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Weak Type Estimate of Singular Integral Operators on Variable Weak Herz–Type Hardy Spaces

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Abstract. This paper is concerned with the boundedness properties of singular integral operators on variable weak Herz spaces and variable weak Herz-type Hardy spaces. Allowing our parameters to vary from point to point will raise extra difficulties, which, in general, are overcome by imposing regularity assumptions on these exponents, either at the origin or at infinity. Our results cover the classical results on weak Herz-type Hardy spaces with fixed exponents.

Key Words: Herz-Type Hardy Space, Weak Herz Spaces, Weak Herz-Type Hardy Spaces, Atom, Variable Exponent, Singular Intergral Operators
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1 Introduction

It is well known that Herz spaces play an important role in Harmonic Analysis. Since they were introduced in [9], the theory of these spaces has had a remarkable development in part due to their usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [3], in the summability of Fourier transforms [8] and in regularity theory for elliptic equations in divergence form [22].

In recent years, there was growing interest in generalizing classical spaces such as Lebesgue, Herz spaces and Sobolev spaces to the case with either variable integrability or variable smoothness. The motivation for the increasing interest in such spaces comes not only from theoretical purposes but also from applications to fluid dynamics [23], image restoration [4] and PDE with non-standard growth conditions.

Herz spaces $K^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)$ and $\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)$ with variable exponent p but fixed α and q were recently studied by Izuki [10, 11]. These spaces with variable exponents $\alpha(\cdot)$ and $p(\cdot)$ were studied in [2], where the boundedness results for a wide class of classical operators on these function spaces were given. The spaces $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ first were introduced by Izuki and Noi in [12].

Rafeiro and Samko [21] introduced generalized local and global Herz spaces, all of whose characteristics are variable. They have proved that variable Morrey-type spaces and complementary variable Morrey-type spaces are included into the scale of these generalized variable Herz spaces. Also, boundedness of a class of sublinear operators in generalized variable Herz spaces was given. In [20], an embeddings results for Herz spaces into weighted Lebesgue spaces are established. The boundedness of singular integral operators on variable function spaces were widely studied by many authors (see, for example, [2, 11, 17, 18] and references therein).

The variable weak Herz spaces $W\dot{K}_{p(\cdot),q}^{\alpha}(\mathbb{R}^n)$ and variable weak Herz-type Hardy spaces $WH\dot{K}_{p(\cdot),q}^{\alpha}(\mathbb{R}^n)$ were introduced earlier in [27] with variable p but fixed α and q. In the same paper, the boundedness of a class of singular integral operators on such spaces was proved. For fixed exponents, this problem was provided in [16].

Based on the approach of [1] and [12], we introduce the variable weak Herz spaces $W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and the variable weak Herz-type Hardy spaces $WH\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ including the inhomogeneous version, and we prove the boundedness properties of singular integral operators on such spaces. To do these, first we present an equivalent quasi-norm of $W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $WH\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ spaces. Based on these equivalent quasi-norms and the results of [7] and [16], we obtain our results. Allowing α and q to vary from point to point will raise extra difficulties, which, in general, are overcome by imposing regularity assumptions on this exponent, either at the origin or at infinity.

2 Preliminaries

As usual, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ is the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. The expression $f \lesssim g$ means that $f \leq cg$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$. For any $x \in \mathbb{R}$, $\lfloor x \rfloor$ stands for the largest integer smaller than or equal to x.

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball in $\mathbb{R}^n$ with center at x and radius r. By $\operatorname{supp} f$ we denote the support of the function f, i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function.

The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$, and we denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$, namely, the set of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$.

The variable exponents that we consider are always measurable functions on $\mathbb{R}^n$ with range in $[c,\infty[$ for some $c>0$. We denote the set of such functions by $\mathrm{P}_0(\mathbb{R}^n)$. The subset of variable exponents with range $[1,\infty)$ is denoted by $\mathrm{P}(\mathbb{R}^n)$. For $p\in \mathrm{P}_0(\mathbb{R}^n)$, we use the notations $p^+=\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)$ and $p^-=\operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$. Everywhere below, we shall consider bounded exponents.

Let $p\in \mathrm{P}_0(\mathbb{R}^n)$. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ is the class of all measurable functions f on $\mathbb{R}^n$ such that

$$\varrho_{p(\cdot)}(f)=\int_{\mathbb{R}^n}\Big|\frac{f(x)}{\lambda}\Big|^{p(x)}dx<\infty$$

for some $\lambda>0$. This is a quasi-Banach function space equipped with the norm

$$\|f\|_{p(\cdot)}=\inf\Big\{\mu>0:\varrho_{p(\cdot)}\Big(\frac{1}{\mu}f\Big)\le 1\Big\}.$$

If $p(x)\equiv p$ is constant, then $L^{p(\cdot)}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ is the classical Lebesgue space.

A useful property is that $\varrho_{p(\cdot)}(f)\leqslant 1$ if and only if $\|f\|_{p(\cdot)}\leqslant 1$ (*unit ball property*). This property is clear for constant exponents due to the obvious relation between the norm and the modular in that case.

We say that a function $g:\mathbb{R}^n\to\mathbb{R}$ is *locally* log-*Hölder continuous* if there exists a constant $c_{\log}>0$ such that

$$|g(x)-g(y)|\le\frac{c_{\log}}{\ln(e+1/|x-y|)}$$

for all $x,y\in\mathbb{R}^n$. If

$$|g(x)-g(0)|\le\frac{c_{\log}}{\ln(e+1/|x|)}$$

for all $x\in\mathbb{R}^n$, then we say that g is log-*Hölder continuous at the origin* (or has a log *decay at the origin*). If for some $g_\infty\in\mathbb{R}$ and $c_{\log}>0$, there holds

$$|g(x)-g_\infty|\le\frac{c_{\log}}{\ln(e+|x|)}$$

for all $x\in\mathbb{R}^n$, then we say that g is log-*Hölder continuous at infinity* (or has a log *decay at infinity*).

By $\mathcal{P}^{\ln}_0(\mathbb{R}^n)$ and $\mathcal{P}^{\ln}_\infty(\mathbb{R}^n)$ we denote the class of all exponents $p\in\mathrm{P}(\mathbb{R}^n)$ which have a log decay at the origin and at infinity, respectively.

The notation $\mathcal{P}^{\ln}(\mathbb{R}^n)$ is used for all those exponents $p \in \mathrm{P}(\mathbb{R}^n)$ which are locally log-Hölder continuous and have a log decay at infinity, with $p_\infty := \lim_{|x|\to\infty} p(x)$. Obviously, we have $\mathcal{P}^{\ln}(\mathbb{R}^n) \subset \mathcal{P}_0^{\ln}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\ln}(\mathbb{R}^n)$. Note that $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ if and only if $p' \in \mathcal{P}^{\ln}(\mathbb{R}^n)$, and since $(p')_\infty = (p_\infty)'$, we write only p'_∞ for any of these quantities. Here p' denotes the conjugate exponent of p given by $1/p(\cdot) + 1/p'(\cdot) = 1$.

Let $p, q \in \mathrm{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf\Big\{\lambda_v > 0 : \varrho_{p(\cdot)}\Big(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\Big) \leq 1\Big\}.$$

The (quasi)-norm is defined from this as usual:

$$\|\, (f_v)_v \,\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\Big\{\mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(\frac{1}{\mu}(f_v)_v\Big) \leq 1\Big\}. \tag{1}$$

Since $q^+ < \infty$, we can replace (1) by the simpler expression

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \big\| |f_v|^{q(\cdot)} \big\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$. It is known, cf. [1] and [13], that $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a norm if $q(\cdot) \geq 1$ is constant almost everywhere (a.e.) on $\mathbb{R}^n$ and $p(\cdot) \geq 1$, or if $1/p(x) + 1/q(x) \leq 1$ a.e. on $\mathbb{R}^n$, or if $1 \leq q(x) \leq p(x) < \infty$ a.e. on $\mathbb{R}^n$.

Very often we have to deal with the norm of characteristic functions on balls (or cubes) when studying the behavior of various operators in harmonic analysis. In classical L^p spaces, the norm of such functions is easily calculated, but this is not the case when we consider variable exponents. Nevertheless, it is known that for $p \in \mathcal{P}^{\log}$, we have

$$\big\|\chi_B\big\|_{p(\cdot)} \big\|\chi_B\big\|_{p'(\cdot)} \approx |B|. \tag{2}$$

Also,

$$\big\|\chi_B\big\|_{p(\cdot)} \approx |B|^{1/p(x)}, \quad x \in B \tag{3}$$

for small balls $B \subset \mathbb{R}^n$ ($|B| \leq 2^n$), and

$$\big\|\chi_B\big\|_{p(\cdot)} \approx |B|^{1/p_\infty} \tag{4}$$

for large balls ($|B| \geq 1$), with constants only depending on the log-Hölder constant of p (see, for example, [6, Section 4.5]).

We refer the reader to the recent monograph [6, Section 4.5] for further details, historical remarks and more references on variable exponent spaces.

We end this section by technical lemmas which will be used throughout this paper. The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 1 *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers, such that*

$$\left\| \{\varepsilon_k\}_{k\in\mathbb{Z}} \right\|_{\ell^q} = I < \infty.$$

Then the sequences

$$\Big\{\delta_k : \delta_k = \sum_{j\leq k} a^{k-j}\varepsilon_j\Big\}_{k\in\mathbb{Z}} \quad \text{and} \quad \Big\{\eta_k : \eta_k = \sum_{j\geq k} a^{j-k}\varepsilon_j\Big\}_{k\in\mathbb{Z}}$$

belong to ℓ^q, and

$$\left\| \{\delta_k\}_{k\in\mathbb{Z}} \right\|_{\ell^q} + \left\| \{\eta_k\}_{k\in\mathbb{Z}} \right\|_{\ell^q} \leq cI,$$

with $c > 0$ only depending on a and q.

The proof of the following results are given in [2], where the second lemma is a generalization of (2), (3) and (4) to the case of dyadic annuli.

Lemma 2 *Let $\alpha \in L^\infty(\mathbb{R}^n)$ and $r_1 > 0$. If α is* log*-Hölder continuous both at the origin and at infinity, then*

$$r_1^{\alpha(x)} \lesssim r_2^{\alpha(y)} \times \begin{cases} (r_1/r_2)^{\alpha^+} & \text{if} \quad 0 < r_2 \leq r_1/2, \\ 1 & \text{if} \quad r_1/2 < r_2 \leq 2r_1, \\ (r_1/r_2)^{\alpha^-} & \text{if} \quad r_2 > 2r_1 \end{cases}$$

for any $x \in B(0, r_1) \setminus B(0, r_1/2)$ and $y \in B(0, r_2) \setminus B(0, r_2/2)$, with the implicit constant not depending on x, y, r_1 and r_2.

Lemma 3 *Let $p \in \mathcal{P}^{\ln}_\infty(\mathbb{R}^n)$ and let $R = B(0, r) \setminus B(0, r/2)$. If $|R| \geq 2^{-n}$, then*

$$\left\|\chi_R\right\|_{p(\cdot)} \approx |R|^{1/p(x)} \approx |R|^{1/p_\infty}$$

with the implicit constants independent of r and $x \in R$. The left-hand side equivalence remains true for every $|R| > 0$ if we assume, additionally, $p \in \mathcal{P}^{\ln}_0(\mathbb{R}^n) \cap \mathcal{P}^{\ln}_\infty(\mathbb{R}^n)$.

3 Sublinear operators on variable weak Herz spaces

For convenience, we set

$$B_k = B(0, 2^k), \quad R_k = B_k \setminus B_{k-1} \quad \text{and} \quad \chi_k = \chi_{R_k}, \quad k \in \mathbb{Z}.$$

The definition of Herz spaces of variable smoothness and integrability were introduced and investigated in [7].

Definition 1 *Let $p,q \in \mathrm{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The* inhomogeneous Herz space $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *consists of all $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ such that*

$$\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|f\chi_{B_0}\|_{p(\cdot)} + \big\| \big(2^{k\alpha(\cdot)} f\chi_k\big)_{k\geq 1} \big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Similarly, the homogeneous Herz space $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ such that*

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \big\| \big(2^{k\alpha(\cdot)} f\chi_k\big)_{k\in\mathbb{Z}} \big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

If α and p,q are constant, then $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = K^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ are the classical Herz spaces.

Let us denote

$$\|\{g_k\}\|_{\ell^q_>(L^{p(\cdot)})} = \Big(\sum_{k=0}^{\infty} \|g_k\|^q_{p(\cdot)}\Big)^{1/q}$$

and

$$\|\{g_k\}\|_{\ell^q_<(L^{p(\cdot)})} = \Big(\sum_{k=-\infty}^{-1} \|g_k\|^q_{p(\cdot)}\Big)^{1/q}$$

for sequences $\{g_k\}_{k\in\mathbb{Z}}$ of measurable functions (with the usual modification when $q=\infty$).

The following proposition plays a fundamental role in this paper.

Proposition 1 *Let $\alpha \in L^\infty(\mathbb{R}^n)$ and $p,q \in \mathrm{P}_0(\mathbb{R}^n)$. If α and q are* log*-Hölder continuous at infinity, then*

$$K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = K^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n).$$

Additionally, if α and q have a log *decay at the origin, then*

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \|\{2^{k\alpha(0)} f\chi_k\}\|_{\ell^{q(0)}_<(L^{p(\cdot)})} + \|\{2^{k\alpha_\infty} f\chi_k\}\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}.$$

For the proof see [7, Proposition 1]. We refer the reader to the paper [24], where the continuous variable Herz spaces are studied.

Let $k \in \mathbb{Z}$ and $\lambda > 0$. We set $A_k(\lambda,f) = \{x \in R_k : |f(x)| > \lambda\}$ and $\tilde{A}_0(\lambda,f) = \{x \in B(0,1) : |f(x)| > \lambda\}$.

Now we will give the definition of the weak Herz space.

Definition 2 *Let $p,q \in \mathrm{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The* inhomogeneous weak Herz space $WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *consists of all measurable functions f such that*

$$\|f\|_{WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \sup_{\lambda>0} \lambda \big\| \big(2^{k\alpha(\cdot)} \chi_{A_k(\lambda,f)}\big)_{k\geq 0} \big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty,$$

where A_0 is replaced by $\tilde{A}_0$. Similarly, the homogeneous weak Herz space $W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all measurable functions f such that*

$$\|f\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \sup_{\lambda>0} \lambda \left\| \left(2^{k\alpha(\cdot)} \chi_{A_k(\lambda,f)}\right)_{k\in\mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

If α, p and q are constant, then $WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} = WK^{\alpha}_{p,q}$ and $W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = W\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ are the classical weak Herz spaces. Now we recall the variable weak $L^{p(\cdot)}(\mathbb{R}^n)$ spaces.

Definition 3 *Let $p \in \mathrm{P}_0(\mathbb{R}^n)$. The weak Lebesgue space $L^{p(\cdot),\infty}(\mathbb{R}^n)$ with variable exponent consists of all Lebesgue measurable function f satisfying*

$$\|f\|_{L^{p(\cdot),\infty}} = \sup_{\lambda>0} \lambda \left\| \chi_{\{x\in\mathbb{R}^n : |f(x)|>\lambda\}} \right\|_{p(\cdot)} < \infty.$$

It is obvious that if p is constant, then

$$W\dot{K}^{0}_{p,p}(\mathbb{R}^n) = WK^{0}_{p,p}(\mathbb{R}^n) = L^{p,\infty}(\mathbb{R}^n).$$

The following proposition is one of the main tools of this paper.

Proposition 2 *Let $\alpha \in L^\infty(\mathbb{R}^n)$ and $p, q \in \mathrm{P}_0(\mathbb{R}^n)$. If α and q are* log-*Hölder continuous at infinity, then*

$$WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = WK^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n).$$

Additionally, if α and q have a log *decay at the origin, then*

$$\|f\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \sup_{\lambda>0} \left(\lambda \left\| \{2^{k\alpha(0)} \chi_{A_k(\lambda,f)}\} \right\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} + \lambda \left\| \{2^{k\alpha_\infty} \chi_{A_k(\lambda,f)}\} \right\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \right). \quad (5)$$

Proof. Our arguments are based on [7, Proposition 1]. We will do the proof in four steps.

Step 1. We will prove that

$$WK^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n) \hookrightarrow WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

We need to show that

$$\|f\|_{WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{WK^{\alpha_\infty}_{p(\cdot),q_\infty}}$$

for any $f \in WK^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n)$. By the scaling argument, we see that it suffices to consider the case $\|f\|_{WK^{\alpha_\infty}_{p(\cdot),q_\infty}} = 1$ and show that

$$\sum_{k=0}^{\infty} \left\| \left| c\lambda 2^{k\alpha(\cdot)} \chi_{A_k(\lambda,f)} \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1 \quad (6)$$

for any $\lambda > 0$ and some constant $c > 0$ independent of λ. Since α has logarithmic decay at infinity, for $k \in \mathbb{N}$ and $x \in R_k$, we have

$$k|\alpha(x) - \alpha_\infty| \lesssim \frac{k}{\ln(e + |x|)} \lesssim 1.$$

Therefore, $2^{k\alpha(x)} \approx 2^{k\alpha_\infty}$, $x \in R_k$ with constants independent of k and x, and hence,

$$\sum_{k=1}^{\infty} \left\| \left|c\lambda 2^{k\alpha(\cdot)} \chi_{A_k(\lambda,f)}\right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \approx \sum_{k=1}^{\infty} \left\| \left|c\lambda 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)}\right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

In addition, $\left\|\lambda \chi_{A_0(\lambda,f)}\right\|_{p(\cdot)} \leq 1$ for any $\lambda > 0$, which yields by the unit ball property that

$$\left\| |\lambda \chi_{A_0(\lambda,f)}|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1$$

for any $\lambda > 0$. Observe that (6) clearly follows if we prove the estimate

$$\left\| \left|c\lambda 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)}\right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq \left\|\lambda 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)}\right\|_{p(\cdot)}^{q_\infty} + 2^{-k} = \delta, \quad k \in \mathbb{N}. \tag{7}$$

This claim can be reformulated as showing that

$$\left\| \delta^{-1} \left|c\lambda 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)}\right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1,$$

which, by the unit ball property, is equivalent to

$$\left\| c\lambda \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)} \right\|_{p(\cdot)} \leq 1.$$

Since q has logarithmic decay at infinity, for $k \in \mathbb{N}$ and $x \in R_k$, we have

$$\frac{k|q(x) - q_\infty|}{q_\infty q(x)} \leq \frac{k|q(x) - q_\infty|}{q_\infty q^-} \lesssim \frac{k}{\ln(e + |x|)} \lesssim 1.$$

Therefore, $2^{k(\frac{1}{q_\infty} - \frac{1}{q(x)})} \approx 1$ with constants independent of k and x. Also, since $1 < 2^k\delta < 2^{k+1}$,

$$(2^k\delta)^{\frac{1}{q_\infty} - \frac{1}{q(x)}} \leq (2^{k+1})^{|\frac{1}{q_\infty} - \frac{1}{q(x)}|} \lesssim 1.$$

Using the fact that

$$\delta^{-\frac{1}{q(x)}} = (2^k\delta)^{\frac{1}{q_\infty} - \frac{1}{q(x)}} 2^{k(\frac{1}{q(x)} - \frac{1}{q_\infty})} \delta^{-\frac{1}{q_\infty}} \quad \text{and} \quad \left\|\lambda 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)}\right\|_{p(\cdot)} \leq \delta^{\frac{1}{q_\infty}},$$

with an appropriate choice of $c > 0$, we obtain

$$\left\| c\lambda \delta^{-1/q(\cdot)} 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)} \right\|_{p(\cdot)} \leq \left\| \lambda \delta^{-1/q_\infty} 2^{k\alpha_\infty} \chi_{A_k(\lambda,f)} \right\|_{p(\cdot)} \leq 1.$$

Step 2. In this step, we prove that

$$WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow WK^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n),$$

which is equivalent to

$$\|f\|_{WK^{\alpha_\infty}_{p(\cdot),q_\infty}} \lesssim \|f\|_{WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$$

for any $f \in WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Again, by the scaling argument, we see that it suffices to consider the case $\|f\|_{WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = 1$ and show that

$$\sum_{k=0}^{\infty} \left\|c\lambda 2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\right\|_{p(\cdot)}^{q_\infty} \lesssim 1 \tag{8}$$

for some constant $c > 0$. As before, for $k \in \mathbb{N}$, we have

$$\left\|\lambda 2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\right\|_{p(\cdot)}^{q_\infty} \lesssim \left\|\lambda 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right\|_{p(\cdot)}^{q_\infty}.$$

Now, our estimate (8), clearly follows from the inequality

$$\left\|c\lambda 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right\|_{p(\cdot)}^{q_\infty} \le \left\| \left|\lambda 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-k} = \delta. \tag{9}$$

This claim can be reformulated as showing that

$$\left\|c\lambda\delta^{-1/q_\infty} 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right\|_{p(\cdot)} \le 1.$$

From above, $\delta^{-1/q_\infty} \lesssim \delta^{-1/q(x)}$, then, with an appropriate choice of $c > 0$,

$$\left\|c\lambda\delta^{-1/q_\infty} 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right\|_{p(\cdot)} \le \left\|\lambda\delta^{-1/q(\cdot)} 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right\|_{p(\cdot)}.$$

The left-hand side is less than or equal to 1 if and only if

$$\left\|\left|\lambda\delta^{-1/q(\cdot)} 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \le 1.$$

We see that the right-hand side can be rewritten as

$$\delta^{-1}\left\| \left|\lambda 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \le 1,$$

which follows immediately from the definition of δ.

Step 3. Let us prove that

$$\left\|\{\lambda 2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\}\right\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} + \left\|\{\lambda\ 2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\}\right\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \lesssim \|f\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

for any $\lambda > 0$. We suppose that $\big\|f\big\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq 1$. If, in addition, α has a log decay at the origin, then we also have $2^{k\alpha(x)} \approx 2^{k\alpha(0)}$ for $k < 0$ and $x \in R_k$. Thus,

$$\big\|\{\lambda 2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\}\big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} \approx \big\|\{\lambda 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\}\big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}.$$

As in Step 2, we can prove that

$$\Big\|c\lambda 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\Big\|^{q(0)}_{p(\cdot)} \leq \Big\|\,\big|\lambda 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\big|^{q(\cdot)}\,\Big\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^k$$

for any $k < 0$ and for some constant $c > 0$. Then

$$\big\|\{\lambda 2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\}\big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} \lesssim 1.$$

Using estimate (9), we obtain

$$\big\|\{\lambda 2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\}\big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \lesssim 1.$$

Therefore,

$$\big\|\{\lambda 2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\}\big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} + \big\|\{\lambda 2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\}\big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \lesssim 1.$$

The desired estimate can be obtained by the scaling argument.

Step 4. Let

$$\big\|\{\lambda 2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\}\big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} \leq 1$$

and

$$\big\|\{\lambda\ 2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\}\big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \leq 1.$$

Similarly, to Steps 1 and 3, for any $k < 0$ and for some constant $c > 0$, we have

$$\Big\|\,\big|c\lambda 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\big|^{q(\cdot)}\,\Big\|_{\frac{p(\cdot)}{q(\cdot)}} \leq \Big\|\lambda 2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\Big\|^{q(0)}_{p(\cdot)} + 2^k$$

and using (7), we obtain

$$\sum_{k=-\infty}^{\infty}\Big\|\,\big|\lambda 2^{k\alpha(\cdot)}\chi_{A_k(\lambda,f)}\big|^{q(\cdot)}\,\Big\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim 1.$$

Therefore, $\big\|f\big\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim 1$. The desired estimate follows by the scaling argument. $\square$

Consider sublinear operators satisfying the size condition

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}dy, \quad x \notin \operatorname{supp} f, \tag{10}$$

for integrable and compactly supported functions f. Condition (10) first appeared in [25], and it is satisfied by several classical operators in harmonic analysis such as Calderón-Zygmund operators, the Carleson maximal operator and the Hardy-Littlewood maximal operator (see [25, 15]).

We have the following results:

Theorem 1 *Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $q \in \mathrm{P}_0(\mathbb{R}^n)$ be* log-*Hölder continuous at infinity, $p \in \mathcal{P}^{\ln}_{\infty}(\mathbb{R}^n), 0 < q^- \leq q^+ < 1$ and $1 < p^- \leq p^+ < \infty$ with*

$$- n/p_\infty < \alpha_\infty < n\big(1 - 1/p_\infty\big). \tag{11}$$

Suppose that T is a sublinear operator satisfying estimate (10). *If T is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot),\infty}(\mathbb{R}^n)$, then T is bounded from $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

For homogeneous spaces we have the following statement:

Theorem 2 *Let $q \in \mathrm{P}_0(\mathbb{R}^n)$, $p \in \mathcal{P}^{\ln}_0(\mathbb{R}^n) \cap \mathcal{P}^{\ln}_{\infty}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < \infty$. Let $\alpha \in L^\infty(\mathbb{R}^n)$ and q be* log-*Hölder continuous both at the origin and at infinity and such that*

$$- n/p^+ < \alpha^- \leq \alpha^+ < n\big(1 - 1/p^-\big). \tag{12}$$

Then every sublinear operator T satisfying (10) *which is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot),\infty}(\mathbb{R}^n)$ is also bounded from $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

Since the proofs are similar, we will prove only Theorem 2.

Proof of Theorem 2. We will partially use some decomposition techniques already used in [16], where the constant exponent case was studied. In view of Proposition 2, we use the equivalent quasi-norm (5). Split the operator T into

$$|Tf| \leq \big|T(f\chi_{B_{k-2}})\big| + \big|T(f\chi_{\widetilde{R}_k})\big| + \big|T(f\chi_{\mathbb{R}^n\setminus B_{k+2}})\big|$$

with $k \in \mathbb{Z}$, where $\widetilde{R}_k := \big\{x \in \mathbb{R}^n : 2^{k-2} \leq |x| < 2^{k+2}\big\}$.
Estimate of $T(f\chi_{B_{k-2}})$. Let $\lambda > 0$,

$$I_1 = \lambda\big\|\{2^{k\alpha(0)}\chi_{A_k(\lambda,T(f\chi_{B_{k-2}}))}\}\big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}$$

and

$$I_2 = \lambda\big\|\{2^{k\alpha_\infty}\chi_{A_k(\lambda,T(f\chi_{B_{k-2}}))}\}\big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}.$$

Given $k \leq -1$ and $x \in R_k$, we can write

$$\big|T\big(f\chi_{B_{k-2}}\big)(x)\big| \lesssim \int_{B_{k-2}} |x-y|^{-n}|f(y)|dy = c\sum_{j=-\infty}^{k-2}\int_{R_j}|x-y|^{-n}|f(y)|dy.$$

Since $|x-y|>2^{k-2}$ for $x\in R_k$ and $y\in R_j$, by Hölder's inequality,

$$\left|T\left(f\chi_{B_{k-2}}\right)(x)\right|\lesssim 2^{-kn}\sum_{j=-\infty}^{k-2}\int_{R_j}|f(y)|dy\lesssim 2^{-kn}\sum_{j=-\infty}^{k-2}\left\|f\chi_j\right\|_{p(\cdot)}\left\|\chi_j\right\|_{p'(\cdot)}.$$

Observe that $2^{j\alpha(x)}\approx 2^{j\alpha(0)}$, $x\in R_j$, which yields that

$$2^{k\alpha(0)}\left|T\left(f\chi_{B_{k-2}}\right)(x)\right|\lesssim 2^{-kn}\sum_{j=-\infty}^{k-2}2^{(k-j)\alpha(0)}\left\|2^{j\alpha(\cdot)}f\chi_j\right\|_{p(\cdot)}\left\|\chi_j\right\|_{p'(\cdot)}.$$

Therefore,

$$\lambda 2^{k\alpha(0)}\left\|\chi_{A_k(\lambda,T(f\chi_{B_{k-2}}))}\right\|_{p(\cdot)}$$

can be estimated by

$$\begin{aligned}&c2^{-kn}\sum_{j=-\infty}^{k-2}2^{(k-j)\alpha(0)}\left\|2^{j\alpha(\cdot)}f\chi_j\right\|_{p(\cdot)}\left\|\chi_j\right\|_{p'(\cdot)}\left\|\chi_k\right\|_{p(\cdot)}\\ \lesssim\,&2^{-kn}2^{jn/p'(y)}2^{kn/p(x)}\sum_{j=-\infty}^{k-2}2^{(k-j)\alpha(0)}\left\|2^{j\alpha(\cdot)}f\chi_j\right\|_{p(\cdot)},\end{aligned}\tag{13}$$

where we used Lemma 3, and the positive constant c is independent of k, $x\in R_k$ and $y\in R_j$. Since p has a log decay at the origin, we also have

$$2^{jn/p'(y)}\approx 2^{jn/p'(0)},\quad 2^{kn/p(x)}\approx 2^{kn/p(0)},\quad x\in R_k, y\in R_j.$$

Hence, we can estimate (13) by

$$c\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha(0)-n+\frac{n}{p(0)})}\left\|2^{j\alpha(\cdot)}f\chi_j\right\|_{p(\cdot)}.$$

Since $\alpha(0)-n+n/p(0)<0$, applying Lemma 1 and Proposition 1, we get

$$I_1\lesssim\left\|f\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Now we estimate I_2. In view of the estimation of I_1,

$$\lambda 2^{k\alpha_\infty}\left\|\chi_{A_k(\lambda,T(f\chi_{B_{k-2}}))}\right\|_{p(\cdot)}$$

can be estimated by

$$c\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha^+-n)}\left\|2^{j\alpha(\cdot)}f\chi_j\right\|_{p(\cdot)}2^{-jn/p(y)}2^{kn/p(x)},\quad k\in\mathbb{N}_0.$$

If $0 \le j \le k-2$, by Lemma 3, we get

$$2^{-jn/p(y)}2^{kn/p(x)} \approx 2^{-jn/p_\infty}2^{kn/p_\infty} \approx 2^{(k-j)n/p_\infty} \lesssim 2^{(k-j)n/p^-}.$$

In the case $j < 0 \le k-2$, we obtain

$$2^{-jn/p(y)}2^{kn/p(x)} \lesssim 2^{-jn/p^-}2^{kn/p^-} \lesssim 2^{(k-j)n/p^-}.$$

Finally, for $j \le k-2 < 0$, we have

$$2^{-jn/p(y)}2^{kn/p(x)} \approx 2^{(k-j)n/p(y)}2^{k(n/p(x)-n/p(y))} \lesssim 2^{(k-j)n/p^-}.$$

Indeed, since $2^{k-1} \le |x| < 2^k$, $|y| < 2^j < 2^{k-2}$, we make use of local log-Hölder continuity of p at the origin to get for $k \le 0$,

$$(-kn)\Big|\frac{1}{p(y)} - \frac{1}{p(x)}\Big| \lesssim \frac{-k}{\log\left(e + \frac{1}{2^k}\right)} \le c$$

with $c > 0$ independent of k, j, x, y. Therefore,

$$\lambda 2^{k\alpha_\infty}\big\|\chi_{A_k(\lambda, T(f\chi_{B_{k-2}}))}\big\|_{p(\cdot)}$$

can be estimated by

$$c\sum_{j=-\infty}^{k-2} 2^{(k-j)(\alpha^+ - n + \frac{n}{p^-})}\big\|2^{j\alpha(\cdot)}f\chi_j\big\|_{p(\cdot)}, \quad k \ge 0.$$

Since $\alpha^+ - n + n/p^- < 0$, applying Lemma 1 and Proposition 1, we get

$$I_2 \lesssim \|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Estimate of $T(f\chi_{\widetilde{R}_k})$. Let $\lambda > 0$,

$$V_1 = \lambda\big\|\{2^{k\alpha(0)}\chi_{A_k(\lambda, T(f\chi_{\widetilde{R}_k}))}\}\big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}$$

and

$$V_2 = \lambda\big\|\{2^{k\alpha_\infty}\chi_{A_k(\lambda, T(f\chi_{\widetilde{R}_k}))}\}\big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}.$$

We estimate only V_1. Since T is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot),\infty}(\mathbb{R}^n)$, it follows that

$$\lambda\big\|\chi_{A_k(\lambda, T(f\chi_{\widetilde{R}_k}))}\big\|_{p(\cdot)} \lesssim \big\|f\chi_{\widetilde{R}_k}\big\|_{p(\cdot)},$$

where the implicit constant is independent of λ and k. Consequently,

$$V_1 \lesssim \big\|\{2^{k\alpha(0)}f\chi_{\widetilde{R}_k}\}\big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} \lesssim \|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

where we used Proposition 1. Similarly,

$$V_2 \lesssim \big\|\{2^{k\alpha_\infty} f\chi_{\widetilde{R}_k}\}\big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \lesssim \|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Estimate of $T(f\chi_{\mathbb{R}^n\setminus B_{k+2}})$. Let $\lambda>0$,

$$J_1 = \lambda\Big\|\{2^{k\alpha(0)}\chi_{A_k(\lambda,T(f\chi_{\mathbb{R}^n\setminus B_{k+2}}))}\}\Big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}$$

and

$$J_2 = \lambda\Big\|\{2^{k\alpha_\infty}\chi_{A_k(\lambda,T(f\chi_{\mathbb{R}^n\setminus B_{k+2}}))}\}\Big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}.$$

We estimate only J_1, the estimation for J_2 can be obtained in the similar way. Given $k\le -1$ and $x\in R_k$, we can write

$$\begin{aligned}\big|T\big(f\chi_{\mathbb{R}^n\setminus B_{k+2}}\big)(x)\big| &\lesssim \int_{\mathbb{R}^n\setminus B_{k+2}} |x-y|^{-n}|f(y)|dy\\ &= c\sum_{j=k+3}^{\infty}\int_{R_j}|x-y|^{-n}|f(y)|dy.\end{aligned}$$

Noting that $|x-y|>2^j$ for $x\in R_k$ and $y\in R_j$, we use successively Hölder's inequality to obtain the estimate

$$2^{k\alpha(0)}\big|T\big(f\chi_{\mathbb{R}^n\setminus B_{k+2}}\big)(x)\big| \lesssim \sum_{j=k+3}^{\infty}2^{-jn}2^{(k-j)\alpha^-}\big\|2^{j\alpha(\cdot)}f\chi_j\big\|_{p(\cdot)}\big\|\chi_j\big\|_{p'(\cdot)}.$$

Therefore,

$$\lambda 2^{k\alpha(0)}\big\|\chi_{A_k(\lambda,T(f\chi_{B_{k+2}}))}\big\|_{p(\cdot)}$$

can be estimated by

$$\begin{aligned}& c\sum_{j=k+3}^{\infty}2^{-jn}2^{(k-j)\alpha^-}\big\|2^{j\alpha(\cdot)}f\chi_j\big\|_{p(\cdot)}\big\|\chi_j\big\|_{p'(\cdot)}\big\|\chi_k\big\|_{p(\cdot)}\\ &\lesssim \sum_{j=k+3}^{\infty}2^{-jn}2^{(k-j)\alpha^-}\big\|2^{j\alpha(\cdot)}f\chi_j\big\|_{p(\cdot)}2^{jn/p'(y)}2^{kn/p(x)},\end{aligned} \tag{14}$$

again, by Lemma 3, where the positive constant c is independent of k, $x\in R_k$ and $y\in R_j$. Due to Lemma 2, we can estimate (14) by

$$c\sum_{j=k+3}^{\infty}2^{(k-j)(\alpha^-+\frac{n}{p^+})}\big\|2^{j\alpha(\cdot)}f\chi_j\big\|_{p(\cdot)}.$$

Since $\alpha^-+n/p^+>0$, applying Lemma 1 and Proposition 1, we get

$$J_1\lesssim \|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

The proof is complete. □

Remark 1 *Theorems 1 and 2 are proved in [27] for the case of constant $\alpha > 0$. In [27], the author has assumed the boundedness of the maximal operator $\mathcal{M}$ on $L^{p(\cdot)}(\mathbb{R}^n)$ both in the homogeneous and in the inhomogeneous case. Although we assume the slightly stronger assumption $p \in \mathcal{P}_0^{\ln}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\ln}(\mathbb{R}^n)$ in the homogeneous case, we stress that only the* log*-decay of p is assumed when we deal with inhomogeneous spaces. In the classical Herz spaces, Theorems 1 and 2 can be found in [16]. We refer the reader to [19] for the boundedness of Riesz potential operator in continual variable exponents Herz spaces.*

Since the Hardy-Littlewood maximal operator $\mathcal{M}$ is sublinear, satisfies the size condition (10) and is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot),\infty}(\mathbb{R}^n)$ if $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ and $1 \leq p^- \leq p^+ \leq \infty$ (see [6, Theorem 4.3.8]), from Theorems 1 and 2, we immediately arrive at the following result.

Corollary 1 *Let $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 \leq p^- \leq p^+ < \infty$, and $\alpha \in L^\infty(\mathbb{R}^n)$.*

(i) Let $q \in \mathcal{P}_0^{\ln}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\ln}(\mathbb{R}^n)$ with $0 < q^- \leq q^+ < \infty$. If (11) *holds and α satisfies the* log*-Hölder decay condition, then $\mathcal{M}$ is bounded from $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

(ii) Let $q \in \mathcal{P}_0^{\ln}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\ln}(\mathbb{R}^n)$ with $0 < q^- \leq q^+ < \infty$. If (12) *holds and α has a* log *decay both at the origin and at infinity, then $\mathcal{M}$ is bounded from $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

Remark 2 *The boundedness of the maximal operator in variable exponent Lebesgue spaces was first proved by Diening [5]. Further details and references on subsequent contributions, can be found in [6, Chapter 4]. We refer the reader to the paper [14] for more results about the boundedness of sublinear operators on Herz spaces of fixed exponents.*

4 Variable Herz-type Hardy space

The main goal of this section is to recall an atomic decomposition result for variable Herz-type Hardy space. First we introduce the basic notations. Let $G_N f$ be the grand maximal function of f defined by

$$G_N f(x) = \sup_{\varphi \in \mathcal{A}_N} |\varphi_N^*(f)(x)|,$$

where $\mathcal{A}_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha| \leq N, |\beta| \leq N} |x^\alpha \partial^\beta \varphi(x)| \leq 1\}$ and $\varphi_N^*(f)(x) = \sup_{t>0} |\varphi_t * f(x)|$ with $\varphi_t = t^{-n}\varphi(\cdot/t)$.

Definition 4 *Let $p, q \in \mathrm{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$ and $N > n+1$. The* inhomogeneous Herz-type Hardy space $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $G_N f \in K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and we define*

$$\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|G_N f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Similarly, the homogeneous Herz-type Hardy space $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $G_N f \in \dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and*

$$\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|G_N f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Note that G_N satisfies the size condition (10). Let $q \in \mathrm{P}_0(\mathbb{R}^n)$, $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$, and let α and q be log-Hölder continuous at infinity, with $\alpha \in L^\infty(\mathbb{R}^n)$ and $-n/p_\infty < \alpha_\infty < \big(1 - 1/p_\infty\big)$. Then

$$HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \cap L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) = K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

Let $q \in \mathrm{P}_0(\mathbb{R}^n)$, $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$, and let α and q be log-Hölder continuous both at the origin and at infinity and such that $\alpha \in L^\infty(\mathbb{R}^n)$, $-n/p^+ < \alpha^- \leq \alpha^+ < n\big(1 - 1/p^-\big)$. Then

$$H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \cap L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) = \dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

To prove our results, we use the atomic decomposition of Herz-type Hardy space.

Definition 5 *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p \in \mathrm{P}(\mathbb{R}^n)$, $q \in \mathrm{P}_0(\mathbb{R}^n)$ and $s \in \mathbb{N}_0$. A function a is said to be a central $(\alpha(\cdot), p(\cdot))$-atom if*

(i) $\operatorname{supp} a \subset \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| \leq r\}$, $r > 0$;

(ii) $\|a\|_{p(\cdot)} \leq |\overline{B(0,r)}|^{-\alpha(0)/n}$, $0 < r < 1$;

(iii) $\|a\|_{p(\cdot)} \leq |\overline{B(0,r)}|^{-\alpha_\infty/n}$, $r \geq 1$;

(iv) $\int_{\mathbb{R}^n} x^\beta a(x)dx = 0$, $|\beta| \leq s$.

A function a on $\mathbb{R}^n$ is said to be a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type if it satisfies conditions (iii), (iv) above and $\operatorname{supp} a \subset \overline{B(0,r)}, r \geq 1$.

Now we come to the atomic decomposition theorems.

Theorem 3 *Let α and q be* log-*Hölder continuous at infinity and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. For any $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k, \tag{15}$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type with $\operatorname{supp} a \subset B_k$*, and*

$$\Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty} \leq c\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely, if $\alpha_\infty \geq n(1-1/p_\infty)$, $s \geq [\alpha_\infty + n(1/p_\infty - 1)]$ and (15) *holds, then $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and*

$$\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\Big\{\Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\Big\},$$

where the infimum is taken over all decompositions of f as above.

Theorem 4 *Let α and q be* log*-Hölder continuous both at the origin and at infinity, and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. For any $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \tag{16}$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with $\operatorname{supp} a \subset B_k$*, and*

$$\Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty} \leq c\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely, if $\alpha(\cdot) \geq n(1-1/p^-)$, $s \geq [\alpha^+ + n(1/p^- - 1)]$ and (16) *holds, then $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and*

$$\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\Big\{\Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\Big\},$$

where the infimum is taken over all decompositions of f as above.

For the proof, see [7], Theorem 4.

Remark 3 *In the necessity part of Theorems 3-4, the atoms in the decompositions* (15) *and* (16) *can be taken to be supported in dyadic annuli.*

Remark 4 *In comparison with the results of Liu and Wang [26], the above described method requires the log-Hölder continuity at two points only (zero and infinity).*

Next, we will give the definition of the weak Herz-type Hardy space with variable exponents.

Definition 6 *Let $p, q \in \mathrm{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$ and $N > n+1$. The* inhomogeneous weak Herz-type Hardy space $WHK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $G_N f \in WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and we define*

$$\|f\|_{WHK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|G_N f\|_{WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Similarly, the homogeneous weak Herz-type Hardy space $WH\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $G_N f \in W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and we define*

$$\|f\|_{WH\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|G_N f\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

5 Boundedness of some singular integral operators

Let K be a measurable function defined on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{x = y\}$ that satisfies the size condition

$$|K(x,y) - K(x,0)| \le \frac{|y|^\delta}{|x|^{n+\delta}} \tag{17}$$

if $2|y| < |x|$ for some $\delta \in (0,1]$. Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a linear operator such that for any $f \in L^2(\mathbb{R}^n)$ with compact support and almost all $x \in \operatorname{supp} f$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy, \tag{18}$$

which is more general than the standard Calderón-Zygmund operator.

This section is devoted to the study of the behavior of T on variable Herz-type Hardy space.

Theorem 5 *Let α and q be* log-*Hölder continuous at infinity, and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$ and $0 < q^- \le q^+ < \infty$ be such that*

$$n\Big(1 - \frac{1}{p_\infty}\Big) \le \alpha_\infty < n\Big(1 - \frac{1}{p_\infty}\Big) + \delta.$$

Then every linear operator T satisfying (18) *which is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded from $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

The counterpart for homogeneous Herz spaces runs as follows:

Theorem 6 *Let α and q be* log-*Hölder continuous both at the origin and at infinity, and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$ and $0 < q^- \le q^+ < \infty$ be such that*

$$\alpha(\cdot) \ge n\Big(1 - \frac{1}{p^-}\Big), \quad \alpha(0) < n\Big(1 - \frac{1}{p(0)}\Big) + \delta \quad \textit{and} \quad \alpha_\infty < n\Big(1 - \frac{1}{p_\infty}\Big) + \delta.$$

Then every linear operator T satisfying (18) *which is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded from $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

Since the proofs are similar, we will prove only Theorem 6.

Proof of Theorem 6. Let $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. Then

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \tag{19}$$

where the series converges in the sense of distributions, $\lambda_k \ge 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with $\operatorname{supp} a_k \subset R_k$ and

$$\Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty} \le c\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{20}$$

We will prove that

$$\big\|Tf\big\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \le c\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Decompose f into two parts:

$$f = \sum_{k=-\infty}^{j-4} \lambda_k a_k + \sum_{k=j-3}^{\infty} \lambda_k a_k, \quad j \in \mathbb{Z}. \tag{21}$$

Define

$$\begin{aligned}
I_1 &= \Big\|\Big\{2^{j\alpha(0)} T\Big(\sum_{k=-\infty}^{j-4} \lambda_k a_k\Big)\chi_j\Big\}\Big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}, \\
I_2 &= \Big\|\Big\{2^{j\alpha_\infty} T\Big(\sum_{k=-\infty}^{j-4} \lambda_k a_k\Big)\chi_j\Big\}\Big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}, \\
I_3 &= \Big\|\Big\{2^{j\alpha(0)} T\Big(\sum_{k=j-3}^{\infty} \lambda_k a_k\Big)\chi_j\Big\}\Big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}
\end{aligned}$$

and

$$I_4 = \Big\|\Big\{2^{j\alpha_\infty} T\Big(\sum_{k=j-3}^{\infty} \lambda_k a_k\Big)\chi_j\Big\}\Big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}.$$

We will estimate each term separately.

Estimate of I_1. Let $x \in R_j, k+4 \leq j \leq -1$. From (17) and Hölder's inequality, we obtain

$$\begin{aligned} |Ta_k(x)| &\leq \int_{\mathbb{R}^n} \big|k(x,y) - k(x,0)\big| \, |a_k(y)| \, dy \\ &\leq \int_{R_k} \frac{|y|^\delta}{|x|^{n+\delta}} \, |a_k(y)| \, dy \\ &\leq c 2^{k\delta - j(n+\delta)} \big\|a_k\big\|_{p(\cdot)} \big\|\chi_k\big\|_{p'(\cdot)}, \end{aligned}$$

where the positive constant c is independent of x, k and j. Hence, we obtain

$$\begin{aligned} \big\|(Ta_k)\chi_j\big\|_{p(\cdot)} &\lesssim 2^{k\delta - j(n+\delta)} \big\|a_k\big\|_{p(\cdot)} \big\|\chi_k\big\|_{p'(\cdot)} \big\|\chi_j\big\|_{p(\cdot)} \\ &\lesssim 2^{(k-j)\delta - jn - k\alpha(0)} \big\|\chi_k\big\|_{p'(\cdot)} \big\|\chi_j\big\|_{p(\cdot)}. \end{aligned}$$

Observe that

$$2^{-jn} \big\|\chi_k\big\|_{p'(\cdot)} \big\|\chi_j\big\|_{p(\cdot)} \lesssim 2^{-jn} 2^{kn/p'(y)} 2^{jn/p(x)}, \quad x \in R_j, y \in R_k,$$

where the positive constant c is independent of k, $x \in R_j$ and $y \in R_k$. Since p has a log decay at the origin, we have

$$2^{kn/p'(y)} \approx 2^{kn/p'(0)}, \quad 2^{jn/p(x)} \approx 2^{jn/p(0)}, \quad x \in R_j, y \in R_k.$$

Using the fact that $\alpha(0) < n(1 - 1/p(0)) + \delta$, we obtain by Lemma 1 that

$$\begin{aligned} I_1^{q(0)} &\lesssim \sum_{j=-\infty}^{-1} \Big(\sum_{k=-\infty}^{j-4} |\lambda_k| \, 2^{(k-j)(\delta + n - \frac{n}{p(0)} - \alpha(0))} \Big)^{q(0)} \\ &\lesssim \sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \\ &\lesssim \big\|f\big\|^{q(0)}_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \end{aligned}$$

Estimate of I_2. Decompose the sum $\sum_{k=-\infty}^{j-4} \lambda_k a_k$ into two parts

$$\sum_{k=-\infty}^{-1} \lambda_k a_k + \sum_{k=0}^{j-4} \lambda_k a_k, \quad j \in \mathbb{N}_0.$$

Put $\sum_{k=0}^{j-4} \lambda_k a_k = 0$ if $j = 0, 1, 2, 3$. Let $k \leq -1$. From the estimation of I_1, we get

$$\begin{aligned} \big\|(Ta_k)\chi_j\big\|_{p(\cdot)} &\lesssim 2^{(k-j)\delta - jn - k\alpha(0)} \big\|\chi_k\big\|_{p'(\cdot)} \big\|\chi_j\big\|_{p(\cdot)} \\ &\lesssim 2^{k(\delta + n - \frac{n}{p(0)} - \alpha(0))} 2^{-j(\delta + n - \frac{n}{p_\infty})}, \end{aligned}$$

where we have used Lemma 3. Therefore,

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{j\alpha_\infty q_\infty}\Big(\sum_{k=-\infty}^{-1}|\lambda_k|\big\|(Ta_k)\chi_j\big\|_{p(\cdot)}\Big)^{q_\infty}\\
&\lesssim \sum_{j=0}^{\infty} 2^{-j(\delta+n-\frac{n}{p_\infty}-\alpha_\infty)q_\infty}\Big(\sum_{k=-\infty}^{-1} 2^{k(\delta+n-\frac{n}{p(0)}-\alpha(0))}|\lambda_k|\Big)^{q_\infty}\\
&\lesssim \sup_{k\le -1}|\lambda_k|^{q_\infty}\\
&\lesssim \big\|f\big\|^{q_\infty}_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.
\end{aligned}$$

Now let $k \ge 0$. Again, from the estimation of I_1, we get

$$\begin{aligned}
\big\|(Ta_k)\chi_j\big\|_{p(\cdot)} &\lesssim 2^{(k-j)\delta-jn-k\alpha_\infty}\big\|\chi_k\big\|_{p'(\cdot)}\big\|\chi_j\big\|_{p(\cdot)}\\
&\lesssim 2^{k(\delta+n-\frac{n}{p_\infty}-\alpha_\infty)}2^{-j(\delta+n-\frac{n}{p_\infty})}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\Big(\sum_{j=0}^{\infty} 2^{j\alpha_\infty q_\infty}\Big(\sum_{k=0}^{j-4}|\lambda_k|\big\|(Ta_k)\chi_j\big\|_{p(\cdot)}\Big)^{q_\infty}\Big)^{1/q_\infty}\\
&\lesssim \Big(\sum_{j=0}^{\infty}\Big(\sum_{k=0}^{j-4} 2^{(k-j)(\delta+n-\frac{n}{p_\infty}-\alpha_\infty)}|\lambda_k|\Big)^{q_\infty}\Big)^{1/q_\infty}\\
&\lesssim \Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\\
&\lesssim \big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},
\end{aligned}$$

where the second inequality follows from Lemma 1. Hence, $I_2 \lesssim \big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$.

Estimate of I_3 and I_4. Using the boundedness of T on $L^{p(\cdot)}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
I_3^{q(0)} &\le \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)}\Big(\sum_{k=j-3}^{\infty}|\lambda_k|\big\|(Ta_k)\chi_j\big\|_{p(\cdot)}\Big)^{q(0)}\\
&\le \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)}\Big(\sum_{k=j-3}^{\infty}|\lambda_k|\big\|a_k\big\|_{p(\cdot)}\Big)^{q(0)}.
\end{aligned}$$

Further,

$$\sum_{k=j-3}^{\infty}|\lambda_k|\big\|a_k\big\|_{p(\cdot)} = \sum_{k=j-3}^{-1}|\lambda_k|\big\|a_k\big\|_{p(\cdot)} + \sum_{k=0}^{\infty}|\lambda_k|\big\|a_k\big\|_{p(\cdot)}.$$

If $k \leq -1$, then $\big\|a_k\big\|_{p(\cdot)} \lesssim 2^{-k\alpha(0)}$, and Lemma 1 yields that

$$\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \Big(\sum_{k=j-3}^{-1} |\lambda_k| \big\|a_k\big\|_{p(\cdot)} \Big)^{q(0)}$$

is bounded by

$$c \sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \lesssim \big\|f\big\|^{q(0)}_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Now if $k \geq 0$, then $\big\|a_k\big\|_{p(\cdot)} \lesssim 2^{-k\alpha_\infty}$, and

$$\begin{aligned}\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \Big(\sum_{k=0}^{\infty} |\lambda_k| \big\|a_k\big\|_{p(\cdot)} \Big)^{q(0)} &\lesssim \sup_{k\in\mathbb{N}_0} |\lambda_k|^{q(0)} \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \\ &\lesssim \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \Big)^{q(0)/q_\infty} \\ &\lesssim \big\|f\big\|^{q(0)}_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.\end{aligned}$$

Finally, for estimate I_4, we easily obtain

$$\begin{aligned} I_4^{q_\infty} &\lesssim \sum_{j=0}^{\infty} 2^{jq_\infty\alpha_\infty} \sum_{k=j-3}^{\infty} 2^{-kq_\infty\alpha_\infty} |\lambda_k|^{q_\infty} \\ &\lesssim \sum_{i=-3}^{-1} |\lambda_i|^{q_\infty} + \sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \\ &\lesssim \big\|f\big\|^{q_\infty}_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.\end{aligned}$$

This completes the proof. $\square$

Theorem 7 *Let α and q be* log*-Hölder continuous at infinity, and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < 1$ be such that*

$$\alpha_\infty = n\Big(1 - \frac{1}{p_\infty}\Big) + \delta. \tag{22}$$

Then every linear operator T satisfying (18) *which is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded from $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

For homogeneous spaces, we have the following statement:

Theorem 8 *Let α and q be* log-*Hölder continuous both at the origin and at infinity, and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$ and $0 < q^- \le q^+ < 1$ be such that*

$$\alpha(\cdot) \ge n\Big(1-\frac{1}{p^-}\Big), \alpha(0) = n\Big(1-\frac{1}{p(0)}\Big)+\delta \text{ and } \alpha_\infty = n\Big(1-\frac{1}{p_\infty}\Big)+\delta. \quad (23)$$

Then every linear operator T satisfying (18) *which is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded from $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

Since the proofs are similar, we will prove only Theorem 8.

Proof of Theorem 8. Our arguments are based on [16], where the constant exponent case was studied. Let $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. Due to Theorem 4, we have decomposition (19) and estimate (20). We will show that

$$\big\|Tf\big\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$$

for any $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. Decompose f as in (21). We will use the equivalent quasi-norm (5). Let $\lambda > 0$,

$$\begin{aligned} E_1 =& \lambda\Big\|\Big\{2^{j\alpha(0)}\chi_{A_j(\frac{\lambda}{2},\sum_{k=-\infty}^{j-4}\lambda_k(Ta_k))}\Big\}\Big\|_{\ell^{q(0)}_<(L^{p(\cdot)})}, \\ E_2 =& \lambda\Big\|\Big\{2^{j\alpha_\infty}\chi_{A_j(\frac{\lambda}{2},\sum_{k=-\infty}^{j-4}\lambda_k(Ta_k))}\Big\}\Big\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}, \\ E_3 =& \lambda\Big\|\Big\{2^{j\alpha(0)}\chi_{A_j(\frac{\lambda}{2},\sum_{k=j-3}^{\infty}\lambda_k T(a_k))}\Big\}\Big\|_{\ell^{q(0)}_<(L^{p(\cdot)})} \end{aligned}$$

and

$$E_4 = \lambda\Big\|\Big\{2^{j\alpha_\infty}\chi_{A_j(\frac{\lambda}{2},\sum_{k=j-3}^{\infty}\lambda_k(Ta_k))}\Big\}\Big\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}.$$

Estimate of E_1. Let $x \in R_j$ and $k \le j-4$, $j \le -1$. By (17) and (18), we have the estimate

$$|Ta_k(x)| \le \int_{R_k} \frac{|y|^\delta}{|x|^{n+\delta}}\,|a_k(y)|\,dy.$$

Using Hölder's inequality, we estimate the last integral by

$$\begin{aligned} c2^{k\delta-j(n+\delta)}\big\|a_k\big\|_{p(\cdot)}\big\|\chi_k\big\|_{p'(\cdot)} \lesssim& 2^{k\delta-j(n+\delta)-k\alpha(0)}\big\|\chi_k\big\|_{p'(\cdot)} \\ \lesssim& 2^{k\left(n-\frac{n}{p(0)}+\delta-\alpha(0)\right)}2^{-j(n+\delta)} \\ \lesssim& 2^{-j(n+\delta)}, \end{aligned}$$

where the implicit constant is independent of j and k. Therefore,

$$|Ta_k(x)| \le C2^{-j(n+\delta)} \quad (24)$$

for any $x \in R_j$ and any $k \leq j-4, j \leq -1$, where the positive constant C is independent of j, k and x. Observe that (24) is true even for $j \in \mathbb{N}_0$. Consequently,

$$\begin{aligned}\Big|\sum_{k=-\infty}^{j-4} \lambda_k T(a_k)(x)\Big| \leq & |Ta_k(x)| \sum_{k=-\infty}^{j-4} |\lambda_k| \\ \leq & C2^{-j(n+\delta)}\Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\Big)^{1/q(0)} \\ \leq & C2^{-j(n+\delta)}\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},\end{aligned}$$

where we used the embedding $\ell^{q(0)}_{<} \hookrightarrow \ell^1_{<}$. Assume that

$$\Big|\Big\{x \in R_j : \Big|\sum_{k=-\infty}^{j-4} \lambda_k Ta_k(x)\Big| > \frac{\lambda}{2}\Big\}\Big| \neq 0.$$

Then

$$\lambda \leq 2C2^{-j(n+\delta)}\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}, \quad j \leq -1.$$

For any fixed $\lambda > 0$, put

$$j_\lambda = \lfloor \frac{1}{n+\delta} \log_2(2C\lambda^{-1}\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}})\rfloor.$$

The advantage of this choice consists in the fact that

$$\Big\{x \in R_j : \Big|\sum_{k=-\infty}^{j-4} \lambda_k Ta_k(x)\Big| > \frac{\lambda}{2}\Big\} = \emptyset$$

if $-1 \geq j \geq j_\lambda + 1$. Hence, we obtain

$$\begin{aligned}E_1 \leq & C\lambda\Big(\sum_{j=-\infty}^{j_\lambda} 2^{jq(0)\alpha(0)}\big\|\chi_{\{x\in R_j : |\sum_{k=-\infty}^{j-4}\lambda_k(Ta_k)(x)|>\frac{\lambda}{2}\}}\big\|^{q(0)}_{p(\cdot)}\Big)^{1/q(0)} \\ \leq & C\lambda\Big(\sum_{j=-\infty}^{j_\lambda} 2^{jq(0)\alpha(0)}\big\|\chi_j\big\|^{q(0)}_{p(\cdot)}\Big)^{1/q(0)} \\ \leq & C\lambda\Big(\sum_{j=-\infty}^{j_\lambda} 2^{jq(0)(\alpha(0)+\frac{n}{p(0)})}\Big)^{1/q(0)} \\ \leq & C\lambda 2^{j_\lambda(n+\delta)} \\ \lesssim & \big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.\end{aligned}$$

Estimate of E_2. Observe that $\sum_{k=-\infty}^{j-4}|\lambda_k| \lesssim \|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$, which yields that

$$\Big|\sum_{k=-\infty}^{j-4}\lambda_k Ta_k(x)\Big| \le \sum_{k=-\infty}^{j-4}|\lambda_k|\,|Ta_k(x)| \le C2^{-j(n+\delta)}\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}, \quad x\in R_j,$$

where we used estimate (24). Further,

$$\begin{aligned}
E_2 \le& C\lambda\Big(\sum_{j=0}^{j_\lambda}2^{jq_\infty\alpha_\infty}\big\|\chi_{\big\{x\in R_j:\,\big|\sum_{k=-\infty}^{j-4}\lambda_k(Ta_k)(x)\big|>\frac{\lambda}{2}\big\}}\big\|_{p(\cdot)}^{q_\infty}\Big)^{1/q_\infty}\\
\le& C\lambda\Big(\sum_{j=0}^{j_\lambda}2^{jq_\infty\alpha_\infty}\|\chi_j\|_{p(\cdot)}^{q_\infty}\Big)^{1/q_\infty}\\
\le& C\lambda\Big(\sum_{j=0}^{j_\lambda}2^{jq_\infty(\alpha_\infty+\frac{n}{p_\infty})}\Big)^{1/q_\infty}\\
\lesssim& \|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.
\end{aligned}$$

Estimate of E_3. From the boundedness of T on $L^{p(\cdot)}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
\sum_{k=j-3}^{\infty}|\lambda_k|\|(Ta_k)\chi_j\|_{p(\cdot)} &\lesssim \sum_{k=j-3}^{-1}|\lambda_k|\|(Ta_k)\chi_j\|_{p(\cdot)} + \sum_{k=0}^{\infty}|\lambda_k|\|(Ta_k)\chi_j\|_{p(\cdot)}\\
&\lesssim \sum_{k=j-3}^{-1}|\lambda_k|\|a_k\|_{p(\cdot)} + \sum_{k=0}^{\infty}|\lambda_k|\|a_k\|_{p(\cdot)}\\
&\lesssim \sum_{k=j-3}^{-1}|\lambda_k|2^{-k\alpha(0)} + \sum_{k=0}^{\infty}|\lambda_k|2^{-k\alpha_\infty}.
\end{aligned}$$

Hence,

$$\sum_{j=-\infty}^{-1}2^{jq(0)\alpha(0)}\Big(\sum_{k=j-3}^{-1}|\lambda_k|\|(Ta_k)\chi_j\|_{p(\cdot)}\Big)^{q(0)}$$

can be estimated by

$$c\sum_{j=-\infty}^{-1}2^{jq(0)\alpha(0)}\Big(\sum_{k=j-3}^{-1}|\lambda_k|2^{-k\alpha(0)}\Big)^{q(0)} \lesssim \sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)} \lesssim \|f\|^{q(0)}_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$$

due to Lemma 1. Now

$$\begin{aligned}
\sum_{j=-\infty}^{-1}2^{jq(0)\alpha(0)}\Big(\sum_{k=0}^{\infty}|\lambda_k|\|(Ta_k)\chi_j\|_{p(\cdot)}\Big)^{q(0)} &\lesssim \sum_{j=-\infty}^{-1}2^{jq(0)\alpha(0)}\Big(\sum_{k=0}^{\infty}|\lambda_k|2^{-k\alpha_\infty}\Big)^{q(0)}\\
&\lesssim \sup_{k\in\mathbb{N}_0}|\lambda_k|^{q(0)}\\
&\lesssim \|f\|^{q(0)}_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.
\end{aligned}$$

Estimate of E_4. By Lemma 1, we obtain

$$
\begin{aligned}
E_4 \lesssim & \Big(\sum_{j=0}^{\infty} 2^{jq_\infty \alpha_\infty} \sum_{k=j-3}^{\infty} 2^{-kq_\infty \alpha_\infty} |\lambda_k|^{q_\infty} \Big)^{1/q_\infty} \\
\lesssim & \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \Big)^{1/q_\infty} + \sum_{k=-3}^{-1} |\lambda_k| \\
\lesssim & \|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.
\end{aligned}
$$

This completes the proof. □

Remark 5 *Statements similar to Theorems 7 and 8, with α, p and q constants, can be found in [16], while with α and q constants, Theorems 7 and 8 are proved in [27] under the assumption that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (both in the homogeneous and the inhomogeneous situation). Here we are requiring the log-Hölder continuity at two points only (zero and infinity). We also note that our conditions* (22) *and* (23) *are more explicit than those used in [27], hence allowing better comparison with the already known constant exponent setting.*

If the operator T in Theorems 7 and 8 is of convolution type, we can obtain the following results:

Theorem 9 *Let α and q be* log-*Hölder continuous at infinity, and let $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < 1$ be such that*

$$
\alpha_\infty = n\Big(1 - \frac{1}{p_\infty}\Big) + \delta.
$$

Let T be defined by

$$
Tf(x) = \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} K(x-y) f(y)\, dy,
$$

where the kernel K satisfies

$$
|K(x-y) - K(x)| \leq \frac{|y|^\delta}{|x|^{n+\delta}}
$$

if $2|y| < |x|$ for some $\delta \in (0,1]$. Assume that T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Then T is also bounded from $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $WHK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

The counterpart for homogeneous Herz spaces runs as follows:

Theorem 10 *Let α and q be* log-*Hölder continuous, both at the origin and at infinity, and let $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$ and $0 < q^- \le q^+ < 1$ be such that*

$$\alpha(\cdot) \ge n\Big(1 - \frac{1}{p^-}\Big), \quad \alpha(0) = n\Big(1 - \frac{1}{p(0)}\Big) + \delta, \quad \text{and} \quad \alpha_\infty = n\Big(1 - \frac{1}{p_\infty}\Big) + \delta.$$

Let T be as in Theorem 9. Assume that T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Then T is also bounded from $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $WH\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Since the proofs are similar, we will prove only Theorem 10.

Proof of Theorem 10. Let $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. Then we have decomposition (19) and estimate (20). As before, we decompose f as in (21). Let $\lambda > 0$,

$$\begin{aligned}
F_1 =&\lambda\Big\|\Big\{2^{j\alpha(0)}\chi_{A_j(\frac{\lambda}{2},\sum_{k=-\infty}^{j-4}\lambda_k G_N(Ta_k))}\Big\}\Big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})},\\
F_2 =&\lambda\Big\|\Big\{2^{j\alpha_\infty}\chi_{A_j(\frac{\lambda}{2},\sum_{k=-\infty}^{j-4}\lambda_k G_N(Ta_k))}\Big\}\Big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})},\\
F_3 =&\lambda\Big\|\Big\{2^{j\alpha(0)}\chi_{A_j(\frac{\lambda}{2},\sum_{k=j-3}^{\infty}\lambda_k G_N(Ta_k))}\Big\}\Big\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}
\end{aligned}$$

and

$$F_4 = \lambda\Big\|\Big\{2^{j\alpha_\infty}\chi_{A_j(\frac{\lambda}{2},\sum_{k=j-3}^{\infty}\lambda_k G_N(Ta_k))}\Big\}\Big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}.$$

Estimates of F_3 and F_4. The boundedness of G_N and T on $L^{p(\cdot)}(\mathbb{R}^n)$ yields that

$$\begin{aligned}
F_3 \le&\Big(\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)}\Big(\sum_{k=j-3}^{\infty}|\lambda_k|\big\|G_N(Ta_k)\chi_j\big\|_{p(\cdot)}\Big)^{q(0)}\Big)^{1/q(0)}\\
\le&\Big(\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)}\sum_{k=j-3}^{\infty}|\lambda_k|^{q(0)}\big\|(Ta_k)\chi_j\big\|_{p(\cdot)}^{q(0)}\Big)^{1/q(0)}\\
\le&\Big(\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)}\sum_{k=j-3}^{\infty}|\lambda_k|^{q(0)}\big\|a_k\big\|_{p(\cdot)}^{q(0)}\Big)^{1/q(0)}.
\end{aligned} \tag{25}$$

Decompose the second sum in (25) into two parts

$$\sum_{k=j-3}^{-1}|\lambda_k|^{q(0)}\big\|a_k\big\|_{p(\cdot)}^{q(0)} + \sum_{k=0}^{\infty}|\lambda_k|^{q(0)}\big\|a_k\big\|_{p(\cdot)}^{q(0)} = P_{1,j} + P_{2,j}.$$

If $k \leq -1$, then $\|a_k\|_{p(\cdot)} \lesssim 2^{-k\alpha(0)}$, and Lemma 1 yields

$$\begin{aligned}\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} P_{1,j} &\lesssim \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \sum_{k=j-3}^{-1} |\lambda_k|^{q(0)} 2^{-k\alpha(0)q(0)} \\ &\lesssim \sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \\ &\lesssim \|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}^{q(0)}.\end{aligned}$$

Now if $k \geq 0$, then $\|a_k\|_{p(\cdot)} \lesssim 2^{-k\alpha_\infty}$ and

$$\begin{aligned}\sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} P_{2,j} &\lesssim \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \sum_{k=0}^{\infty} |\lambda_k|^{q(0)} 2^{-k\alpha_\infty q(0)} \\ &\lesssim \sup_{k\in\mathbb{N}_0} |\lambda_k|^{q(0)} \sum_{j=-\infty}^{-1} 2^{jq(0)\alpha(0)} \\ &\lesssim \|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}^{q(0)}.\end{aligned}$$

By Lemma 1, we easily obtain that

$$\begin{aligned}F_4 &\lesssim \Big(\sum_{j=0}^{\infty} 2^{jq_\infty\alpha_\infty} \sum_{k=j-3}^{\infty} 2^{-kq_\infty\alpha_\infty} |\lambda_k|^{q_\infty} \Big)^{1/q_\infty} \\ &\lesssim \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \Big)^{1/q_\infty} + \sum_{k=-3}^{-1} |\lambda_k| \\ &\lesssim \|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.\end{aligned}$$

Estimate of F_1. Let $x \in R_j$, $k \leq j-4$, $|x-y| < t$ and $\varphi \in \mathcal{A}_N$. We have

$$\begin{aligned}(Ta_k * \varphi_t)(y) &= \int_{\mathbb{R}^n} Ta_k(z)\big(\varphi_t(y-z) - \varphi_t(y)\big)dz \\ &= t^{-n} \int_{\mathbb{R}^n} Ta_k(z)\big(\varphi(\frac{y-z}{t}) - \varphi(\frac{y}{t})\big)dz \\ &= P_3(y) + P_4(y) + P_5(y),\end{aligned}$$

since $\int_{\mathbb{R}^n} Ta_k(z)dz = 0$, where

$$P_3(y) = t^{-n} \int_{|z|<2^{k+1}} Ta_k(z)\psi_t(y,z)dz,$$

$$P_4(y) = t^{-n} \int_{2^{k+1}\leq|z|<|x|/2} Ta_k(z)\psi_t(y,z)dz$$

and

$$P_5(y) = t^{-n} \int_{|z|\geq|x|/2} Ta_k(z)\psi_t(y,z)dz,$$

with $\psi_t(y,z) = \varphi((y-z)/t) - \varphi(y/t), |x-y| < t, z \in \mathbb{R}^n$. Let us estimate each term separately. By the mean value theorem,

$$\begin{aligned} \big|\psi_t(y,z)\big| \leq & t^{-1}|z| \sup_{|\beta|=1} |\partial^\beta \varphi(\frac{y}{t} - \theta\frac{z}{t})| \\ \lesssim & t^n|z|(t+|y-\theta z|)^{-n-1}, \quad 0<\theta<1. \end{aligned} \tag{26}$$

Observe that

$$t + |y-\theta z| \geq |x-y| + |y-\theta z| \geq |x-\theta z| \geq \frac{1}{2}|x|, \quad z \in B_{k+1}.$$

This together with the boundedness of T on $L^{p(\cdot)}(\mathbb{R}^n)$ yields that

$$\begin{aligned} |P_3(y)| \leq & Ct^{-n}\big\|Ta_k\big\|_{p(\cdot)}\big\|\psi_t(y,\cdot)\chi_{B_{k+1}}\big\|_{p'(\cdot)} \\ \leq & C\frac{2^k}{|x|^{n+1}}\big\|a_k\big\|_{p(\cdot)}\big\|\chi_{B_{k+1}}\big\|_{p'(\cdot)} \\ \leq & C\frac{2^{k(1-\alpha(0)+n-\frac{n}{p(0)})}}{|x|^{n+1}} \\ \leq & \frac{C}{|x|^{n+\delta}}, \end{aligned}$$

where the positive constant C is independent of x. Further,

$$P_4(y) = t^{-n} \int_{2^{k+1}\leq|z|<|x|/2} \int_{R_k} a_k(h)(K(z-h)-K(z))dh\psi_t(y,z)dz$$

and

$$\int_{R_k} |a_k(h)||K(z-h)-K(z)|dh \leq \int_{R_k} |a_k(h)|dh\frac{|h|^\delta}{|z|^{n+\delta}} \leq \frac{2^{k\delta}}{|z|^{n+\delta}}\big\|a_k\big\|_1.$$

Therefore, due to (26),

$$\begin{aligned} |P_4(y)| \leq & C2^{k\delta}|x|^{-1-n}\big\|a_k\big\|_1 \int_{|z|<|x|/2} \frac{1}{|z|^{n+\delta-1}}dz \\ \leq & C2^{k\delta}\frac{1}{|x|^{n+\delta}}\big\|a_k\big\|_{p(\cdot)}\big\|\chi_{B_k}\big\|_{p'(\cdot)} \\ \leq & \frac{C}{|x|^{n+\delta}}, \end{aligned}$$

where the positive constant C is independent of x. By the vanishing condition of a_k, we have

$$P_5(y)=t^{-n}\int_{|z|\geq|x|/2}\int_{R_k}a_k(h)(K(z-h)-K(z))dh\psi_t(y,z)dz.$$

Observe that

$$\begin{aligned}
&\int_{|z|\geq|x|/2}|z|^{-n-\delta}\left|\psi_t(y,z)\right|dz\\
\leq&\int_{|z|\geq|x|/2}|z|^{-n-\delta}|\varphi(\frac{y-z}{t})|dz+\int_{|z|\geq|x|/2}|z|^{-n-\delta}|\varphi(\frac{y}{t})|dz\\
\leq&\frac{C_1t^n}{|x|^{n+\delta}}+\int_{|z|\geq|x|/2}|z|^{-n-\delta}|\varphi(\frac{y}{t})|dz\\
\leq&\frac{C_1t^n}{|x|^{n+\delta}}+C_2t^n\int_{|z|\geq|x|/2}|z|^{-n-\delta}(t+|y|)^{-n}dz\\
\leq&\frac{C_1t^n}{|x|^{n+\delta}}+C_2t^n|x|^{-n}\int_{|z|\geq|x|/2}|z|^{-n-\delta}dz\\
\leq&\frac{Ct^n}{|x|^{n+\delta}},
\end{aligned}$$

since $t+|y|\geq|x-y|+|y|\geq|x|$. Hence, $|P_5(y)|\leq C/|x|^{n+\delta}$, where the positive constant C is independent of x. Therefore, we obtain

$$\sum_{k=-\infty}^{j-4}\lambda_kG_N(Ta_k)(x)\leq C\sum_{k=-\infty}^{j-4}|\lambda_k|\frac{1}{|x|^{n+\delta}}\leq A2^{-j(n+\delta)}\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},\quad x\in R_j.$$

The desired estimate follows by the same arguments as in Theorem 8.

The estimate of F_2 is similar to the estimate of F_1. This completes the proof. □

Remark 6 *Under the assumption that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ spaces both in the homogeneous and the inhomogeneous situation, Theorems 9 and 10 with α and q constants are given in [27]. For the classical Herz-type Hardy spaces, see [16].*

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