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Geometry Associated with the SL$(3, \mathbb{R})$ Action on Homogeneous Space Using the Erlangen Program

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Abstract. We investigate the action of the Lie group SL$(3, \mathbb{R})$ on the two-dimensional homogeneous space. All the one-parameter subgroups (up to conjugacy) of SL$(3, \mathbb{R})$ are considered. We discuss the orbits and curvatures of these one-parameter subgroups. We also classify these subgroups in terms of fixed points.

Key Words: Lie Group SL$(3, \mathbb{R})$, Homogeneous Space, Iwasawa Decomposition, One-Parameter Subgroups, Group Action, Derived Representation, Orbit, Curvature, Fixed Point
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Introduction

In this paper, we have focused on geometry and studied objects in the plane, exploring properties that are invariant under the action of the Lie group SL$(3, \mathbb{R})$. The approach we have followed is in the same line as the 'Erlangen Program' proposed by Felix Klein [12]. In the 'Erlangen Program', geometry is defined as a space of objects along with a group of transformations and is based on the invariants under the transformations (see, for example, [6, 17]). Following this approach, Kisil [15] investigated the kind of geometry related to the Lie group SL$(2, \mathbb{R})$ action. In this direction, function theory associated with the representation of SL$(2, \mathbb{R})$ was thoroughly studied by Kisil and Biswas in [1, 2, 16]. Further extensions and investigations of Möbius action of SL$(2, \mathbb{R})$ were done by many authors, for example, in [3, 4, 5, 7, 14]. Therefore, it becomes imperative to examine the underlying geometry related to the higher dimensional groups. However, there is extensive literature devoted to the study of one-parameter subgroups of Lie groups, particularly for the Lie group SL$(2, \mathbb{R})$, the results of which may extend to the group SL$(3, \mathbb{R})$ as well, see [10, 18]. In view of this, we are writing the present paper to carry the exposition further for a deeper understanding.

Here, we consider the transformation group as the Lie group $\mathrm{SL}(3,\mathbb{R})$ and the space as a two-dimensional homogeneous space. The aim of our work is to generalise the theory from the group $\mathrm{SL}(2,\mathbb{R})$ to the group $\mathrm{SL}(3,\mathbb{R})$ and develop the existing theories if possible. To study geometry, we obtain all the one-parameter subgroups (up to conjugacy) of $\mathrm{SL}(3,\mathbb{R})$. Though many of the obtained statements seem to be particular cases of standard general results about Lie groups, explicit description of the one-parameter subgroups of $\mathrm{SL}(3,\mathbb{R})$ up to conjugacy and presentation of some of their geometrical properties have their own charm.

1 Preliminaries

In this section, we review some standard definitions and theorems that are pertinent to our work.

As in [11], we define the matrix Lie group as any subgroup G of $\mathrm{GL}(n,\mathbb{C})$ satisfying the property: If A_m is any sequence of matrices in G and A_m converges to some matrix A, then either $A \in G$ or A is not invertible. Also, the Lie algebra of a matrix Lie group G, denoted by $\mathfrak{g}$, is the collection of all matrices M such that e^{tM} is in G for all real numbers t.

The one-parameter subgroup of the Lie group G is defined as the Lie group homomorphism $f : \mathbb{R} \to G$, i.e., f is smooth and such that $f(0) = 1_G$ and $f(s+t) = f(t)f(s)$ for all $s,\ t \in \mathbb{R}$. In particular, if G is a matrix Lie group, then every one-parameter subgroup $A(t)$ of G is formed by $A(t) = e^{tX}$ where $X \in \mathfrak{g}$ is the Lie algebra of G.

Definition 1 *Transformation group G is a non-void set of mappings of a set X into itself such that*
(i) the identity map is included in G,
(ii) if $g_1 \in G$ and $g_2 \in G$, then $g_1 g_2 \in G$,
(iii) if $g \in G$, then g^{-1} exists and belongs to G.

Definition 2 *A group action $\varphi : G \times X \mapsto X$ is said to be transitive if for every $x,\ y \in X$, there exists $g \in G$ such that $g \cdot x = y$.*

Definition 3 *Homogeneous space is the pair (G, X) such that the action of the group G on X is transitive and X is a topological space.*

1.1 Derived representation

Let G be a Lie group and H be a Banach space. Let $f : G \to H$ be a smooth map. For each $X \in \mathfrak{g}$ and $y \in G$, the Lie derivative, or the derived representation is defined by $\mathcal{L}_X f(y) = \dfrac{d}{dt} f(y\ exp(tX)) \mid_{t=0}$ (see [19]).

1.2 Group action on coset spaces

Let G be a Lie group and H be a closed subgroup of G. Then it follows by Cartan's theorem that there exists a unique differential structure on H making it a Lie group such that the natural inclusion $\iota : H \to G$ is a Lie group homomorphism (see [20]).

Let $G/H = \{gH : g \in G\}$ denote the space of left cosets of H.

Definition 4 *The projection map $p : G \to G/H$ is given by sending $g \in G$ to its equivalence class $[g]$ and is defined as $p(g) = gH = [g]$.*

Theorem 1 *[13] Let G be a Lie group and H a closed subgroup of G. Let G/H have the quotient topology. Then G/H has a unique smooth manifold structure such that the projection map $p : G \to G/H$ is a smooth submersion and G acts smoothly on G/H.*

Definition 5 *A section s of a projection map p is defined as the right inverse of p such that $s : G/H \to G$ and $p(s(x)) = x$ for all $x \in G/H$.*

Remark 1 We consider the action $G \times G/H \to G/H$ given by $(a, gH) \mapsto agH$. The action can be viewed as a composition of smooth maps as follows:

$$\begin{gathered}\phi : G \times G/H \to G/H \\ \phi(g, x) := g \cdot x = p(g * s(x)),\end{gathered}$$

where $*$ denotes the group operation on G (see Lemma 1 for more details).

1.3 Iwasawa decomposition of $\mathbf{SL}(3,\mathbb{R})$

The Iwasawa decomposition of the group $\mathrm{SL}(3,\mathbb{R})$ is defined by $\mathrm{SL}(3,\mathbb{R}) = ANK$, where A is the subgroup of diagonal matrices, N is a unipotent subgroup and K is the maximal compact subgroup. Explicitly,

$$A = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix} : a,\ b \in \mathbb{R},\ a,\ b > 0 \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x,\ y,\ z \in \mathbb{R} \right\} \qquad \text{and} \qquad K = \mathrm{SO}(3,\mathbb{R}),$$

see [19] for more details.

1.4 Real projective plane $\mathbb{RP}^2$

The real projective plane $\mathbb{RP}^2$ consists of points which are equivalence classes of the set $\mathbb{R}^3 \smallsetminus \{(0,0,0)\}$ modulo the equivalence relation $x \sim \lambda x$, for all λ in $\mathbb{R} \smallsetminus \{0\}$. A point in $\mathbb{RP}^2$ has homogeneous coordinates (X, Y, Z), where the coordinates (X, Y, Z) and $(\lambda X, \lambda Y, \lambda Z)$ are considered to represent the same point for all $\lambda \neq 0$ in $\mathbb{R}$. For more details, we refer to [9].

1.5 The Jordan normal form

Let V be a finite-dimensional vector space of dimension n over the algebraically closed field $\mathbb{F}$, and let T be a linear operator on V. Then there exists a basis $\mathcal{B}$ of V for which

$$[\mathrm{T}]_{\mathcal{B}} = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & J_k \end{pmatrix}, \tag{1}$$

where each 0 is a zero matrix and each J_i $(i = 1, 2.., k \leq n)$ is a square matrix of the form (λ_j) or

$$\begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_j \end{pmatrix}$$

for some eigenvalue λ_j of T. Such a matrix J_i is called a Jordan block corresponding to λ_j, and the matrix $[\mathrm{T}]_{\mathcal{B}}$ is called a Jordan normal form of T. In particular, when $V = \mathbb{F}^n$, every $A \in \mathbb{F}^{n\times n}$ is similar to a matrix having the form (1) (see [8] for details).

Theorem 2 *[8] Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. If v_i is an eigenvector of T corresponding to λ_i $(1 \leq i \leq k)$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.*

2 Two-dimensional homogeneous space

Consider the matrix Lie group SL$(3, \mathbb{R})$=$\{A \in M(3, \mathbb{R}) : \det(A) = 1\}$ and the action on the space of left cosets $X = G/H$, where $G = \mathrm{SL}(3, \mathbb{R})$ and

$$H = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \middle| a_{22}a_{33} - a_{23}a_{32} = \frac{1}{a_{11}}, a_{11} \neq 0 \right\}.$$

Here H is a closed subgroup of $\mathrm{SL}(3,\mathbb{R})$, and hence, a matrix Lie group (see [20]). By the dimension theorem of the quotient space,

$$\dim X = \dim(G) - \dim(H) = 2.$$

Since X is a two-dimensional space, each element of X is parametrized by the pair (x, y). This parametrization allows one to express the set theoretic action of $\mathrm{SL}(3,\mathbb{R})$ on $\mathrm{SL}(3,\mathbb{R})/H$ as a composition of smooth maps as follows:

$$g : x \mapsto g \cdot x = p(g * s(x)).$$

We can define another map $r : G \to H$ such that $r(g) = h$, where $h = s(p(g))^{-1}g$. Hence, g can be uniquely written as $g = s(p(g))r(g)$.

Consequently, we have a decomposition $g = s(p(g))r(g)$ of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11}/a_{31} & 0 & 1 \\ a_{21}/a_{31} & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ 0 & \dfrac{a_{21}}{a_{31}}a_{32} - a_{22} & \dfrac{a_{21}}{a_{31}}a_{33} - a_{23} \\ 0 & a_{12} - \dfrac{a_{11}}{a_{31}}a_{32} & a_{13} - \dfrac{a_{11}}{a_{31}}a_{33} \end{pmatrix},$$

where $a_{31} \neq 0$.

In this set up, we define $p(g) = \left(\dfrac{a_{11}}{a_{31}}, \dfrac{a_{21}}{a_{31}}\right)$, where $\begin{pmatrix} a_{11}/a_{31} & 0 & 1 \\ a_{21}/a_{31} & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is the matrix representation of the equivalence class of $[g]$.

We define section $s : X \to \mathrm{SL}(3,\mathbb{R})$ as

$$s(x, y) = \begin{pmatrix} x & 0 & 1 \\ y & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

such that

$$p(s(x, y)) = p\begin{pmatrix} x & 0 & 1 \\ y & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (x, y).$$

Then the $\mathrm{SL}(3,\mathbb{R})$ action takes the form

$$(x, y) \mapsto \left(\frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}}\right), \tag{2}$$

provided $a_{31}x + a_{32}y + a_{33} \neq 0$.

Remark 2 If we allow $a_{31}x + a_{32}y + a_{33} = 0$, then action (2)) gives us the following projective transformation of the space $\mathbb{RP}^2$.

Let any point $[p] \in \mathbb{RP}^2$ be represented by three-dimensional column vector $(X, Y, Z)^t$ in homogeneous coordinates, and let the action of $g = (a_{ij})$ be defined by matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} a_{11}X + a_{12}Y + a_{13}Z \\ a_{21}X + a_{22}Y + a_{23}Z \\ a_{31}X + a_{32}Y + a_{33}Z \end{pmatrix}.$$

This $\mathrm{SL}(3, \mathbb{R})$ action on $\mathbb{RP}^2$ is denoted as $g : [p] \mapsto [g \cdot p]$.

Let ϕ be the action. We consider the projective transformation $\phi_g : \mathbb{RP}^2 \to \mathbb{RP}^2$ such that $\phi_g([p]) = [g \cdot p]$ for all $g \in \mathrm{SL}(3, \mathbb{R})$, see [9].

We state some elementary but essential observations on the action of G on G/H defined in terms of maps s and p.

Lemma 1 *The action $\phi : G \times G/H \to G/H$ given by $\phi(g, x) = p(g * s(x))$ is a transitive group action.*

Remark 3 Let s_1 and s_2 be two sections such that $p(s_1(x)) = p(s_2(x)) = x$ for any $x \in X$. Then $p(g * s_1(x)) = p(g * s_2(x))$, for all $g \in G$. Also, for any $g \in G$, we have $s(p(g)) = gh$ for some $h \in H$ depending on g.

3 One-parameter subgroups of SL$(3, \mathbb{R})$

In this section, we describe the one-parameter subgroups of $\mathrm{SL}(3, \mathbb{R})$.

Theorem 3 *Any continuous one-parameter subgroup of $SL(3, \mathbb{R})$ is conjugate to one of the following subgroups:*

$$A_1 = \left\{ \begin{pmatrix} e^{at} & 0 & 0 \\ 0 & e^{bt} & 0 \\ 0 & 0 & e^{-(a+b)t} \end{pmatrix} : t \in \mathbb{R} \right\},$$

$$B = \left\{ \begin{pmatrix} e^{at} & te^{at} & 0 \\ 0 & e^{at} & 0 \\ 0 & 0 & e^{-2at} \end{pmatrix} : t \in \mathbb{R} \right\},$$

$$C = \left\{ \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\},$$

$$D = \left\{ \begin{pmatrix} e^{at}\cos bt & -e^{at}\sin bt & 0 \\ e^{at}\sin bt & e^{at}\cos bt & 0 \\ 0 & 0 & e^{-2at} \end{pmatrix} : t \in \mathbb{R} \right\},$$

where $a,\ b \in \mathbb{R}$.

Proof. Any one-parameter subgroup is obtained through the exponential map

$$e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n$$

of an element X of the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$ of $\mathrm{SL}(3,\mathbb{R})$. The Lie algebra is given by

$$\mathfrak{sl}(3,\mathbb{R}) = \{X \in M(3,\mathbb{R}) : \mathrm{tr}(X) = 0\}$$

We prove this result by characterising the elements of the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$.

Let

$$X = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathfrak{sl}(3,\mathbb{R}),$$

then the characteristic polynomial is

$$P_X(\lambda) = \lambda^3 - \mathrm{tr}(X)\lambda^2 + (a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{12}a_{21} - a_{23}a_{32} - a_{13}a_{31})\lambda - \det(X).$$

The nature of the λ_i's will determine the nature of the elements up to similarity as similar matrices have the same characteristic polynomial. We have two main cases.

Case 1. All three roots are real.

i) We assume that all roots are distinct, i.e., $\lambda_1 \neq \lambda_2 \neq \lambda_3$. Therefore, eigenvectors are linearly independent (cf. Theorem 2) and up to similarity matrices are diagonalizable. Then the Jordan normal matrix takes the form

$$X = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -(\lambda_1 + \lambda_2) \end{pmatrix} \text{ as } \mathrm{tr}(X) = 0. \quad (3)$$

ii) Next we assume that $\lambda_1 = \lambda_2 \neq \lambda_3$ and $\lambda_1 = \lambda_2 = \lambda$. Then the algebraic multiplicity of λ is 2.

If the geometric multiplicity of λ is 2, then the maximum number of linearly independent eigenvectors corresponding to λ is 2. Consequently, all the eigenvectors of X are linearly independent (cf. Theorem 2). Thus, X is diagonalizable, and the Jordan matrix takes the form

$$X = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} \text{ as } \mathrm{tr}(X) = 0. \quad (4)$$

If the geometric multiplicity of λ is 1, then the Jordan block corresponding to λ is $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, and Jordan matrix takes the form

$$X = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} \text{ as } \mathrm{tr}(X) = 0. \quad (5)$$

iii) Finally, we assume that all roots are equal, i.e., $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. Then the algebraic multiplicity of λ is 3.

If the geometric multiplicity of λ is 3, then X is diagonalizable, and the Jordan matrix takes the form

$$X = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ as } \operatorname{tr}(X) = 0. \tag{6}$$

If the geometric multiplicity of λ is 2, then the Jordan blocks corresponding to λ are $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ and (λ). Thus, the Jordan matrix takes the form

$$X = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ as } \operatorname{tr}(X) = 0. \tag{7}$$

If the geometric multiplicity of λ is 1, then the Jordan block corresponding to λ is 3×3 matrix, and the Jordan form is

$$X = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ as } \operatorname{tr}(X) = 0.$$

Case 2. Two roots are complex conjugate and one real.

Let $\lambda_1 = a + ib$, $\lambda_2 = a - ib$ and λ_3 is real, $a,\ b \in \mathbb{R}$, $b \neq 0$. Then the Jordan form is

$$X = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & -2a \end{pmatrix} \text{ as } \operatorname{tr}(X) = 0.$$

Thus, we can combine equations (3), (4) and (6) to have the following form

$$X_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -(\lambda_1 + \lambda_2) \end{pmatrix}. \tag{8}$$

Further, equations (5) and (7) can be combined to

$$X_2 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}, \tag{9}$$

and we have the other two forms

$$X_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

$$X_4 = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & -2a \end{pmatrix}. \tag{11}$$

Using the fact that every one-parameter subgroup of matrix Lie group $\mathrm{SL}(3,\mathbb{R})$ can be expressed as e^{tX} for some $X \in \mathfrak{sl}(3,\mathbb{R})$ and $e^{t(MXM^{-1})} = Me^{tX}M^{-1}$ (see [11]), we have four types of one-parameter subgroups up to conjugacy (cf. [10]), and they are

$$A_1 = \exp(tX_1) = \sum_{n=0}^{\infty} \frac{t^n X_1^n}{n!} = \begin{pmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{-t(\lambda_1+\lambda_2)} \end{pmatrix},$$

$$B = \exp(tX_2) = \sum_{n=0}^{\infty} \frac{t^n X_2^n}{n!} = \begin{pmatrix} e^{t\lambda} & te^{t\lambda} & 0 \\ 0 & e^{t\lambda} & 0 \\ 0 & 0 & e^{-t\lambda} \end{pmatrix},$$

$$C = \exp(tX_3) = \sum_{n=0}^{\infty} \frac{t^n X_3^n}{n!} = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix},$$

$$D = \exp(tX_4) = \sum_{n=0}^{\infty} \frac{t^n X_4^n}{n!} = \begin{pmatrix} e^{ta}\cos bt & e^{ta}\sin bt & 0 \\ -e^{ta}\sin bt & e^{ta}\cos bt & 0 \\ 0 & 0 & e^{-2ta} \end{pmatrix}.$$

Hence, the result follows. □

Remark 4 An alternative proof of the Theorem 3 can be deduced from the characterisation of the elements of the Lie group $\mathrm{SL}(3,\mathbb{R})$. Let a, b, $(ab)^{-1}$ be three eigenvalues of any matrix K in $\mathrm{SL}(3,\mathbb{R})$. Based on their Jordan normal form, these matrices can be written in the general form as follows:

$$K_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix}, \qquad K_2 = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1/a^2 \end{pmatrix},$$

$$K_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad K_4 = \begin{pmatrix} \lambda_1 & \lambda_2 & 0 \\ -\lambda_2 & \lambda_1 & 0 \\ 0 & 0 & (\lambda_1^2+\lambda_2^2)^{-1} \end{pmatrix},$$

where $a = \lambda_1 + i\lambda_2$ and $b = \lambda_1 - i\lambda_2$ in the case of complex conjugate eigenvalues. Thus, up to conjugacy, these are the one-parameter subgroups of $\mathrm{SL}(3,\mathbb{R})$. In particular, we get the complete list of different forms of elements (up to conjugacy) of $\mathrm{SL}(3,\mathbb{R})$. Here, we can see that although these subgroups do not exactly match the subgroups obtained in Theorem 3, their forms are the same.

Hence, using two different approaches, i.e., using Lie algebra as well as Lie group approach, we can show that up to conjugacy, there are only four types of one-parameter subgroups of $\mathrm{SL}(3,\mathbb{R})$.

4 Action of the one-parameter subgroups of $\mathbf{SL}(3,\mathbb{R})$

We now consider the action of the four types of one-parameter subgroups of $\mathrm{SL}(3,\mathbb{R})$.

Theorem 4 *1. The action of the subgroup A_1 defines orbit passing through (α,β) as power curve of the form $y = cx^n$, where $n = (a+2b)/(2a+b)$ and $c = \beta/\alpha^n$. The derived representation is $((2a+b)\alpha, (a+2b)\beta)$. The curvature of A_1-orbit at $t = 0$ is given by*

$$\kappa_{A_1}\big|_{t=0} = \frac{(a+2b)(2a+b)(b-a)\alpha\beta}{((2a+b)^2\alpha^2 + (a+2b)^2\beta^2)^{3/2}}.$$

2. Under the action of the subgroup B, orbit passing through (α,β) satisfies the equation $x = (R+Y)y$, where $R = \alpha/\beta$ and $Y = (3a)^{-1}\log(y/\beta)$. The derived representation is $((3a\alpha+\beta), 3a\beta)$. The curvature of B-orbit at $t = 0$ is equal to

$$\kappa_B\big|_{t=0} = \frac{9a^2\beta^2}{\left((3a\alpha+\beta)^2 + 9a^2\beta^2\right)^{3/2}}.$$

3. The action of the subgroup C defines orbit passing through (α,β) as parabola of the form $y^2 = 2x + c$, where $c = \beta^2 - 2\alpha$. The derived representation is $(\beta, 1)$. The curvature of C-orbit is

$$\kappa_C\big|_{t=0} = \frac{1}{\left(1+\beta^2\right)^{3/2}}.$$

4. Under the action of D, orbit passing through the point (α,β) satisfies the equation $\dfrac{x}{y} = \dfrac{\alpha\cos\theta - \beta\sin\theta}{\alpha\sin\theta + \beta\cos\theta}$, where $\theta = \dfrac{b}{6a}\log\left(\dfrac{x^2+y^2}{\alpha^2+\beta^2}\right)$. The derived representation is $((3a\alpha - b\beta), (3a\beta + b\alpha))$. For D-orbit, the curvature is

$$\kappa_D\big|_{t=0} = \frac{b}{\sqrt{(9a^2+b^2)(\alpha^2+\beta^2)}}.$$

Proof. Here, we demonstrate the proof for the one-parameter subgroup A_1 only, as the proofs for the remaining one-parameter subgroups are similar.

The A_1-orbit passing through (α,β) is defined as $O = \{g\cdot(\alpha,\beta) : g \in A_1\}$. Let the position vector at time t be (x, y). Then, under the action of A_1 on (α,β), we get

$$x = e^{t(2a+b)}\alpha \qquad \text{and} \qquad y = e^{t(a+2b)}\beta.$$

Therefore, eliminating t, we have

$$y = \frac{\beta}{\alpha^n}x^n = cx^n, \qquad \text{where} \quad n = \frac{a+2b}{2a+b} \text{ and } c = \frac{\beta}{\alpha^n}.$$

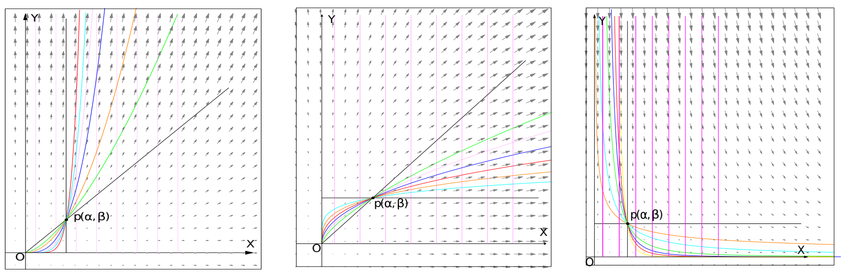
The derived representation (see Subsection 1.1) is obtained as

$$\begin{aligned} dA_1(\alpha,\beta) &= \frac{\partial}{\partial t}\bigg|_{t=0}\left\{\begin{pmatrix} e^{at} & 0 & 0 \\ 0 & e^{bt} & 0 \\ 0 & 0 & e^{-(a+b)t}\end{pmatrix}\cdot(\alpha,\beta)\right\} \\ &= \frac{\partial}{\partial t}\bigg|_{t=0}[e^{(2a+b)t}\alpha, e^{(a+2b)t}\beta] \\ &= ((2a+b)\alpha,(a+2b)\beta). \end{aligned}$$

Using the formula for the curvature of the parametric function $f(t) = (e^{t(2a+b)}\alpha, e^{t(a+2b)}\beta)$, we obtain

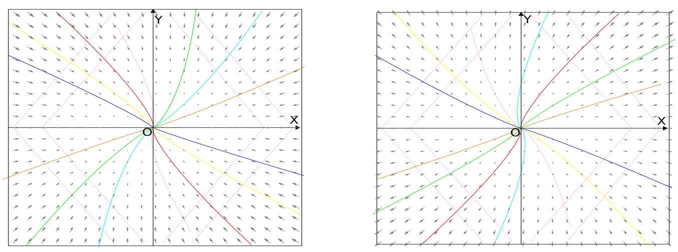
$$\kappa_{A_1}\big|_{t=0} = \frac{(2a+b)(a+2b)(b-a)\alpha\beta}{((2a+b)^2\alpha^2+(a+2b)^2\beta^2)^{3/2}}.$$

□



(a) A_1-orbit when $n > 1$ (b) A_1-orbit when $0 < n < 1$ (c) A_1-orbit when $n < 0$

Figure 1: A_1-Orbit passing through the point (α,β)



(a) B-orbit when $a > 0$ (b) B-orbit when $a < 0$

Figure 2: Orbits for subgroup B

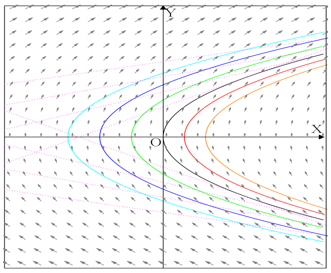


Figure 3: Orbits for subgroup C

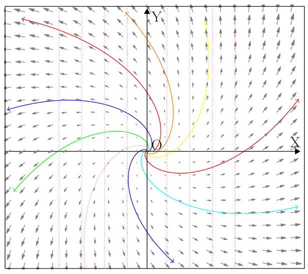


Figure 4: Orbits for subgroup D

Remark 5 1. If $a = b$, then the A_1-orbit passing through (α, β) is a straight line of the form

$$y = \frac{\beta}{\alpha} x.$$

2. If $\beta = 0$, then the B-orbits are straight lines of the form $y = 0$.

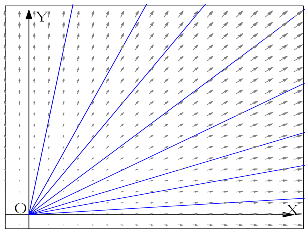


Figure 5: A_1-orbit when $a = b$

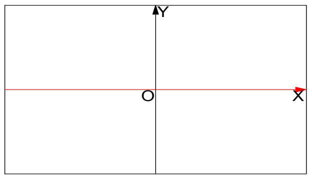


Figure 6: B-Orbit passing through $(\alpha, 0)$

3. If $a = 0$, then the D-orbit passing through the point (α, β) is a circle centered at the origin with radius $\sqrt{\alpha^2 + \beta^2}$. However, the D-orbit passing through the point $(s, 0)$ is a Logarithmic spiral with curvature

$$\kappa_D\big|_{t=0} = \frac{b}{s\sqrt{(9a^2 + b^2)}}.$$

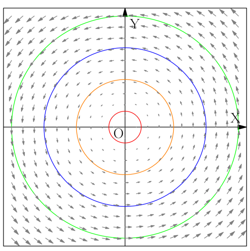


Figure 7: D-orbits when $a = 0$

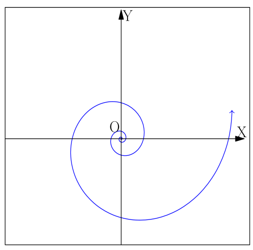


Figure 8: D-orbit passing through $(s, 0)$

5 One-parameter subgroups and fixed point

The one-parameter subgroups A_1, B, C, D of $\mathrm{SL}(3,\mathbb{R})$ possess fixed points, and we are going to characterise these subgroups up to conjugacy in terms of fixed points.

A point $[x]$ is called a fixed point under the $\mathrm{SL}(3,\mathbb{R})$ action if $A[x]=[x]$ in $\mathbb{RP}^2$, $A\in\mathrm{SL}(3,\mathbb{R})$, see [9]. As $x\sim\lambda x$ in projective space $\mathbb{RP}^2$, the fixed point $[x]\in\mathbb{RP}^2$ can be viewed as an eigenvector in $\mathbb{R}^3$ due to the fact that if $[x]\in\mathbb{RP}^2$ is a fixed point of $A\in\mathrm{SL}(3,\mathbb{R})$, then $A[x]=[x]=[\lambda x]=\lambda[x]$ for some non zero $\lambda\in\mathbb{R}$.

Lemma 2 *The number of fixed points for the one-parameter subgroups A_1, B, C, D are three, two, one, and one, respectively.*

Proof. Based upon the eigenvalues and the eigenvectors of these four one-parameter subgroups of $\mathrm{SL}(3,\mathbb{R})$, there are four cases to follow.

The subgroup A_1 has three distinct eigenvalues and the corresponding three eigenvectors are linearly independent (see equation (8)). Thus, the number of fixed points for A_1 is three.

The subgroup B has two real eigenvalues λ_1 with algebraic multiplicity 2 and geometric multiplicity 1 and another real eigenvalue λ_2. Hence, B obtains two linearly independent real eigenvectors, one corresponding to λ_1 and the other corresponding to λ_2 (see equation (9)). Therefore, there are two fixed points for B.

The subgroup C has real eigenvalue λ with algebraic multiplicity 3 and geometric multiplicity 1 (see equation (10)). Hence, C obtains only one real eigenvector, and thus, the number of fixed points for C is one.

The subgroup D has one real eigenvalue and two complex conjugate eigenvalues (see equation (11)), which implies that D has only one real eigenvector and two complex conjugate eigenvectors. As the space is considered to be $\mathbb{RP}^2$, the number of fixed points for D is one. □

Conclusion

We have discussed the action of the transformation group $\mathrm{SL}(3,\mathbb{R})$ on the two-dimensional homogeneous space. Here, we have mainly focused on the one-parameter subgroups (up to conjugacy), which were originally introduced to define infinitesimal transformations that generate the Lie algebra. Under the one-parameter subgroup action, we get different orbits such as power curves, parabolas, logarithmic spirals, etc. These are the natural invariants for the respective subgroups since orbits remain invariant under the group action. We have also obtained the derived representations of the vector fields. Curvatures and fixed points of these one-parameter subgroups are also studied.

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