ARMENIAN JOURNAL OF MATHEMATICS
Volume 3, Number 1, 2010, 14–21

On injectors of finite groups

Shitian Liu* and Runshi Zhang**

* *School of Science, Sichuan University of Science & Engineering, Zigong 643000 P.R.China.*
liust@suse.edu.cn

** *School of Science, Sichuan University of Science & Engineering Zigong 643000 P.R.China.*
zhangrs-75@163.com

Received by the editors May 07, 2010; accepted for publication February 15, 2010.

Abstract

If $\mathscr{F}$ is a non-empty Fitting class, $\pi = \pi(\mathscr{F})$ and G a group such that every chief factor of $G/G_{\mathscr{F}}$ is an C_π^s-group. Then G has at least one $\mathscr{F}$-injector. This result is used to resolve an open problem and generalize some known results.

Key Words: Fitting class, Soluble, Chief factor, $\mathscr{F}$-injector.
Mathematics Subject Classification 2000: 20D10, 20F17

1 Introduction

All groups in this paper are finite. Fischer, Gaschütz, and Hartley [1] proved that for any Fitting class $\mathscr{F}$ and any finite solvable group G there exist $\mathscr{F}$-injectors and any two of them are conjugate in G. A class $\mathscr{F}$ of groups is a Fitting class if (i) $G \in \mathscr{F}$, $N \lhd G$ implies that $N \in \mathscr{F}$ and (ii) N_1, $N_2 \lhd G$, $N_1, N_2 \in \mathscr{F}$ implies that $N_1N_2 \in \mathscr{F}$. Fitting classes were introduced by Fischer, Gaschütz, and Hartley [1]. If $\mathscr{F}$ is a Fitting class, each group G possesses a unique maximal normal $\mathscr{F}$-subgroup called the $\mathscr{F}$-radical of G and denoted by $G_{\mathscr{F}}$, which contains each subnormal $\mathscr{F}$-subgroup of G. Furthermore if N is subnormal in G, then $N_{\mathscr{F}} = N \cap G_{\mathscr{F}}$. A subgroup V of G is called an $\mathscr{F}$-injector of G if $V \cap N$ is $\mathscr{F}$-maximal in N for each subnormal subgroup N of G. In particular, an $\mathscr{F}$-injector V of G lies in $\mathscr{F}$ and contains $G_{\mathscr{F}}$. Tomkinson in [2] proved that let $\mathscr{F}$ be a Fitting class, then the $\mathcal{G}$-group G possesses $\mathscr{F}$-injectors, where $\mathcal{G}$ is the class of periodic locally soluble FC-groups. Flavell in **[3]** proved that if G is a group whose local subgroups are $\mathscr{N}$-constrained, then all nilpotent injectors of G are conjugate. Guo, and Vorob'ev [4] described the $\mathscr{H}$-injectors associated with a Hartley class $\mathscr{H}$. And some good results are given by some authors (see [5, 6, 7, 8]).

In this note, we give the following notations (see [9, p386-387]):
π: any set of primes.
E_π: G has at least one Hall π-subgroup;
C_π: G satisfies E_π and any two Hall π-subgroups of G are conjugate in G.
E_π^n: G has a nilpotent Hall π-subgroup.
C_π^s: G satisfies C_π and its Hall π-subgroup are soluble.

Guo, and Li [10] gave that, let $\mathscr{F}$ be a non-empty Fitting class, $\pi = \pi(G)$ and G a group such that every chief factor of $G/G_{\mathscr{F}}$ is an E_π^n-group, then G has at most one $\mathscr{F}$-injector and any two $\mathscr{F}$-injectors are conjugate in G.

Concerning Fitting classes and $\mathscr{F}$-injectors, the following problem arose:

Problem (see [11]). Let $\mathscr{F}$ is a local Fitting class. Could we describe the $\mathscr{F}$-injectors of a group?

In this note, we will partially deal with the problem and prove the following main theorem.

Theorem 1.1. *Let $\mathscr{F}$ be a non-empty Fitting class, $\pi = \pi(\mathscr{F})$ and G a group such that every chief factor of $G/G_{\mathscr{F}}$ is an C_π^s-group. Then*
(1) G has at least one $\mathscr{F}$-injector.
(2) Any two $\mathscr{F}$-injector are exactly all the $\mathscr{F}$-maximal subgroups which contain the $\mathscr{F}$-radical $G_{\mathscr{F}}$.
(3) In any G there exist $\mathscr{F}$-injectors and any two of them are conjugate in G.

For some notion and notations, the reader is referred to Ballester-Bolinches and Ezquerro [12], and Doerk and Hawkes [13].

2 Preliminaries

Definition 2.1 ([1])**.** *A class $\mathscr{F}$ of groups is a Fitting class if*
(i) $G \in \mathscr{F}$, $N \lhd G$ implies that $N \in \mathscr{F}$ and
(ii) $N_1, N_2 \lhd G$, $N_1, N_2 \in \mathscr{F}$ implies that $N_1N_2 \in \mathscr{F}$.

Definition 2.2 (see [12] or [13])**.** *A subgroup V of G is called an $\mathscr{F}$-injector of G if $V \cap N$ is $\mathscr{F}$-maximal in N for each subnormal subgroup N of G.*

Lemma 2.1 ([1])**.** *Let $\mathscr{F}$ be a Fitting class. Then a soluble group G has at most one $\mathscr{F}$-injector and any two $\mathscr{F}$-injectors of G are conjugate in G.*

Lemma 2.2 ([10]). *Let $\mathscr{F}$ be a Fitting class, and H an $\mathscr{F}$-injector of G. Then the following statements hold:*
(1) H is an $\mathscr{F}$-maximal subgroup of G;
(2) $G_{\mathscr{F}} \leq H$;
(3) For every $x \in H$, H^x is also an $\mathscr{F}$-injector of G;
(4) If K is subnormal subgroup of G, then $H \cap K$ is an $\mathscr{F}$-injector of K.

Lemma 2.3 ([1]). *If V is an $\mathscr{F}$-injector of G and $V \leq H \leq G$, then V is an $\mathscr{F}$-injector of H.*

Lemma 2.4 ([1]). *Let $N \lhd G$ with G/N nilpotent. If V_1 and V_2 are $\mathscr{F}$-maximal in G and $V_1 \cap N = V_2 \cap N$ is $\mathscr{F}$-maximal in N, then V_1 and V_2 are conjugate in G.*

Lemma 2.5 ([15, p334], or [14, Corollary 7.3.12]). *Let $\mathscr{F}$ be a Fitting class, if H is subnormal subgroup of G, then $H_{\mathscr{F}} = H \cap G_{\mathscr{F}}$.*

Lemma 2.6 ([13, IX-Lemma 1.6]). *Let $\mathscr{F}$ be a Fitting class, and let G be a finite soluble group. Let $N \lhd G$, and let L be a subgroup of G such that $L \cap N$ is an $\mathscr{F}$-injector of N. Assume that either*
(1) G/N is nilpotent, and L is $\mathscr{F}$-maximal in G, or
(2) $L \in \mathscr{F}$ and $LN = G$.
Then L is an $\mathscr{F}$-injector of G.

Lemma 2.7 ([9, Theorem C1]). *If G has a series in which every factor is a C_π^s-group, then G is a C_π^s-group and every Hall π-subgroup of G is solvable.*

Lemma 2.8. *Let $\mathscr{F}$ be a Fitting class and G a group. Suppose that $G/G_{\mathscr{F}}$ is soluble and G/N is soluble. If V is an $\mathscr{F}$-maximal subgroup of G and $V \cap N$ is an $\mathscr{F}$-injector of N, then V is an $\mathscr{F}$-injector of G*

Proof. Assume that the Lemma is not true and G is a minimal-order-counter-example. By **[12, Theorem 2.4.27]**, G has a unique conjugate class of $\mathscr{F}$-injectors. Let V_0 be a $\mathscr{F}$-injector of G, then, by Lemma 2.2(1), V_0 is a maximal $\mathscr{F}$-subgroup of G.

Cases 1. $NV < G$.

Since G/N is soluble, VN/N is soluble. Obviously V is also a maximal $\mathscr{F}$-subgroup of VN. By Lemma 2.5, $(VN)_{\mathscr{F}} = NV \cap G_{\mathscr{F}}$. Hence the quotient $NV/(NV)_{\mathscr{F}} = NV/(NV \cap G_{\mathscr{F}}) \cong NVG_{\mathscr{F}}/G_{\mathscr{F}} \leq G/G_{\mathscr{F}}$ is soluble. Thus, the minimal choice of G implies that V is an $\mathscr{F}$-injector of G. By **[12, Theorem 2.4.27]**, G has a unique conjugate class of $\mathscr{F}$-injectors, there exist an element $x \in NV$ such that $(V_0 \cap NV)^x = V$, and so $V \leq V_0^x$. Since an $\mathscr{F}$-maximal subgroup of G, $V = V_0^x$, and so V is an $\mathscr{F}$-injector of G by virtue of Lemma 2.2(3), a contradiction.

Cases 2. $NV = G$.

Let M be a maximal normal subgroup of G containing N. Since G/N is soluble, M/N is soluble. It is easy to see that $V \cap M \lhd V$. Let V_1 be an maximal $\mathscr{F}$-subgroup of M with $V \cap N \leq V_1$. Since $V \cap N = (V \cap M) \cap N \leq V_1 \cap N$ and $V_1 \cap N = (V_1 \cap N) \cap (NV) = (V \cap N) \cap (V_1 \cap N) \leq V \cap N$, $V_1 \cap N = V \cap N$ is an $\mathscr{F}$-injector of N by hypotheses, and, by Lemma 2.5, the quotient $M/M_{\mathscr{F}} = M/(M \cap G_{\mathscr{F}}) \cong MG_{\mathscr{F}}/G_{\mathscr{F}} \leq G/G_{\mathscr{F}}$ is soluble. The minimal choice of G implies that V_1 is an $\mathscr{F}$-injector of M. Since $M = (NV) \cap M = N(V \cap M) \leq NV_1 \leq G$, $NV_1 = G$ or M. If the former, then, $G/N = NV/N \cong V/V \cap N \cong V_1N/N \cong V_1/V_1 \cap N$. Comparing the order, we have $V_1 = V^x$ for some $x \in G$. By Lemma 2.2(3), V_1 is an $\mathscr{F}$-injector of G, a contradiction. So have $M = NV_1$, and $|N(V \cap M)| = |NV_1| = |M|$. This shows $|N||V \cap M|/|V \cap N| = |N||V_1|/|N \cap V_1|$ and hence $(V \cap M)^x = V_1$ for some $x \in G$. By Lemma 2.2(3), $V \cap M$ is an $\mathscr{F}$-injector of M. On the other hand, by Lemma 2.2(4), $V_0 \cap M$ is an $\mathscr{F}$-injector of M. By **[12, Theorem 2.4.27]**, there exists an $x \in M$ such that $V_0 \cap M = (V \cap M)^x = V^x \cap M$. Moreover, by Lemma 2.2(1), V_0, V^x are $\mathscr{F}$-maximal subgroup of M and $G_{\mathscr{F}} \leq V \cap V_0$. By **[16, Lemma 2.3]**, V, V_0 are conjugate in G, and so V is an $\mathscr{F}$-injector of G, a contradiction.

This completes the proof. □

Lemma 2.9. *Let $\mathscr{F}$ be non-empty Fitting class and G a group. If every chief factor of $G/G_{\mathscr{F}}$ is an C_π^s-group and $N \lhd G$, then every chief factor of $N/N_{\mathscr{F}}$ is also an C_π^s-group.*

Proof. By hypotheses, there exists a series

$$G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_n = G$$

Such that every chief factor are C_π^s-group. Since $N_{\mathscr{F}} = N \cap G_{\mathscr{F}}$ by Lemma 2.5, $N/N_{\mathscr{F}} = N/(N \cap G_{\mathscr{F}}) \cong NG_{\mathscr{F}}/G_{\mathscr{F}} \leq G/G_{\mathscr{F}}$. It follows that, intersection of N and the above series is the series such that every chief factor of $N/N_{\mathscr{F}}$ is also an C_π^s-group.

This completes the proof. □

3 Some results

In this section, we will give the proof of the main theorem 1.1 and some applications.

The proof of Theorem 1.1

Proof. Our proof proceeds via a number of steps.

Step 1. G is a C_π-group and if H is a Hall π-subgroup of G, then $H/H_{\mathscr{F}}$ is soluble.

By Lemma 2.7, we have G is a C_π-group and, since $G_{\mathscr{F}} \leq H_{\mathscr{F}} \leq H$, $H/H_{\mathscr{F}}$ is soluble.

Step 2. If G has an $\mathscr{F}$-injector, then an $\mathscr{F}$-injector of G is also an $\mathscr{F}$-injector of some Hall π-subgroup of G.

Let V be an $\mathscr{F}$-injector of G. Assume that V is a π-group of G. Without loss of generality, assume that $V \leq H$. Now prove that V is also an $\mathscr{F}$-injector of H.

Denote $N = G_{\mathscr{F}}$. Then set
$\mathscr{F}^* = \{M/N : M \in \mathscr{F}, N \leq M\}$ is a Fitting set of the soluble group G/N.
Moreover, by **[13, VIII-2.17(a)]**, have that

$$\mathscr{F}_0 = \{S \leq G : SN/N \in \mathscr{F}^* \text{and } S \text{ is subnormal in } SN\}$$

is a Fitting set of G. Observe that $\mathscr{F}_0 \subset \mathscr{F}$ and, for any subnormal subgroup S of G, $S_{\mathscr{F}_0} = S_{\mathscr{F}}$. By hypotheses of the theorem, G has a subnormal series

$$1 \leq N = G_{\mathscr{F}} = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_n = G$$

such that G_i/G_{i-1} is a G-chief factor and is an E^s_π-group. By **[13, VIII-2.17(b)]**, if V/N is an $\mathscr{F}^*$-injector of H/N, then V is an $\mathscr{F}_0$-injector of H. Since $G_{\mathscr{F}} \leq H$, H has the subnormal series

$$1 \leq G_{\mathscr{F}} = H \cap G_0 \leq G_1 \cap H \leq G_2 \cap H \cdots \leq G_{n-1} \cap H \leq G_n \cap H = H.$$

such that $G_i \cap H/G_{i-1} \cap H$ is an E^s_π-group by Lemma 2.6. And also V is an $\mathscr{F}$-injector of H. To see that, we prove that, for any subnormal subgroup $H_i = G_i \cap H$ of H, the subgroup $V \cap H_i$ is $\mathscr{F}$-maximal in H_i. Suppose that there exists $W \in \mathscr{F}$ such that $V \cap H_i \leq W \leq H_i$. Then $(V \cap H_i)N/N = (V/N) \cap (SN/N) \leq WN/N \leq H_iN/N$. Since $H_{i\mathscr{F}0} = H_{i\mathscr{F}} \leq V \cap H_i \in \text{Inj}_{\mathscr{F}0}(H_i)$, then $H_{i_{\mathscr{F}}} \leq W$. By Lemma 2.5, $N \cap H_i = H_{i\mathscr{F}}$. Therefore $W(N \cap H_i) = WH_{i\mathscr{F}} = W$, W is subnormal in WN, and so $WN \in \mathscr{F}$. Thus, $WN/N \in \mathscr{F}^*$. Since $(V/N) \cap (H_iN/N)$ is $\mathscr{F}^*$-maximal in H_iN/N, $(V \cap H_i)N = WN$, This means that $V \cap H_i = (V \cap H_i)(N \cap H_i) = WN \cap H_i = W$, and $V \cap H_i$ is $\mathscr{F}$-maximal in H_i. Therefore, have that $V \in \text{Inj}(G)$.

Step 3. If G have $\mathscr{F}$-injectors, then any two $\mathscr{F}$-injectors are conjugate in G.

By **[13, VIII-2.15]**, if $V \in \text{Inj}(G)$, then V/N is an $\mathscr{F}^*$-injector of the soluble group G/N. By Lemma 2.1, the $\mathscr{F}^*$-injectors of G/N are conjugate in G/N. And so any two $\mathscr{F}$-injectors are conjugate in G.

Step 4. G has an $\mathscr{F}$-injector.

Let H be a Hall π-subgroup of G, Then $H/H_{\mathscr{F}}$ is soluble by step 1, and hence H has an $\mathscr{F}$-injector by **[12, Theorem 2.4.27]**. In order to prove that G has an $\mathscr{F}$-injector, only needs to prove an arbitrary $\mathscr{F}$-injector of H is an $\mathscr{F}$-injector of G.

Let V be an $\mathscr{F}$-injector of H. Let K be an subnormal subgroup of G. Then $V \cap K$ is a subnormal subgroup of H. By Lemma 2.2(4), the subgroup $V \cap K = (V \cap H) \cap K = V \cap (H \cap K)$ is an $\mathscr{F}$-injector of $H \cap K$. Since $|K : H \cap K| = |KH : H|$ is a π'-number, $H \cap K$ is a Hall π-subgroup of K. So we need to deal with the following cases: $K = G$ or $K < G$.

Case 1: $K < G$. Then by induction, $V \cap K$ is an $\mathscr{F}$-injector of K, and $V \cap K$ is an $\mathscr{F}$-maximal subgroup of K. Since K is arbitrary, V is also an $\mathscr{F}$-injector of G.

Case 2: $K = G$. Let W be an maximal $\mathscr{F}$-subgroup of G with $V \leq W \leq G$. Since for every subnormal subgroup M of G, $W \cap M = V \cap M$ is an $\mathscr{F}$-maximal subgroup of M

by case 1. And so W is an $\mathscr{F}$-injector of G. Since $G \in C_\pi^s$, there exists an element $x \in G$ such that $V \leq W \leq H^x$. But, by Lemma 2.2(3), V^x is also an $\mathscr{F}$-injector of H^x. By step 2, W is also an $\mathscr{F}$-injector of H. Since $H/H_{\mathscr{F}}$ is soluble, by **[12, Theorem 2.4.27]**, W and V are conjugate in G, and so $V = W$.

This completes the proof. □

Remark 3.1. *This Theorem 1.1 is comparing the Theorem 2.4.27 of [12].*

Corollary 3.1. *Let $\mathscr{F}$ be a non-empty Fitting class and $\pi = \pi(\mathscr{F})$.If every chief factor for every maximal subgroup of G is an C_π^s-group, then G has an $\mathscr{F}$-injector.*

Proof. 1. Let M_1, M_2 be maximal subgroups of G such that M_1, M_2 are not conjugate in G. Then $G = M_1 M_2$. By Theorem 3.1, M_1, M_2 have $\mathscr{F}$-injectors V_1, V_2.

If $M_1 \cap M_2 = 1$, then $G = M_1 \times M_2$, and, by **[17, Lemma 1]**, G contains $\mathscr{F}$-injectors which are the product of the $\mathscr{F}$-injectors of the factors, M_1, M_2.

If $M_1 \cap M_2 \neq 1$, so there exists a prime p dividing the order of $M_1 \cap M_2$. And so assume that $|G : M_1| = p$ or q, where $p \neq q$.

Case 1: If $|G : M_1| = q$, then $M_1 \lhd G$, and V_1, which is an $\mathscr{F}$-injector of M_1, is also an $\mathscr{F}$-injector of G. To see this. Only needs to prove every subnormal subgroup K of G, $V_1 \cap K$ is an $\mathscr{F}$-injector of K. By Lemma 2.5, $M_{1\mathscr{F}} = G_{\mathscr{F}} \cap M_1$. By hypotheses, there exists a series

$$1 \leq W_0 = M_{1\mathscr{F}} = G_{\mathscr{F}} \cap M_1 \leq W_1 \leq W_2 \leq \cdots \leq W_{n-1} = M_1 \leq W_n = G$$

such that every chief factor of G is E_π^s-group, then by Theorem 1.1, V_1 is an $\mathscr{F}$-injector of G.

Case 2: If $|G : M_1| = p$, then, for a Sylow p-subgroup P_1 of M_1, there exists a p-subgroup P_2 such that $P = P_1 P_2$ is a Sylow p-subgroup of G and $|P : P_1| = p$. If $p \notin \pi(\mathscr{F})$, by case 1, V_1 is an $\mathscr{F}$-injector of G. If $p \in \pi(\mathscr{F})$, then there exists a Hall subgroup H such that $H/H_{\mathscr{F}}$ is soluble. So by Theorem 1.1, G has an $\mathscr{F}$-injector.

2. Let M_1, M_2 be maximal subgroups of G such that M_1, M_2 are conjugate in G. Then $M_1 M_2 = M_2^g M_2 = M_2 \leq G$, for some $g \in G$. Then, if $M_2 < G$, by case 2, G also has an $\mathscr{F}$-injector. If $M_2 = G$, by Lemma 2.6, and Theorem 1.1, G has an $\mathscr{F}$-injector.

This completes the proof. □

Remark 3.2. *If the condition of Corollary 3.1 is that, every chief factor is an E_π^n, we also can get the same result .*

Corollary 3.2. *Let $\mathscr{F}$ be a non-empty Fitting class. If every chief factor of G is an E_π^s-group. then A is an $\mathscr{F}$-injector of G if and only if A is a maximal $\mathscr{F}$-subgroup of G containing $G_{\mathscr{F}}$.*

Acknowledgment.

The authors would like to thank the referee for the valuable suggestions and comments. This object is partial supported by Scientific Research Fund of School of Science of SUSE.

References

[1] B. Fischer, W. Gaschütz, and B. Hartley, Injectoren endlicher auflösbarer Gruppen, *Math.Z.* 102, 1967, pp. 337–339.

[2] M. J. Tomkinson, $\mathscr{F}$-injectors of locally soluble groups, *Glas. J. Math*, 10(2), 1968, pp. 130–136.

[3] P. Flavell, Nilpotent injectors in finite groups all of whose local subgroups are $\mathscr{N}$-constrained, *J. Algebra*, 149, 1992, pp. 405–418.

[4] W. Guo, and N. T. Vorob'ev, On injectors of finite soluble groups, *Comm. Algebra*, 36, 2008, pp. 3200–3208.

[5] J. C. Beidlema, and M. J. Karbe, Injectors of locally soluble FC-groups, *Mh. Math*, 103, 1987, 3–13.

[6] A. Neumann, Nilpotent injectors in finite groups, *Arch. Math.*, 71, 1998, pp. 337–340.

[7] M. J. Asiáin, Fitting classes and injectors, *Rend. Ciro. Mate. Palerma Series II*, 48, 1999, pp. 93–100.

[8] J. Cossey, A note on injectors in finite soluble groups, *Bull. Austral. Math. Soc.*, 17, 1977, 419–421.

[9] P. Hall, Theorem like Sylow's, *Proc. London Math. Soc.*, 6(3), 1956, pp. 286–304.

[10] W. Guo, and B. Li, On the injectors of finite groups, *J. Group Theory*, 10, pp. 2007, pp. 849–858.

[11] B. Hartley, On Fischer's dualization of formation theory, *Proc. London Math. Soc.*, 3(2), 1969, pp. 193–207.

[12] A. Ballester-Bolinches, and L. M. Ezquerro, Classes of finite groups, Springer, Netherlands, 2006.

[13] K. Doerk, and T. Hawkes, Finite soluble groups, Walter de Gruyter, New York, 1992.

[14] M. R. Dixon, Sylow theory, formations and Fitting classes in locally finite groups, World Scientific, London, 1994.

[15] W. Anderson, Injectors in finite solvable groups, *J. Algebra*, 36, 1975, pp. 333–338.

[16] A. A. Klimowicz, Injectors in certain classes of locally π-soluble groups, *Arkiv För Mathematik*, 14(1), 1976, pp. 119–124.

[17] M. J. Iranzo, and F. P. Monasor, Fitting classes $\mathscr{F}$ such that all finite groups have $\mathscr{F}$-injectors, *Isr. J. Math.*, 56(1), 1986, pp. 97–101.