tangent to* $\tilde M_{\bar t}$ *at* $\bar x$, *and* $\mathcal{I}$ *is the time interval* $\mathcal{I} = (\max\{\bar t - \eta, 0\}, \min\{\bar t + \eta, \tilde T_{\max}\})$. *In addition, the* C^2 *norm of* $\tilde u$ *is uniformly bounded* (*by a constant depending only on* $\max \tilde H$).

Next we want to prove the existence of the limiting hypersurface (see [2]). To do this, an essential step is the derivation of some kind of estimate on the curvature (e.g. a Harnack inequality, or a Hölder estimate) which is uniform in time. Such an estimate allows us to say that, if the curvature is positive at a given point of our hypersurface, then it also satisfies a uniform lower bound in a whole neighborhood. However, there is a difficulty in deriving this type of inequalities, which has been pointed out in [15] and is related to the fact that the speed we are considering has a homogeneity degree greater than one in the curvatures; namely, we cannot ensure a condition that the evolution equations for the curvatures are uniformly parabolic. In fact, the operators $\mathcal{L}_{\tilde c}$ and $\mathcal{L}_{\tilde F}$ which appear in the equations, in contrast with the standard Laplacian $\triangle$, become degenerate if the curvatures go to zero, and this is exactly the behavior we are not able to exclude at this stage.

Consequently, we will make use of regularity theory for degenerate parabolic equations. Similar as in [15], we will prove a uniform C^α-estimate for the m^{th} mean curvature $\tilde H_m$ by means of Lemma 2.5, valid for equations of porous medium type. The procedure here is more complicate than in [15]; in particular, it is necessary to rewrite down the evolution equation for $\tilde H_m$ in a particular form which suits to the hypotheses of the regularity theorem.

Lemma 5.7. *In a local coordinate system, the evolution equation for the* m^{th} *mean curvature* $\tilde H_m$ *under rescaled flow* (1.3) *can be written as*

$$\frac{\partial}{\partial \tilde t} \tilde H_m = D_i\Big(\frac{\gamma}{d} \tilde H_m^{\frac{1-m}{m}} \tilde c^{ij} D_j \tilde H_m^d\Big) + \tilde\Gamma^j_{jl} \tilde c^{li} D_i \tilde H_m^\gamma + \big(\tilde H_m^\gamma - \tilde h\big) \tilde c^{ij} \tilde h_{ik} \tilde h^k_j - \frac{m}{n} \tilde\theta \tilde H_m, \tag{5.10}$$

where $d = \gamma + \frac{m-1}{m}$ *and* D_i *denote derivatives with respect to the coordinates.*

Proof. By the definition of c^{ij}, we can write down $\tilde F^{ij} = \gamma \tilde H_m^{\gamma-1} \tilde c^{ij}$, thus the evolution equation for $\tilde H_m$ becomes

$$\frac{\partial}{\partial \tilde t} \tilde H_m = \mathcal{L}_{\tilde c} \tilde F + (\tilde F - \tilde h) \tilde c^{ij} \tilde h_{ik} \tilde h^k_j - \frac{m}{n} \tilde\theta \tilde H_m. \tag{5.11}$$

As in [2], using part (5) in Lemma 2.1 we have

$$\mathcal{L}_{\tilde c} = D_i\big(\tilde c^{ij} D_j\big) + \tilde\Gamma^i_{il} \tilde c^{lj} D_j.$$

Then, if we set $d = \gamma + \frac{m-1}{m}$, we have

$$\mathcal{L}_{\tilde{c}}\tilde{F} = \frac{\gamma}{d} D_i\Big(\tilde{c}^{ij}\tilde{H}_m^{\frac{1-m}{m}} D_j \tilde{H}_m^d\Big) + \tilde{\Gamma}^i_{il}\tilde{c}^{lj} D_j \tilde{F}.$$

By substitution of this in (5.11), we reach formula (5.10) in the statement. $\square$

Same as in [2], choosing a coordinate system such that $\tilde{g}_{ij}$ is identity and $\tilde{h}_{ij}$ is diagonal at a given point, we can easily estimate the tensor $\tilde{c}^{ij}$. It is easy to see that

$$\tilde{c}^{ij} = \frac{\partial \tilde{H}_m}{\partial \tilde{\lambda}_i}\delta^{ij}.$$

By definition of H_m, the derivative $\partial \tilde{H}_m/\partial\tilde{\lambda}_i$ is a sum of products of $m-1$ principal curvatures. Hence we find, for any vector $\xi \in \mathbb{R}^n$,

$$C'_{m,n}\tilde{\lambda}_1^{m-1}|\xi|^2 \le \tilde{c}^{ij}\xi_i\xi_j \le C''_{m,n}\tilde{\lambda}_n^{m-1}|\xi|^2$$

for suitable constants $C'_{m,n}$, $C''_{m,n}$ depending only on m, n. It follows

$$C'_{m,n}(\varepsilon\tilde{H})^{m-1}|\xi|^2 \le \tilde{c}^{ij}\xi_i\xi_j \le C''_{m,n}\tilde{H}^{m-1}|\xi|^2.$$

For short, we express a double bound like the above one by writing down $\tilde{c}^{ij} \approx \tilde{H}^{m-1}\tilde{g}^{ij}$. With this notation, we also obtain

$$\tilde{H}_m^{\frac{1-m}{m}}\tilde{c}^{ij} \approx \tilde{g}^{ij}, \tag{5.12}$$

because $\tilde{H} \approx \tilde{H}_m^{1/m}$ by part (4) of Lemma 2.1 and (5.7). Another useful inequality related to $\tilde{c}$ is

$$\tilde{c}^{ij}\tilde{h}_{ik}\tilde{h}^k_j = \tilde{c}^{ii}\tilde{\lambda}_i^2 \le C''_{m,n}\tilde{H}^{m-1}\sum_i \tilde{\lambda}_i^2 = C''_{m,n}\tilde{H}^{m-1}|\tilde{A}|^2 \le B''_{m,n}C_9^{m+1}, \tag{5.13}$$

which is true thanks to convexity and Lemma 5.4.

Similar as in [2, 15], we also need the following lemma.

Lemma 5.8. *There is a constant $C_{13} > 0$ depending on n, m, γ and M_0 such that*

$$\int_{\tilde{t}_1}^{\tilde{t}_2}\int_{\tilde{M}} |\nabla \tilde{H}_m^d|^2 d\tilde{\mu}_{\tilde{t}} d\tilde{t} \le C_{13}(1 + \tilde{t}_2 - \tilde{t}_1).$$

Proof. Using (5.12), by direct computation we have

$$\begin{aligned}\int_{\tilde{M}} |\nabla \tilde{H}_m^d|^2 d\tilde{\mu}_{\tilde{t}} &\approx \int_{\tilde{M}} \tilde{H}_m^{\frac{1-m}{m}} |\nabla \tilde{H}_m^d|^2_{\tilde{c}} d\tilde{\mu}_{\tilde{t}} \\ &= \frac{d}{\gamma}\int_{\tilde{M}} \big(\nabla\tilde{F}, \nabla\tilde{H}_m^d\big)_{\tilde{c}} d\tilde{\mu}_{\tilde{t}} = -\frac{d}{\gamma}\int_{\tilde{M}} \tilde{H}_m^d \mathcal{L}_{\tilde{c}}\tilde{F} d\tilde{\mu}_{\tilde{t}},\end{aligned}$$

where the last equality follows using integration by parts and Lemma 2.1 (5). Next, we can use the evolution equation (5.11) to deduce

$$\begin{aligned}\int_{\tilde{M}} |\nabla \tilde{H}_m^d|^2 d\tilde{\mu}_{\tilde{t}} &\approx -\frac{d}{\gamma(d+1)}\int_{\tilde{M}} \frac{\partial \tilde{H}_m^{d+1}}{\partial \tilde{t}} d\tilde{\mu}_{\tilde{t}} + \frac{d}{\gamma}\int_{\tilde{M}} (\tilde{F}-\tilde{h})\tilde{H}_m^d \tilde{c}^{ij}\tilde{h}_{ik}\tilde{h}_j^k d\tilde{\mu}_{\tilde{t}} - \frac{md}{n\gamma}\int_{\tilde{M}} \tilde{\theta}\tilde{H}_m^{d+1} d\tilde{\mu}_{\tilde{t}} \\ &\leq C_{14}\frac{d}{d\tilde{t}}\int_{\tilde{M}} \tilde{H}_m^{d+1} d\tilde{\mu}_{\tilde{t}} + C_{15}.\end{aligned}$$

Notice that C_{15} comes from (4.5), (5.8), (5.9), (5.13) and the bounds in Lemma 5.4. Finally, recall $0 < \tilde{H}_m \leq C_8$ and, by Proposition 4.3, $\tilde{M}_{\tilde{t}}$ is contained in a ball of radius C_7; these facts can be applied (after integrating the above inequality on $[\tilde{t}_1, \tilde{t}_2]$) to achieve the estimate in the statement. □

Proposition 5.9. *There are constants $C_{16}(n, m, \gamma, M_0) > 0$ and $0 < \alpha < 1$ such that for every $(\bar{x}, \bar{t}) \in \tilde{M} \times (0, \tilde{T}_{\max})$, the α-Hölder norm in space-time of $\tilde{H}_m$ on a neighborhood, say $\tilde{\mathcal{U}} \subset \tilde{M} \times (0, \tilde{T}_{\max})$ is bounded by C_{16}, i.e.*

$$\|\tilde{H}_m\|_{C^\alpha(\tilde{\mathcal{U}})} \leq C_{16}.$$

Proof. We use the local parametrization of $\tilde{M}_{\tilde{t}}$ as the graph of a function $\tilde{u} : B_{\tilde{r}} \times \mathcal{I} \to \mathbb{R}^{n+1}$ coming from Corollary 5.6, where $\mathcal{I} = (\max\{\bar{t}-\tau, 0\}, \bar{t}+\tau)$ for τ not depending on $\bar{t}$. Using D_i to denote the derivatives with respect to these local coordinates, and choosing as positive normal the one which points below, we have (see [2, 5, 17])

$$\tilde{g}_{ij} = \delta_{ij} + D_i\tilde{u}D_j\tilde{u}, \qquad \tilde{g}^{ij} = \delta^{ij} - \frac{D^i\tilde{u}D^j\tilde{u}}{1+|D\tilde{u}|^2},$$

and

$$\tilde{h}_{ij} = \frac{D_{ij}\tilde{u}}{\left(1+|D\tilde{u}|^2\right)^{\frac{1}{2}}}.$$

In addition, the Christoffel symbols have the expression

$$\tilde{\Gamma}_{ij}^k = \Big(\delta^{kl} - \frac{D^k\tilde{u}D^l\tilde{u}}{1+|D\tilde{u}|^2}\Big) D_{ij}\tilde{u}D_l\tilde{u}. \tag{5.14}$$

It is not restrictive to assume that $\bar{t} > \tau$, since the C^α norm of $\tilde{H}_m$ on $\tilde{M} \times [0, \tau]$ is clearly finite by the compactness of $\tilde{M}$.

If we consider $\tilde{c}^{ij}\tilde{h}_{ik}\tilde{h}_j^k$ as a function of $x \in B_{\tilde{r}}$, (5.10) can be regarded as an equation of the form (2.2) for $\nu = \tilde{H}_m$, with

$$a^{ij} = \frac{\gamma}{d}\tilde{H}_m^{\frac{1-m}{m}}\tilde{c}^{ij}, \quad \text{and} \quad f(x,t,\nu,D\nu) = \tilde{\Gamma}_{jl}^j\tilde{c}^{li}D_i\tilde{H}_m^\gamma + (\tilde{H}_m^\gamma - \tilde{h})\tilde{c}^{ij}\tilde{h}_{ik}\tilde{h}_j^k - \frac{m}{n}\tilde{\theta}\tilde{H}_m.$$

Next, notice that

$$a(\xi,\xi) = \frac{\gamma}{d}\tilde{H}_m^{\frac{1-m}{m}}\tilde{c}(\xi,\xi) \approx \frac{\gamma}{d}\tilde{g}^{ij}(\xi,\xi) = \frac{\gamma}{d}|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n, \tag{5.15}$$

combination of (5.8), (5.9), (5.13) and the bounds in Lemma 5.4 yields

$$|f| \leq \frac{\gamma}{d}|\tilde{\Gamma}^j_{jl}\tilde{c}^{li}\tilde{H}_m^{\frac{1-m}{m}} D_i\tilde{H}^d_m| + b_2 \approx \frac{\gamma}{d}|\tilde{\Gamma}^j_{jl}\tilde{g}^{li}D_i\tilde{H}^d_m| + b_2 \leq b_1|D\tilde{H}^d_m| + b_2,$$

where b_1 comes from (5.14) and the fact that $\tilde{u}$ is C^2-uniformly bounded. Moreover, Lemma 5.8 implies

$$\int\int_{B_{\tilde{r}}\times\mathcal{I}} |D\tilde{H}^d_m|^2 d\tilde{\mu}_{\tilde{t}} d\tilde{t} \leq C_{17}(\tau).$$

Therefore, we are in position to apply Lemma 2.5 with $r' = \tilde{r}/2$ and $\delta = \tau/2$ to deduce that

$$\|\tilde{H}_m\|_{C^\alpha\left(B_{\frac{\tilde{r}}{2}}\times[\bar{t}-\frac{\tau}{2},\bar{t}+\frac{\tau}{2}]\right)} \leq C_{16},$$

for suitable $0 < \alpha < 1$ and $C_{16} > 0$ depending on n, m, γ and M_0. □

With the above result, we have

Proposition 5.10. *Let M^n be a compact n-dimensional smooth manifold and $\tilde{X} : M^n \to \mathbb{R}^{n+1}$ be an immersion pinched in the sense that $Q \geq C_0$. If $\tilde{M}_{\tilde{t}} = \tilde{X}_{\tilde{t}}(M^n)$ is the solution to the rescaled flow (1.3), then $\tilde{M}_{\tilde{t}}$ exists on $[0, \infty)$.*

Proposition 5.11. *The quotient $\tilde{Q}$ converges to $\frac{1}{n^n}$ uniformly on $\tilde{M}$ as $\tilde{t} \to \infty$.*

Proof. Let us set $\rho = \frac{1}{n^n} - \frac{\tilde{K}}{\tilde{H}^n}$. By (3.2) we have the following evolution equation

$$\begin{aligned}
\frac{\partial}{\partial\tilde{t}}\rho \;=\;& \mathcal{L}_{\tilde{F}}\rho + \frac{n+1}{n\tilde{H}^n}\big(\nabla\tilde{H}^n, \nabla\rho\big)_{\tilde{F}} - \frac{n-1}{n\tilde{K}}\big(\nabla\tilde{K}, \nabla\rho\big)_{\tilde{F}} + \frac{\tilde{H}^n}{n\tilde{K}}\big|\nabla\rho\big|^2_{\tilde{F}} \\
& - \frac{\tilde{Q}}{\tilde{H}^2}\big|\tilde{H}\nabla_i\tilde{h}_{pq} - \tilde{h}_{pq}\nabla_i\tilde{H}\big|^2_{\tilde{F},\tilde{b}} - \tilde{Q}\Big(\tilde{b}^{ij} - \frac{n}{\tilde{H}}\tilde{g}^{ij}\Big)\tilde{F}^{pq,rs}\nabla_i\tilde{h}_{pq}\nabla_j\tilde{h}_{rs} \\
& - \big(\tilde{h} + (m\gamma-1)\tilde{F}\big)\frac{\tilde{Q}}{\tilde{H}}\big(n|\tilde{A}|^2 - \tilde{H}^2\big).
\end{aligned} \tag{5.16}$$

Applying the maximum principle to equation (5.16), we obtain

$$\max_{\tilde{M}_{\tilde{t}}} \rho \leq \max_{\tilde{M}_0} \rho,$$

where we have used (see Corollary 3.6)

$$\frac{\tilde{Q}}{\tilde{H}^2}\big|\tilde{H}\nabla_i\tilde{h}_{pq} - \tilde{h}_{pq}\nabla_i\tilde{H}\big|^2_{\tilde{F},\tilde{b}} + \tilde{Q}\Big(\tilde{b}^{ij} - \frac{n}{\tilde{H}}\tilde{g}^{ij}\Big)\tilde{F}^{pq,rs}\nabla_i\tilde{h}_{pq}\nabla_j\tilde{h}_{rs} \geq 0.$$

Therefore we have shown that the function $\max_{\tilde{M}_{\tilde{t}}}\rho$ is strictly decreasing unless ρ converges to 0 uniformly on $\tilde{M}$. This implies that the quotient $\tilde{Q}$ converges to $\frac{1}{n^n}$ uniformly on $\tilde{M}$ as $\tilde{t} \to \infty$. □

Theorem 5.12. *Let M^n be a compact n-dimensional smooth manifold and $\tilde{X} : M^n \to \mathbb{R}^{n+1}$ be an immersion pinched in the sense that $Q \geq C_0$. If $\tilde{M}_{\tilde{t}} = \tilde{X}_{\tilde{t}}(M^n)$ is the solution to rescaled flow (1.3), then $\tilde{M}_{\tilde{t}}$ converges exponentially as $\tilde{t} \to \infty$ to a round sphere in the C^∞-topology.*

Proof. Let us take any sequence $\{\tau_j\} \subset [0, \infty)$ with $\tau_j \to \infty$. The uniform bounds on the curvatures imply that there exists a subsequence (again denoted by τ_j) such that, up to translations,

$$\tilde{X}(\cdot, \tau_j) \to \tilde{X}_\infty(\cdot),$$

in the $C^{1,\alpha}$-topology for any $\alpha \in (0,1)$, and $\tilde{M}_\infty = \tilde{X}_\infty(M^n)$ is a convex $C^{1,1}$-hypersurface. By Proposition 4.3, at each time τ_j we can find a point $p_j \in \tilde{M}$ such that

$$\tilde{H}(p_j, \tau_j) \geq \frac{n}{C_7}.$$

Then (5.7) yields

$$\tilde{H}_m(p_j, \tau_j) \geq \varepsilon^m \tilde{H}^m(p_j, \tau_j) \geq \left(\frac{\varepsilon n}{C_7}\right)^m = C_{18} > 0 \tag{5.17}$$

for each fixed j. Proposition 5.9 implies that $\tilde{H}_m$ cannot decrease too fast in the sense that we can find a $\delta > 0$ (independent of (p_j, τ_j)) satisfying

$$\tilde{H}_m|_{B_\delta(p_j) \times [\tau_j - \delta, \tau_j + \delta]} \geq \frac{C_{18}}{2}. \tag{5.18}$$

If δ is small enough, then $\tilde{M}_{\tilde{t}} \cap B_\delta(p_j)$ can be written as the graph of a function $\tilde{u}_j$ for any $\tilde{t} \in [\tau_j - \delta, \tau_j + \delta]$ as in Corollary 5.6. Using the arguments of the proof of Theorem 6.4 in [2] on any neighborhood $B_\delta(p_j) \times [\tau_j - \delta, \tau_j + \delta]$, but with upper bounds independent of time, we can obtain uniform C^∞-estimates on the functions $\tilde{u}_j$ in suitable smaller neighborhood, say of radius $\frac{\delta}{2}$. Therefore, we have that

$$B_{\frac{\delta}{2}}(p_j) \cap \tilde{M}_{\tau_j} \to B_{\frac{\delta}{2}}(p_\infty) \cap \tilde{M}_\infty \quad \text{in } C^\infty,$$

where $\tilde{X}(p_j, \tau_j) \to p_\infty \in \tilde{M}_\infty$.

Recall that, by Proposition 5.11, the limit must be totally umbilic, and therefore is a portion of sphere. By (5.17), the sphere has $\tilde{H}_m$ curvature at least C_{18}. Then, in the neighborhoods $B_{\frac{\delta}{2}}(p_j) \times [\tau_j - \frac{\delta}{2}, \tau_j + \frac{\delta}{2}]$ the $\tilde{H}_m$ curvature becomes arbitrarily close to a constant value not smaller than C_{18}. Using again the uniform Hölder continuity, we deduce that (5.18) holds, for j large, in $B_{\frac{3}{2}\delta}(p_j)$ instead of $B_\delta(p_j)$. Thus we can extend the region where $\tilde{M}_\infty$ is known to be spherical. After a finite number of iterations, we deduce that $\tilde{M}_\infty$ is a sphere, whose radius is uniquely determined by the fixed total area $|M_0|$.

Since the above argument can be applied to any sequence τ_j, we conclude that the whole family $\tilde{M}_{\tilde{t}}$ converges to a sphere as $\tilde{t} \to \infty$, possibly up to a translation in space. This implies that $\tilde{H}_m$ tends to a positive constant as $\tilde{t} \to \infty$, and therefore it is bounded from below by some constant $\tilde{\delta} > 0$ for long times. Consequently, the mean curvature $\tilde{H}$ is bounded from below by $n\tilde{\delta}^{1/m} > 0$ for long times. Thus, we can repeat the argument in the proof of Proposition 5.11 using

$$\frac{\partial}{\partial \tilde{t}}\rho(\tilde{t}) \leq \mathcal{L}_{\tilde{F}}\rho + \frac{n+1}{n\tilde{H}^n}\left(\nabla \tilde{H}^n, \nabla \rho\right)_{\tilde{F}} - \frac{n-1}{n\tilde{K}}\left(\nabla \tilde{K}, \nabla \rho\right)_{\tilde{F}} + \frac{\tilde{H}^n}{n\tilde{K}}\left|\nabla \rho\right|^2_{\tilde{F}} - n(m\gamma - 1)C_0\delta\tilde{\delta}^{\gamma+1/m}\rho.$$

Then there exists a $\delta' > 0$ and a constant C_{19}, such that

$$\rho(\tilde{t}) \leq C_{19} e^{-\delta' \tilde{t}}.$$

In this way we obtain that the rate of convergence of $\tilde{Q}$ to $\frac{1}{n^n}$ is exponential. With this property, we can argue exactly as in [15] to conclude that the second fundamental form of $\tilde{M}_{\tilde{t}}$ converges exponentially in C^∞ to the one of a sphere. The C^∞ convergence of the second fundamental form implies the C^∞ convergence of the immersions and of the metric by standard arguments (see [1, 15]). This implies that $\tilde{M}_{\tilde{t}}$ approaches a sphere as $\tilde{t} \to \infty$. Of course $\tilde{M}_\infty$ has the same total area $|M_0|$. Therefore the proof of Theorem 1.1 (I) is completed. □

Remark 5.13. *It is easy to check that* $0 \leq h < \inf_{x \in M^n} F(x, 0)$ *is of this case, and* T^* *below* (5.1) *is equal to zero.*

6 Case (II) $0 < \Lambda < \infty$

In this section we consider the case $0 < \Lambda < \infty$ and prove the main Theorem 1.1 (II). Since $\tilde{r}_+ = \psi r_+$ and $\tilde{r}_- = \psi r_-$, we have by Proposition 4.3

$$\frac{C_7^{-1}}{\psi} \leq r_- \leq r_+ \leq \frac{C_7}{\psi},$$

which implies for the sequence $\{T_i\}$ in Section 4, there exists a time $T^* < T_{\max}$ such that for any $T_i \geq T^*$,

$$C_{20}^{-1} \leq r_-(T_i) \leq r_+(T_i) \leq C_{20} \tag{6.1}$$

for some constant C_{20}. The following lemma shows that the inner and outer radii of all evolving hypersurfaces M_t are uniformly bounded from below and above.

Lemma 6.1. *There exists a constant* C_{21} *such that*

$$C_{21}^{-1} \leq r_-(t) \leq r_+(t) \leq C_{21}, \qquad \textit{for any } t \in [0, T_{\max}).$$

Proof. We only prove the upper bound, the lower bound is similar. First we claim that $\bar{h} > 0$ in this case, where $\bar{h}$ is the limit in (1.5). Suppose not, we can take any $h^+ > 0$, such that there exists a time $T' < T_{\max}$ and $h(t) < h^+$ for any $t \in [T', T_{\max})$. Then by similar proof as in Lemma 5.1, we prove that M_t is contracting for $t \geq T'$. Therefore $r_+(T_i) \to 0$ as $T_i \to T_{\max}$, which is a contradiction to (6.1). The claim follows.

¿From the claim we know that there must exist a time $T' \in (T^*, T_{\max})$ such that for any $t \in [T', T_{\max})$, $h(t)$ has a positive lower bound $h^- > 0$.

Since M_t for any $t \in [0, T']$ is smooth, compact and convex, the corresponding outer radius is uniformly bounded from above in this time interval. Suppose there is a time

$T'' > T'$ such that $r_+(T'') > C_{21}$. By Lemma 2.3 we can assume C_{21} is large enough so that $r_-(T'') > (1/h^-)^{1/m\gamma}$. Again, we evolve a sphere $\partial B_{r_-(T'')}(O)$ under (1.1). The solution $r_B(t)$ to the differential inequality

$$\begin{cases} \frac{dr_B(t)}{dt} = h - \frac{1}{r_B(t)^{m\gamma}} \geq h^- - \frac{1}{r_B(t)^{m\gamma}}, \quad t \geq T'', \\ r_B(T'') = r_-(T'') > (1/h^-)^{1/m\gamma}, \end{cases} \tag{6.2}$$

is given by

$$r_B(t) + \frac{1}{h^-}\int_{r_-(T'')}^{r_B(t)} \frac{1}{r_B(t)^{m\gamma} - \frac{1}{h^-}} dr_B(t) \geq h^-(t - T'') + r_-(T''),$$

Clearly $r_B(t) \to \infty$ as $t \to \infty$. On the other hand, by containment principle, $\partial B_{r_B(t)}(O)$ is enclosed by M_t for any $t \geq T''$, since $M_{T''}$ encloses $\partial B_{r_B(T'')}(O)$. Therefore there exists some $T_i > T''$ such that $r_+(T_i) \geq r_B(T_i) > C_{20}$, which is a contradiction to (6.1). Combining the case in $[0, T']$, we finish the proof of the lemma. $\square$

Remark 6.2. *Similar as in Remark* 5.2*, by Lemma* 6.1 *and that the hypersurface* M_t *uniformly converges to a round sphere* (*see below for the proof*)*, we have a limit*

$$\lim_{t \to T_{\max}} \psi(t) = \Lambda. \tag{6.3}$$

Following the procedure of [2, 12, 13], we have the following result.

Lemma 6.3. *Given any* $\bar{t} \in [0, T_{\max})$*, let* $\bar{x} \in V_{\bar{t}}$ *be such that* $B(\bar{x}, \bar{r}) \subset V_{\bar{t}}$*, where* $\bar{r} = r_-(\bar{t})$ *is the inner radius of* $M_{\bar{t}}$*. Then we have*

$$B(\bar{x}, \bar{r}/2) \subset V_t \quad \textit{for every}\ t \in [\bar{t}, \min\{\bar{t} + \tau, T_{\max}\}).$$

for some constant τ *depending only on* n, m, γ, $|M_0|$.

Similar as in Section 5, we consider the function Φ defined in (5.4) for $t \in [\bar{t}, \min\{\bar{t} + \tau, T_{\max}\})$, and $\alpha = \frac{1}{4}C_{21}^{-1}$, where C_{21} is given in Lemma 6.1. By using the same method as in [2, 12], we obtain the uniform upper bounds of the m^{th} mean curvature H_m and the mean curvature H.

Proposition 6.4. *There exist positive constants* C_{22}, C_{23} *depending on* n, m, γ, *and* M_0 *such that*

$$H_m(\cdot, t) \leq C_{22} \quad \textit{and} \quad H(\cdot, t) \leq C_{23},$$

hold for every $t \in [0, T_{\max})$.

By Lemma 6.3 and Proposition 6.4, we can prove as in Corollary 5.6 that M_t can be locally written as a graph of a function u, that is, locally, M_t=graph $u(\cdot,t)$, with uniformly bounded C^2 norm. In particular, using D_i to denote the derivatives with respect to the local coordinates, and choosing as positive normal the one which points below, the Christoffel symbols have the expression (see e.g. [2, 15, 17]):

$$\Gamma_{ij}^k = \Big(\delta^{kl} - \frac{D^k u D^l u}{1+|Du|^2}\Big) D_{ij} u D_l u.$$

Same as in Section 5 (also see [2]), we have the following results.

Lemma 6.5. *In a local coordinate system, the evolution equation for the m^{th} mean curvature H_m under unrescaled flow (1.1) can be written as*

$$\frac{\partial}{\partial t} H_m = D_i\Big(\frac{\gamma}{d} H_m^{\frac{1-m}{m}} c^{ij} D_j H_m^d\Big) + \Gamma_{jl}^j c^{li} D_i H_m^\gamma + \big(H_m^\gamma - h\big) c^{ij} h_{ik} h_j^k,$$

where $d=\gamma+\frac{m-1}{m}$ and D_i denote derivatives with respect to the coordinates.

Lemma 6.6. *There is a positive constant C_{24} such that*

$$\int_{t_1}^{t_2}\int_M |\nabla H_m^d|^2 d\mu_t dt \le C_{24}(1+t_2-t_1).$$

Now we can prove a uniform C^α-estimate for the m^{th} mean curvature H_m, that is, we have

Proposition 6.7. *There are constants $C_{25}>0$ and $0<\alpha<1$ such that for every $(\bar{x},\bar{t}) \in M\times(0,T_{\max})$, the α-Hölder norm in space-time of H_m on a neighborhood, say $\mathcal{U}\subset M\times(0,T_{\max})$ is bounded by C_{25}, i.e.*

$$\|H_m\|_{C^\alpha(\mathcal{U})} \le C_{25}.$$

With the above result, we have:

Proposition 6.8. *Let M^n be a compact n-dimensional smooth manifold and $X: M^n \to \mathbb{R}^{n+1}$ be an immersion pinched in the sense of (1.4). If $M_t = X_t(M^n)$ is the solution to flow (1.1), then $T_{\max}=\infty$.*

Lemma 6.9. *We have*

$$\int_0^\infty H_{\min}(t) = +\infty,$$

where $H_{\min}(t)=\min_M H(\cdot,t)$.

Proof. Set $\mathcal{F}(t)=\min_M F(\cdot,t)$. At a point where the minimum is attained, from Lemma 3.1 (4) we get

$$\frac{d}{dt}\mathcal{F} \ge (F-h)F^{pq}h_{pk}h_q^k.$$

As $FF^{pq}h_{pk}h_q^k$ is always nonnegative because M_t is convex, recalling that $h(t)$ has a uniformly upper bound h^+ on $[0, T_{\max})$, we obtain

$$\frac{d}{dt}\mathcal{F} \geq -hF^{pq}h_{pk}h_q^k \geq -h^+ m\gamma H\mathcal{F} \geq -\frac{m\gamma}{\varepsilon}h^+\mathcal{F}^{1+\frac{1}{m\gamma}} = C_{26}\mathcal{F}^{1+\frac{1}{m\gamma}},$$

for some constant C_{26}. Now the application of a maximum principle to the above inequality yields

$$\mathcal{F} \geq \Big(C_{27} + C_{28}\frac{t}{m\gamma}\Big)^{-m\gamma}, \quad \text{with } C_{27} = \mathcal{F}(0)^{-\frac{1}{m\gamma}},$$

which, together with part (4) of Lemma 2.1, leads to

$$\int_0^\infty H_{\min}(t)dt \geq \lim_{s\to\infty}\int_0^s n\mathcal{F}^{\frac{1}{m\gamma}}(t)dt \geq \lim_{s\to\infty}\int_0^s n\Big(C_{27} + C_{28}\frac{t}{m\gamma}\Big)^{-1}dt = \infty.$$

□

Proposition 6.10. *The quotient Q converges to $\frac{1}{n^n}$ uniformly on M_t as $t \to \infty$.*

Proof. We consider the evolution equation for the quotient Q given in (3.2). From Lemma 6.1, we know that $h(t)$ has a positive lower bound $h^- > 0$ as $t > T'$ with T' given in Lemma 6.1. Combining (1.4) and (2.1), we estimate the term containing h as follows:

$$h\frac{Q}{H}\big(n|A|^2 - H^2\big) \geq h^- QH\frac{n|A|^2 - H^2}{H^2} \geq C_{29}H_{\min}(t)(n^{-n} - Q), \quad \forall t > T',$$

with $C_{29} = h^- C_0\delta$. Set $\mathscr{F} = \sup_M (n^{-n} - Q)(\cdot, t)$. Then $\mathscr{F}$ is a locally Lipschitz continuous function and satisfies

$$\frac{d}{dt}\mathscr{F} \leq \sup_{\mathcal{M}(t)}\frac{\partial}{\partial t}(n^{-n} - Q) = \sup_{\mathcal{M}(t)}\frac{\partial}{\partial t}(-Q),$$

where $\mathcal{M}(t) = \{p \in M | \mathscr{F}(t) = n^{-n} - Q(p,t)\}$. Using (3.2), we obtain

$$\frac{d}{dt}\mathscr{F}(t) \leq \sup_{\mathcal{M}(t)}[-C_{29}H_{\min}(t)(n^{-n} - Q)] = -C_{29}H_{\min}(t)\mathscr{F}(t).$$

Since the sum of the gradient terms in the second row of (3.2) is nonnegative, as shown in the proof of Corollary 3.6 (or see Theorem 4.3 in [2]). By the maximum principle, we deduce that

$$\ln\mathscr{F}(t) \leq \ln\mathscr{F}(0) - C_{29}\int_0^t H_{\min}(\tau)d\tau \to -\infty.$$

This allows us to conclude that $\lim_{t\to\infty}\mathscr{F}(t) = 0$, from which the proposition follows. □

Same as in Theorem 5.12 (also see Theorem 7.7 in [2]) we can prove that the solution to (1.1) converges, exponentially in the C^∞−topology, to a round sphere as $t \to \infty$. This finishes the proof of Theorem 1.1 (II).

Remark 6.11. *By the limit (6.3), we easily see that $\tilde{M}_{\tilde{t}}$ converges to a sphere of total area $|M_0|$.*

7 Case (III) $\Lambda = 0$

This section is devoted to discuss the case $\Lambda = 0$ and prove the main Theorem 1.1 (III). Similar to Section 5, we have a limit

$$\lim_{T_i \to T_{\max}} r_-(T_i) = \infty. \tag{7.1}$$

Then there exists a time $T^* < T_{\max}$ such that for any $T_i \geq T^*$, $r_-(T_i)$ is greater than any given positive number N. As before we evolve $\partial B_{r_-(T^*)}(O)$ and $\partial B_{r_+(T^*)}(O)$ under (1.1), respectively. That is to say, they satisfy the following equation

$$\frac{dr_B(t)}{dt} = h(t) - \frac{1}{r_B(t)^{m\gamma}}, \quad t \geq T^*, \tag{7.2}$$

with initial data $r_-(T^*)$ and $r_+(T^*)$ respectively.

First we consider the case $\bar{h} = 0$. Integrating (7.2) from T^* to T_i and using integral mean-value theorem, the outer radius $r_B^+(t)$ of M_t satisfies

$$r_B^+(T_i) - r_+(T^*) = \Big[h(t_i) - \frac{1}{r_B^+(t_i)^{m\gamma}}\Big](T_i - T^*), \tag{7.3}$$

where $t_i \in [T^*, T_i]$.

If we suppose $T_{\max} < \infty$, and take limits of both sides in (7.3), we have $\lim_{t\to T_{\max}} h(t) = \infty$, which contradicts to $\bar{h} = 0$. So $T_{\max} = \infty$.

Next we consider the case $0 < \bar{h} < \infty$. In this case, we choose N greater than $(1/h^-)^{1/m\gamma}$ (now h^- is the uniform positive lower bound of $h(t)$ in $[T^*, T_{\max})$). Therefore by (7.2), the inner radius $r_B^-(t)$ and outer radius $r_B^+(t)$ of M_t satisfy the following inequalities, respectively

$$r_B^-(t) + \frac{1}{h^-}\int_{r_-(T^*)}^{r_B^-(t)} \frac{1}{r_B^-(t)^{m\gamma} - \frac{1}{h^-}} dr_B^-(t) \geq h^-(t - T^*) + r_-(T^*), \tag{7.4}$$

and

$$r_B^+(t) + \frac{1}{h^-}\int_{r_+(T^*)}^{r_B^+(t)} \frac{1}{r_B^+(t)^{m\gamma} - \frac{1}{h^-}} dr_B^+(t) \geq h^-(t - T^*) + r_+(T^*). \tag{7.5}$$

Lemma 7.1. *When $t \geq T^*$, the regions enclosed by the hypersurfaces M_t are increasing. Furthermore $T_{\max} = \infty$, and the solutions to (1.1) expand uniformly to ∞ as $t \to \infty$.*

Proof. For $t \geq T^*$, (7.2) implies that $r_B(t)$ is increasing since $r_B(t) > (1/h^-)^{1/m\gamma}$ initially. By containment principle again, the enclosed regions of M_t are increasing. Moreover, all $M_t\,'s$ are contained in the regions between $\partial B_{r_B^-(t)}(O)$ and $\partial B_{r_B^+(t)}(O)$ for every $t \in [T^*, T_{\max})$.

Suppose $T_{\max}$ is finite. Integrating equation (7.2) from T^* to t, we have

$$r_B^+(t) - r_+(T^*) = \int_{T^*}^t h(\tau)d\tau - \int_{T^*}^t \frac{1}{r_B^+(\tau)^{m\gamma}}d\tau,$$

which implies that $r_B^+(t)$ is uniformly bounded from above in $[T^*, T_{\max})$. This is a contradiction to (7.1). Therefore $T_{\max} = \infty$.

Obviously the diameter of the biggest ball enclosed by M_t tends to ∞ as $t \to \infty$ by (7.4), (7.5) and the containment principle, which implies that M_t expands to ∞ as $t \to \infty$ in this case. The lemma follows. □

Remark 7.2. *Lemma* 7.1 *and Proposition* 4.3 *imply the limit*

$$\lim_{t\to\infty} \psi(t) = 0. \tag{7.6}$$

We do not know whether the rescaled m^{th} mean curvature $\tilde{H}_m$ is uniformly bounded from above or not, but we can prove that if the rescaled hypersurface $\tilde{M}_{\tilde{t}}$ converges to a smooth hypersurface, it must be a sphere. To this end, we need to estimate the lower bound of the rescaled m^{th} mean curvature. Again we consider the function

$$\Phi = \frac{F}{\beta - \mathcal{S}}$$

for some constant β. As in Lemma 5.3 we have the evolution equation of Φ

Lemma 7.3. *For* $t \in [0,\infty)$ *and* $x \in M^n$*,*

$$\begin{aligned}\frac{\partial}{\partial t}\Phi &= \mathcal{L}_F\Phi - \frac{2}{\beta - \mathcal{S}}(\nabla\Phi, \nabla\mathcal{S})_F \\ &\quad + \frac{1}{(\beta - \mathcal{S})^2}\big\{(\beta F^{pq}h_{pk}h_q^k + h)F - [(m\gamma+1)F^2 + h(\beta - \mathcal{S})F^{pq}h_{pk}h_q^k]\big\}.\end{aligned}$$

For any $t_0 \in [T^*, \infty)$, let $\beta = 2r_+(t_0)$ in Lemma 7.3. Then by Lemma 7.1, for any $t \in [T^*, t_0]$,

$$\mathcal{S} = (X, \mathbf{v}) \le r_+(t_0).$$

Applying the maximum principle to the evolution equation of Φ, by the same approach as in the proof of Lemma 5.4 we have

Lemma 7.4. *There are some positive constants* C_{30}*,* C_{31} *and* C_{32} *such that for any* $(x, \tilde{t}) \in M^n \times [0, \infty)$

$$\tilde{F}(x, \tilde{t}) \ge C_{30}.$$

Moreover

$$\tilde{H}_m(x,\tilde{t}) \ge C_{30}^{1/\gamma} = C_{31} \quad \textit{and} \quad \tilde{H}(x,\tilde{t}) \ge nC_{30}^{1/m\gamma} = C_{32}.$$

At last we show that the eigenvalues of the second fundamental form approach to each other, when $\tilde{t} \to \tilde{T}_{\max}$. As before we consider the function defined in Section 5

$$\rho = \frac{1}{n^n} - \frac{\tilde{K}}{\tilde{H}^n}.$$

It is easy to see that ρ ia a scaling invariant. We also have the evolution equation of ρ as in (5.16). By similar discussion as in the proof of Theorem 1.1 (I), the rescaled evolving hypersurfaces $\tilde{M}_{\tilde{t}}$ tends to a sphere as $\tilde{t} \to \infty$. This finishes the proof of Theorem 1.1 (III).

References

[1] B. Andrews, *Contraction of convex hypersurfaces in Euclidean space. Calc. Var. Partial Differential Equations*, 2(2), 1994, pp. 151–171.

[2] E. Cabezas-Rivas and C. Sinestrari, *Volume-preserving folw by powers of the* m^{th} *mean curvature. arXiv: 0902. 2090v1 [math.DG]* 12 Feb 2009.

[3] B. Chow, *Deforming convex hypersurfaces by the nth root of the Gaussian curvature. Invent Math.*, 87, 1987, pp. 63–82.

[4] E. Dibenedetto and A. Friedman, *Hölder estimates for nonlinear degenerate parabolic systems. J. Reine Angew. Math.*, 357, 1985, pp. 1–22.

[5] K. Ecker, *Regularity theory for mean curvature flow. Progress in Nonlinear Differential Equations and their Applications*, 57. Birkhäauser Boston, MA, 2004.

[6] G. Huisken, *Flow by mean curvature of convex surfaces into spheres. J. Differential Geom.*, 20(1), 1984, pp. 237–266.

[7] G. Huisken, *The volume preserving mean curvature flow. J. Reine Angew. Math.*, 382, 1987, pp. 35–48.

[8] G. Huisken and A. Polden, *Geometric evolution equations for hypersurfaces*. Calculus of variations and geometric evolution problems, pp. 45–84, *Lecture Notes in Math.*, 1713, Springer, Berlin, 1999.

[9] G. Li and I. Salavessa, *Forced convex mean curvature flow in Euclidean spaces. Manuscripta Math.*, 126, 2008, pp. 333–351.

[10] G. Li, L. Yu and C. Wu, *Curvature flow with a general forcing term in Euclidean spaces. J. Math. Anal. Appl.*, 353, 2009, pp. 508–520.

[11] J. McCoy, *The surface area preserving mean curvature flow. Asian J. Math.*, 7(1), 2003, pp. 7–30.

[12] J. McCoy, *The mixed volume preserving mean curvature flow. Math. Z.* 246(1-2), 2004, pp. 155–166.

[13] J. McCoy, *Mixed volume preserving curvature flow. Calc.Var.*, 24, 2005, pp. 131-154.

[14] F. Schulze, *Evolution of convex hypersurfaces by powers of the mean curvature. Math. Z.*, 251, 2005, pp. 721–733.

[15] F. Schulze, *Convexity estimates for flows by powers of the mean curvature. Ann. Sc. Norm. Super. Pisc Cl. Sci. (5)*, 5, 2006, pp. 261–277.

[16] K. Tso, *Deforming a hypersurface by its Gauss-Kronecker curvature. Comm. Pure Appl. Math.*, 38(6), 1985, pp. 867–882.

[17] J. I. E. Urbas, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures. Math. Z.*, 205, 1990, pp. 355–372.

[18] X. Zhu, *Lectures on mean curvature flows*. AMS/IP Studies in Advanced Mathematics, 32. *American Mathematical Society, Providence, RI; International Press, Somerville, MA*, 2002.