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A generalization of connectedness via ideals

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Abstract. In this paper, we define and study the $\diamond$-connected spaces as a generalization of the connectedness, and thus of the Ekici-Noiri and Modak-Noiri notions, through ideals.

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Introduction

In 2012, E. Ekici and T. Noiri [1] introduced the notion of connectedness for ideal topological spaces defining $\star$-connectedness, but their concept is not a generalization of connectedness, as it turns out to be a stronger version than that. In 2015, S. Modak and T. Noiri [8] presented new forms of connectedness in ideal topological spaces through the notions of $\star_\star$-connectedness and $\star$-cl-connectedness, but here, too, a generalization is not obtained. We want to point out here that some authors, such as Modak and Noiri [7], have already obtained generalizations of connectedness but without using ideals.

In this work, we introduce and study the $\diamond$-connected spaces. This concept is a generalization of the connectedness, and therefore, of the concepts of Ekici-Noiri and Modak-Noiri. We present several examples, and we also characterize the $\diamond$-connected subspaces of some ideal topological spaces having $\mathbb{R}$ as their underlying set.

1 Preliminaries

The ideal topological spaces have been introduced in Vaidyanathaswamy [9] and Kuratowski [4] books. An *ideal* $\mathcal{I}$ on a set X is a subset of $\mathcal{P}(X)$, the power set of X, such that: if $A \subseteq B \subseteq X$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$; and if $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

Some useful ideals on X are: $\mathcal{P}(A)$, where $A \subseteq X$; the ideal $\mathcal{I}_f\,(X)$ of all finite subsets of X; the ideal $\mathcal{I}_c\,(X)$ of all countable subsets of X; the ideal

$\mathcal{I}_n(X,\tau)$ of all nowhere dense subsets in a topological space (X,τ). When there is no chance for confusion, we write $\mathcal{I}_n$ instead of $\mathcal{I}_n(X,\tau)$.

If (X,τ) is a topological space and $\mathcal{I}$ is an ideal on X, then $(X,\tau,\mathcal{I})$ is called an *ideal topological space*. If $\tau \cap \mathcal{I} = \{\varnothing\}$, then $\mathcal{I}$ is said to be *codense*.

Let $f : X \to Y$ be a function. If $\mathcal{I}$ is an ideal on X, the set $f(\mathcal{I}) = \{f(I) : I \in \mathcal{I}\}$ is an ideal on Y [5]; If f is injective and $\mathcal{J}$ is an ideal on Y, then the set $f^{-1}(\mathcal{J}) = \{f^{-1}(J) : J \in \mathcal{J}\}$ is an ideal on X [5]; If $\mathcal{J}$ is an ideal on Y, the set $\mathcal{I}_{f,\mathcal{J}} = \{A \subseteq X : \text{ there is a } J \in \mathcal{J} \text{ with } A \subseteq f^{-1}(J)\}$ is an ideal on X [6].

Given an ideal space $(X,\tau,\mathcal{I})$ and a set $A \subseteq X$, we denote by $A^*(\mathcal{I})$ the set $\{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau \text{ with } x \in U\}$, written simply as A^* when there is no chance for confusion. It is clear that $A^* \subseteq \overline{A}$, where $\overline{A}$ is the closure of A in (X,τ). Sometimes we will use the notation $adh_\tau(A)$ instead of $\overline{A}$.

A Kuratowski closure operator for a topology $\tau^*(\mathcal{I})$, finer than τ, is defined by $Cl^*(A) = A \cup A^*$ for all $A \subseteq X$ [9]. When there is no chance for confusion, $\tau^*(\mathcal{I})$ is denoted by τ^*. The topology τ^* has as a base $\beta(\tau,\mathcal{I}) = \{V \backslash I : V \in \tau \text{ and } I \in \mathcal{I}\}$ [2].

Two nonempty sets A and B are $\star_\star$-*separated* [8] in the space $(X,\tau,\mathcal{I})$ if $A^\star \cap B = A \cap B^\star = A \cap B = \varnothing$. The nonempty sets A and B are $\star$-*cl-separated* [8] in $(X,\tau,\mathcal{I})$ if $A^\star \cap \overline{B} = \overline{A} \cap B^\star = A \cap B = \varnothing$.

If (X,τ) is a topological space and $A \subseteq X$, then $\overset{0}{A}$ is the interior of A, and A' is the set of accumulation points of A. Furthermore, for $\{A, B\} \subseteq \mathcal{P}(X)$, the sets A and B are *separated* if $\overline{A} \cap B = \varnothing = A \cap \overline{B}$. If B and C are disjoint subsets of A and $A = B \cup C$, we will write $A = B \sqcup C$.

Finally, throughout this work we will use the following topologies in $\mathbb{R}$: $\mathcal{C} = \{\varnothing, \mathbb{R}\} \cup \{(a,\infty) : a \in \mathbb{R}\}$; $\mathcal{L}$ is the (Sorgenfrey) topology of all $V \subseteq \mathbb{R}$ such that for each $a \in V$, there is a number $b > a$ such that $[a, b) \subseteq V$; and γ is the topology in which the neighborhoods of any nonzero point being as in the usual topology $\mathcal{U}$, while neighborhoods of 0 have the form $U \backslash F$, where U is a neighborhood of 0 in $\mathcal{U}$ and $F = \{1/n : n \in \mathbb{Z}^+\}$.

2 The $\diamond$-connected spaces

We begin this section by recalling the concepts defined by Ekici and Noiri [1] and Modak and Noiri [8].

An ideal topological space $(X,\tau,\mathcal{I})$ is said to be $\star$-*connected* [1] if there are no disjoint and nonempty sets $U \in \tau$ and $V \in \tau^*$ such that $X = U \cup V$. A subset A is defined to be $\star$-*connected* if $(A,\tau_A,\mathcal{I}_A)$ is $\star$-connected, where $\tau_A = \{A \cap U : U \in \tau\}$ and $\mathcal{I}_A = \{A \cap I : I \in \mathcal{I}\}$.

The ideal topological space $(X,\tau,\mathcal{I})$ is said to be $\star_\star$-*connected* [8] if there are no nonempty $\star_\star$-separated sets A and B such that $X = A \cup B$.

Further, $(X, \tau, \mathcal{I})$ is said to be $\star$-*cl-connected* [8] if there are no nonempty $\star$-cl-separated sets A and B such that $X = A \cup B$.

It is evident that if $(X, \tau, \mathcal{I})$ is $\star$-connected, then (X, τ) is connected. Moreover, from $\star_\star$-connectedness, it follows the $\star$-cl-connectedness [8], and it is easy to see that $\star$-cl-connectedness implies connectedness.

In this section, we are going to present a definition of connectedness for ideal spaces, which we call $\diamond$-connectedness, that is more general than that of connectedness. We will also present properties and some characterizations of these new spaces, as well as some examples having the set $\mathbb{R}$ as the underlying set in which it is possible to establish necessary and sufficient conditions for a subset to be $\diamond$-connected.

Definition 1 *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\diamond$-connected if for all $\{U, V\} \subseteq \tau$ with $A = (A \cap U) \sqcup (A \cap V)$, we have that $A \cap U \in \mathcal{I}$ or $A \cap V \in \mathcal{I}$. The space $(X, \tau, \mathcal{I})$ is said to be $\diamond$-connected if X is $\diamond$-connected.*

Let us make several remarks. If $I \in \mathcal{I}$ then I is $\diamond$-connected; If A is connected, then A is $\diamond$-connected. In the space $(X, \tau, \{\varnothing\})$, a subset A is $\diamond$-connected if and only if A is connected. If $\mathcal{I}$ is a codense ideal in X then the space $(X, \tau, \mathcal{I})$ is $\diamond$-connected if and only if (X, τ) is connected. The space $(X, \tau, \mathcal{I})$ is $\diamond$-connected if and only if for all disjoint closed sets F and G, from $X = F \cup G$, it follows that $F \in \mathcal{I}$ or $G \in \mathcal{I}$. The space $(X, \tau, \mathcal{I})$ is $\diamond$-connected if and only if for each open and closed subset U, we have $U \in \mathcal{I}$ or $X \backslash U \in \mathcal{I}$. Finally, the set A is $\diamond$-connected if and only if $(A, \tau_A, \mathcal{I}_A)$ is $\diamond$-connected.

Hence we have that if $(X, \tau, \mathcal{I})$ is an ideal space, then

$$(X, \tau, \mathcal{I}) \ \star\text{-connected} \Rightarrow (X,\tau) \text{ connected} \Rightarrow (X, \tau, \mathcal{I}) \ \diamond\text{-connected}.$$

Neither of these implications is reversible. If $X = \{0, 1, 2\}$, $\mathcal{I} = \mathcal{P}(\{1, 2\})$ and $\tau = \{\varnothing, X, \{0\}, \{1, 2\}\}$, then $(X, \tau, \mathcal{I})$ is $\diamond$-connected but (X, τ) is not connected.

Theorem 1 *If A is $\diamond$-connected in the ideal topological space $(X, \tau, \mathcal{I})$ and $I \in \mathcal{I}$, then $A \cup I$ is $\diamond$-connected. In particular, if A is connected in (X, τ) and $I \in \mathcal{I}$, then $A \cup I$ is $\diamond$-connected.*

Proof. We can assume that $I \backslash A \neq \varnothing$. It is clear that $A \cup I = A \cup (I \backslash A)$ and $I \backslash A \in \mathcal{I}$. Suppose that there is a $\{U, V\} \subseteq \tau$ such that $A \cup I = [(A \cup I) \cap U] \sqcup [(A \cup I) \cap V]$. This implies that $A = (A \cap U) \sqcup (A \cap V)$. Since A is $\diamond$-connected, we have that $A \cap U \in \mathcal{I}$ or $A \cap V \in \mathcal{I}$. Suppose, without loss of generality, that $A \cap U \in \mathcal{I}$. Then $(A \cup I) \cap U = (A \cap U) \cup [(I \backslash A) \cap U] \in \mathcal{I}$. □

If A is $\diamond$-connected in $(X, \tau, \mathcal{I})$ and there is no a $\diamond$-connected set $B \subseteq X$ with $A \subseteq B$ and $A \neq B$, then A is said to be *maximal $\diamond$-connected.*

Corollary 1 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then*
1) For each maximal $\diamond$-connected set $A \subseteq X$, it is true that $\bigcup_{I \in \mathcal{I}} I \subseteq A$.
2) If X is not $\diamond$-connected and $A \subseteq X$ is $\diamond$-connected, then $X \backslash A \notin \mathcal{I}$. Hence, if $X \backslash \bigcup_{I \in \mathcal{I}} I \subseteq A$, then $X \backslash A$ is an infinite set.

Below we present a list of examples, in which we characterize the $\diamond$-connected subspaces of various ideal topological spaces.

We start with some trivial examples.

Example 1 *1) If $\mathcal{I}$ is an ideal on X, then in the space $(X, \{\varnothing, X\}, \mathcal{I})$ each $A \subseteq X$ is $\diamond$-connected.*

2) In the space $(X, \mathcal{P}(X), \mathcal{I})$, a subset A is $\diamond$-connected if and only if for each $B \subseteq A$, $B \in \mathcal{I}$ or $A \backslash B \in \mathcal{I}$.

3) If $B \subseteq \mathbb{R}$, then in the space $(\mathbb{R}, \mathcal{L}, \mathcal{P}(B))$, the only $\diamond$-connected subsets are $\varnothing$ and those that have the form $\{a\} \cup D$, where $a \in \mathbb{R}$ and $D \subseteq B$. In fact, according to Theorem 1, any of these sets is $\diamond$-connected. Now, if $A \subseteq X$ and $\{a, b\} \subseteq A \backslash B$ with $a < b$, then $A = [A \cap (-\infty, b)] \sqcup [A \cap [b, \infty)]$ with $\{A \cap (-\infty, b), A \cap [b, \infty)\} \cap \mathcal{P}(B) = \varnothing$, because $a \in A \cap (-\infty, b)$, $b \in A \cap [b, \infty)$ and $\{a, b\} \cap B = \varnothing$. Hence, A is not $\diamond$-connected.

And now we go with some non-trivial examples.

Proposition 1 *In the space $(\mathbb{R}, \mathcal{L}, \mathcal{I}_c(\mathbb{R}))$, a subset A is $\diamond$-connected if and only if A is countable.*

Proof. By Theorem 1, if A is countable then A is $\diamond$-connected.

Now, suppose that A is not countable. Then there exists $r \in \mathbb{R}$ such that $(-\infty, r) \cap A$ and $[r, \infty) \cap A$ are not countable. In fact, given that $A = \bigcup_{n \in \mathbb{Z}} ([n, n+1] \cap A)$, there exists $N \in \mathbb{Z}$ such that the set $A_1 = [N, N+1] \cap A$ is not countable. We define $\alpha = \inf A_1$ and $\beta = \sup A_1$. Consider the sets $B = \{x \in [\alpha, \beta] : [\alpha, x] \cap A_1 \text{ is countable}\}$ and $C = \{x \in [\alpha, \beta] : [x, \beta] \cap A_1 \text{ is countable}\}$. It is clear that $\alpha \in B$, $\beta \in C$ and $B \cap C = \varnothing$. Further, B and C are intervals given that if, for example, $\{u, v\} \subseteq B$ and $u < z < v$, then $z \in [\alpha, \beta]$ and $[\alpha, z] \cap A_1$ is countable, because $[\alpha, v] \cap A_1$ is countable.

Now, if $a = \sup B$, then $a \in B$. This is clear if $B = \{\alpha\}$. If $B \neq \{\alpha\}$ and $\{a_n\}$ is an increasing succesion in $[\alpha, a) \subseteq B$, such that $a_n \to a$ in the space $(\mathbb{R},\mathcal{U})$, then $A_1 \cap [\alpha, a) = \bigcup\limits_{n\geq 1} ([\alpha, a_n] \cap A_1)$, and thus, $A_1 \cap [\alpha, a)$ is countable. This implies that $A_1 \cap [\alpha, a]$ is countable, and hence, $a \in B$. Similarly, it can be verified that if $b = \inf C$, then $b \in C$. On the other hand, since A_1 is not countable, we have that $a < b$. If we put $r = (a+b)/2$, then we have the announced result.

Since $A = [(-\infty, r) \cap A] \sqcup [[r, \infty) \cap A]$, with $\{(-\infty, r), [r, \infty)\} \subseteq \mathcal{L}$, we can conclude that A is not $\diamond$-connected. □

Proposition 2 *In the space $(\mathbb{R}, \mathcal{L}, \mathcal{I}_f(\mathbb{R}))$, a subset A is $\diamond$-connected if and only if A is finite or A is an infinite set such that: a) A is bounded; b) there exists $a \in A$ such that $A' = \{a\}$ in $(\mathbb{R}, \mathcal{L})$; c) for each $r \in \mathbb{R}$, we have $A \cap [r, \infty) \in \mathcal{I}_f(\mathbb{R})$ or $A \cap (-\infty, r) \in \mathcal{I}_f(\mathbb{R})$.*

Proof. Note that, according to Proposition 1, if A is $\diamond$-connected in $(\mathbb{R}, \mathcal{L}, \mathcal{I}_f(\mathbb{R}))$, then A is countable.

Let us prove the sufficient first. Suppose that $A \subseteq \mathbb{R}$ is an infinite set such that the conditions $a)$, $b)$ and $c)$ are satisfied. Put $\alpha = \inf A$ and $\beta = \sup A$. It is clear that $\alpha \leq a < \beta$. Suppose that $\{U, V\} \subseteq \mathcal{L}$ and $A = (A \cap U) \sqcup (A \cap V)$. Without loss of generality we can assume that $a \in U \cap A$. Then there exists $\epsilon \in (0, \beta - a)$ such that $[a, a+\epsilon) \subseteq U$. Further, given that $a \in A'$ and $(\mathbb{R}, \mathcal{L})$ is a T_1 space, then $[a, a+\epsilon) \cap A$ is infinite. The condition $c)$ implies that $[\alpha, a] \cap A$ and $[a+\epsilon, \beta] \cap A$ are finite. Since $V \cap A \subseteq ([\alpha, a] \cap A) \cup ([a+\epsilon, \beta] \cap A)$, the set $V \cap A$ is finite. Thus, A is $\diamond$-connected.

Now, let us prove the necessity. Suppose that $A \subseteq \mathbb{R}$ is $\diamond$-connected and infinite.

Since $A = ((-\infty, 0) \cap A) \sqcup ([0, \infty) \cap A)$, we have $(-\infty, 0) \cap A \in \mathcal{I}_f(\mathbb{R})$ or $[0, \infty) \cap A \in \mathcal{I}_f(\mathbb{R})$, and thus there exists $\inf A$ or there exists $\sup A$.

First, suppose that there exists $a_0 = \inf A$, but A has no upper bounds. Then there is a sequence $\{a_1, a_2, a_3, ...\} \subseteq A$ such that for each $n \geq 0$, $a_{n+1} > \max\{n, a_n\}$. It is clear then that $[a_0, \infty) = \bigcup\limits_{n\geq 0} [a_n, a_{n+1})$ and $A = \left[\bigcup\limits_{n\geq 0} (A \cap [a_{2n}, a_{2n+1}))\right] \sqcup \left[\bigcup\limits_{n\geq 0} (A \cap [a_{2n+1}, a_{2n+2}))\right]$, and hence A is not $\diamond$-connected. Thus, A has upper bounds.

A similar argument allows us to conclude that if there exists $b = \sup A$, then A has lower bounds. We conclude that A is bounded and denote $\alpha = \inf A$ and $\beta = \sup A$.

There is no a strictly increasing sequence in A.

Suppose, towards a contradiction, that $\{u_n\}$ is a strictly increasing sequence in A. We define $\gamma = \sup\{u_n : n \geq 1\}$. Thus, $\alpha \leq u_1 < u_2 < u_3 < \cdots < \gamma \leq \beta$. If we put $U = [\alpha, u_1) \cup [u_2, u_3) \cup [u_4, u_5) \cup \cdots$ and $V = [\gamma, \infty) \cup [u_1, u_2) \cup [u_3, u_4) \cup \cdots$, then $\{U, V\} \subseteq \mathcal{L}$, $A = (A \cap U) \sqcup (A \cap V)$, but $A \cap U \notin \mathcal{I}_f(\mathbb{R})$ and $A \cap V \notin \mathcal{I}_f(\mathbb{R})$. This contradicts that A is $\diamond$-connected.

The set A has at most one accumulation point in $(\mathbb{R}, \mathcal{L})$. If $u_1 < u_2$ are accumulation points of A in $(\mathbb{R}, \mathcal{L})$, then $A = ((-\infty, u_2) \cap A) \sqcup ([u_2, \infty) \cap A)$ with $\{(-\infty, u_2) \cap A, [u_2, \infty) \cap A\} \cap \mathcal{I}_f(\mathbb{R}) = \varnothing$, but this is impossible since A is $\diamond$-connected.

In the space $(\mathbb{R}, \mathcal{U})$, A has an accumulation point. We are going to prove that this point is an accumulation point of A in $(\mathbb{R}, \mathcal{L})$.

Let x_0 be an accumulation point of A in $(\mathbb{R}, \mathcal{U})$. Since there is no a strictly increasing sequence in A, we have that $x_0 < \beta$. We select a $r \in (0, \beta - x_0)$. The set $(x_0 - r, x_0 + r) \cap A$ is infinite, but $(x_0, x_0 + r) \cap A$ cannot be finite since otherwise there would be a strictly increasing sequence in $(x_0 - r, x_0) \cap A$.

Hence, $(x_0, x_0 + r) \cap A$ is infinite. Thus, x_0 is an accumulation point of A in $(\mathbb{R}, \mathcal{L})$.

Let us show that $x_0 \in A$. Suppose the opposite, namely, $x_0 \notin A$. Since x_0 is an accumulation point of A in $(\mathbb{R}, \mathcal{L})$, there exists a strictly decreasing sequence $\{z_n\}$ in A such that $z_n \to x_0$ in $(\mathbb{R}, \mathcal{L})$. If we put $U = [\alpha, x_0) \cup [z_1, \infty) \cup [z_3, z_2) \cup [z_5, z_4) \cup \cdots$ and $V = [z_2, z_1) \cup [z_4, z_3) \cup \cdots$, then $A = (A \cap U) \sqcup (A \cap V)$ with $\{A \cap U, A \cap V\} \cap \mathcal{I}_f(\mathbb{R}) = \varnothing$. This is impossible since A is $\diamond$-connected.

Finally, if $r \in \mathbb{R}$, then $A = ((-\infty, r) \cap A) \sqcup ([r, \infty) \cap A)$, and since A is $\diamond$-connected, we have $(-\infty, r) \cap A \in \mathcal{I}_f(\mathbb{R})$ or $[r, \infty) \cap A \in \mathcal{I}_f(\mathbb{R})$.

From the obtained results it follows that the conditions a), b) and c) are satisfied. □

Proposition 3 *If $\mathcal{U}$ is the usual topology in $\mathbb{R}$, then in the space $(\mathbb{R}, \mathcal{U}, \mathcal{I}_f(\mathbb{R}))$, a subset A is $\diamond$-connected if and only if the following conditions are satisfied:*
a) A^i is bounded, where A^i is the set of isolated points of A;
b) $(A^i)'$ has at most two elements;
c) There is an interval I (eventually $\varnothing$ or a singleton) such that $A = I \cup A^i$ and $(A^i)' \subseteq I$.

Proof. It is observed that for each $B \subseteq \mathbb{R}$, the set B^i is countable, since $(\mathbb{R}, \mathcal{U})$ is a 2-countable space.

Suppose that $A \subseteq \mathbb{R}$ and that the conditions a), b) and c) are satisfied. Let $\{U, V\} \subseteq \mathcal{U}$ be such that $A = (A \cap U) \sqcup (A \cap V)$. Thus, it is clear that $(A \cap V)' \cap U = \varnothing$. On the other hand, given that I is connected in $(\mathbb{R}, \mathcal{U})$ and $I \subseteq A$, we have that $I \subseteq U \cap A$ or $I \subseteq V \cap A$. Without loss of generality, we suppose that $I \subseteq U \cap A$. Then $(V \cap A) \cap I = \varnothing$, and thus $V \cap A \subseteq A^i$. This implies that $A \cap V$ is bounded. If $A \cap V$ were infinite, there would be a $b \in (A \cap V)' \subseteq \mathbb{R} \backslash U \subseteq \mathbb{R} \backslash I \subseteq \mathbb{R} \backslash (A^i)'$, but this is impossible since

$(A \cap V)' \subseteq (A^i)'$. In conclusion, $A \cap V$ is a finite set, that is, $A \cap V \in \mathcal{I}_f(\mathbb{R})$. Therefore A is $\diamond$-connected.

Conversely, let A be $\diamond$-connected.

(i) The set A^i is bounded.

Suppose, for example, that A^i has no upper bounds. Then there is a sequence $\{a_n\}$ in A^i such that for each $n \geq 2$, $a_n > \max\{n, a_{n-1}\}$. For each $n \geq 1$ we can choose an $\alpha_n \in (a_n, a_{n+1}) \setminus A$. If we put $U = (-\infty, \alpha_1) \cup \bigcup_{n \geq 1} (\alpha_{2n}, \alpha_{2n+1})$ and $V = \bigcup_{n \geq 1} (\alpha_{2n-1}, \alpha_{2n})$, then $A = (A \cap U) \sqcup (A \cap V)$ but $A \cap U \notin \mathcal{I}_f(\mathbb{R})$ and $A \cap V \notin \mathcal{I}_f(\mathbb{R})$. This is not possible since A is $\diamond$-connected.

(ii) $(A^i)' \subseteq A$.

Suppose there is a $z \in (A^i)' \setminus A$. Given that $A = [(-\infty, z) \cap A] \sqcup [(z, \infty) \cap A]$ and A is $\diamond$-connected, we have that $(-\infty, z) \cap A \in \mathcal{I}_f(\mathbb{R})$ or $(z, \infty) \cap A \in \mathcal{I}_f(\mathbb{R})$. If, for example, $(z, \infty) \cap A \in \mathcal{I}_f(\mathbb{R})$, we can select a number $r > z$ such that $(r, \infty) \cap A = \varnothing$. Since $z \in (A^i)'$, $(2z - r, r) \cap A^i$ is an infinite set, and hence $(2z - r, z) \cap A^i$ is an infinite set. Furthermore, we can build a sequence $\{v_n\}$ in $(2z - r, z) \cap A^i$ such that $v_n < v_{n+1}$ for each $n \geq 1$ and $v_n \to z$, because $z \in (A^i)'$ and $(z, r) \cap A$ is finite. For each $n \geq 1$, we can choose a $\delta_n \in (v_n, v_{n+1}) \setminus A$ and put $W = (-\infty, \delta_1) \cup \bigcup_{n \geq 1} (\delta_{2n}, \delta_{2n+1})$ and $T = (z, \infty) \cup \bigcup_{n \geq 1} (\delta_{2n-1}, \delta_{2n})$. Then $A = (A \cap W) \sqcup (A \cap T)$ but $A \cap W \notin \mathcal{I}_f(\mathbb{R})$ and $A \cap T \notin \mathcal{I}_f(\mathbb{R})$, which contradicts with the assumption that A is $\diamond$-connected.

(iii) $(A^i)'$ has at most two points.

Suppose that $\{a, b, c\} \subseteq (A^i)'$ with $a < b < c$, and $\varepsilon \in (0, \min\{(b - a)/2, (b - c)/2\})$. The sets $(c - \varepsilon, c + \varepsilon) \cap A^i$ and $(a - \varepsilon, a + \varepsilon) \cap A^i$ are infinite. Let $z \in (b - \varepsilon, b + \varepsilon) \cap A^i$.

Let $\varepsilon_1 \in (0, \min\{\{b + \varepsilon - z, z + \varepsilon - b\}\})$ be such that $(z - \varepsilon_1, z + \varepsilon_1) \cap A = \{z\}$. For $w \in (z - \varepsilon_1, z + \varepsilon_1) \setminus A$, we have $A = [(-\infty, w) \cap A] \sqcup [(w, \infty) \cap A]$ but $(-\infty, w) \cap A \notin \mathcal{I}_f(\mathbb{R})$ and $(w, \infty) \cap A \notin \mathcal{I}_f(\mathbb{R})$, which contradicts the fact that A is $\diamond$-connected.

(iv) $A \setminus A^i$ is an interval.

Suppose that $a < b < c$ with $\{a, c\} \subseteq A \setminus A^i$, and let $\varepsilon \in (0, \min\{(b - a)/2, (c - b)/2\})$. The sets $(c - \varepsilon, c + \varepsilon) \cap A$ and $(a - \varepsilon, a + \varepsilon) \cap A$ are infinite because $A \setminus A^i \subseteq A'$.

If $b \notin A$, then $A = [(-\infty, b) \cap A] \sqcup [(b, \infty) \cap A]$ with $(-\infty, b) \cap A \notin \mathcal{I}_f(\mathbb{R})$ and $(b, \infty) \cap A \notin \mathcal{I}_f(\mathbb{R})$, but this is impossible since A is $\diamond$-connected. Hence $b \in A$.

Now, suppose $b \in A^i$. We can choose $\varepsilon_1 \in (0, \varepsilon)$ such that $(b - \varepsilon_1, b + \varepsilon_1) \cap A = \{b\}$, and let $r \in (b - \varepsilon_1, b + \varepsilon_1) \setminus \{b\}$. Thus, $A = [(-\infty, r) \cap A] \sqcup [(r, \infty) \cap A]$ with $\{(-\infty, r) \cap A, (r, \infty) \cap A\} \cap \mathcal{I}_f(\mathbb{R}) = \varnothing$ given that $(a - \varepsilon, a + \varepsilon) \cap A \subseteq (-\infty, r) \cap A$ and $(c - \varepsilon, c + \varepsilon) \cap A \subseteq (r, \infty) \cap A$. This is not possible since A is $\diamond$-connected. Therefore $b \notin A^i$, and hence $b \in A \setminus A^i$.

(v) If we put $I = A\backslash A^i$, then $A = I \cup A^i$. Obviously, $(A^i)' \cap A^i = \varnothing$. Hence, by (ii), we have that $(A^i)' \subseteq I$, and the proof is complete. □

Example 2 *As a consequence of Proposition 3, we have that the sets* $A_1 = [0,1] \cup \{1+(1/n) : n \in \mathbb{Z}^+\}$, $A_2 = \{-1/n : n \in \mathbb{Z}^+\} \cup [0,1] \cup \{1+(1/n) : n \in \mathbb{Z}^+\}$ *and* $A_3 = I \cup F$, *where* I *is an interval and* $F \subseteq \mathbb{R}$ *is finite, are* $\diamond$*-connected sets in the ideal space* $(\mathbb{R}, \mathcal{U}, \mathcal{I}_f(\mathbb{R}))$. *Note that* $(A_1^i)' = \{1\}$, $(A_2^i)' = \{0,1\}$ *and* $(A_3^i)' = \varnothing$. *On the other hand, the sets* $A_4 = [0,1) \cup \{1+(1/n) : n \in \mathbb{Z}^+\}$ *and* $A_5 = [0,1] \cup \{1+(1/n) : n \in \mathbb{Z}^+\} \cup \mathbb{Z}$ *are not* $\diamond$*-connected.*

Proposition 4 *In the space* $(\mathbb{R}, \mathcal{U}, \mathcal{I}_c(\mathbb{R}))$, *a subset* A *is* $\diamond$*-connected if and only if there is an interval* I *(eventually* $\varnothing$ *or a singleton) and a countable set* J *such that* $A = I \cup J$.

Proof. By Theorem 1, if I is an interval and $J \subseteq \mathbb{R}$ is countable, then $I \cup J$ is $\diamond$-connected.

For the converse, suppose that $A \subseteq \mathbb{R}$ satisfies the following requeriments: 1) A is $\diamond$-connected on $(\mathbb{R}, \mathcal{U}, \mathcal{I}_c(\mathbb{R}))$; and 2) For each interval $I \subseteq A$, we have that $A \backslash I \notin \mathcal{I}_c(\mathbb{R})$. Hence, $A \notin \mathcal{I}_c(\mathbb{R})$ and $A \neq \mathbb{R}$.

In this case, there exists $d \in \mathbb{R}\backslash A$. Thus, $(-\infty, d) \cap A \in \mathcal{I}_c(\mathbb{R})$ or $(d, \infty) \cap A \in \mathcal{I}_c(\mathbb{R})$ because A is $\diamond$-connected. Without loss of generality, we suppose that $(-\infty, d) \cap A \in \mathcal{I}_c(\mathbb{R})$, and thus $(d, \infty) \cap A \notin \mathcal{I}_c(\mathbb{R})$. Since $A \notin \mathcal{I}_c(\mathbb{R})$, we can conclude that the set $D = \{x \in \mathbb{R} : x \notin A \text{ and } (-\infty, x) \cap A \in \mathcal{I}_c(\mathbb{R})\}$ is bounded above. Note that D is not countable since $(-\infty, d) \backslash A \subseteq D$. We define $\alpha = \sup D$. If $\{d_n\}$ is a sequence in D such that $d_n \to \alpha$, we have that $(-\infty, \alpha) \cap A = \bigcup_{n \geq 1} [(-\infty, d_n) \cap A]$ and hence $(-\infty, \alpha) \cap A \in \mathcal{I}_c(\mathbb{R})$. But $A \notin \mathcal{I}_c(\mathbb{R})$, and then $(\alpha, \infty) \cap A \notin \mathcal{I}_c(\mathbb{R})$. Hypothesis 2 about A implies that $(\alpha, \infty) \nsubseteq A$, and therefore, there is a $b > \alpha$ such that $b \notin A$. Consider the set $E = \{x > \alpha : x \notin A\}$. It is clear that $b \in E$, and that if $x \in E$ then $(-\infty, x) \cap A \notin \mathcal{I}_c(\mathbb{R})$. Moreover, if $x \in E$, then $(x, \infty) \cap A \in \mathcal{I}_c(\mathbb{R})$ because A is $\diamond$-connected. Hence, if we put $\beta = \inf E$, then it is easy to see that $(\beta, \infty) \cap A \in \mathcal{I}_c(\mathbb{R})$, and which implies that $\alpha < \beta$ because $A \notin \mathcal{I}_c(\mathbb{R})$. Using Hypothesis 2 again, we have that $(\alpha, \beta) \nsubseteq A$, and thus there is a $u \in (\alpha, \beta)$ such that $u \notin A$. Then $u \in E$ and $\beta \leq u$, and we come to contradiction.

All the above allows us to conclude that if A is $\diamond$-connected in $(\mathbb{R}, \mathcal{U}, \mathcal{I}_c(\mathbb{R}))$, then there is an interval $I \subseteq A$ (eventually $\varnothing$ or a singleton) and $J \in \mathcal{I}_c(\mathbb{R})$ such that $A \backslash I = J$. Hence $A = I \cup J$. □

Proposition 5 *In the space* $(\mathbb{R}, \mathcal{U}, \mathcal{I}_n)$, *a set* A *is* $\diamond$*-connected if and only if there is an interval* E *and* $J \in \mathcal{I}_n$ *such that* $A = E \cup J$. *It is possible that, in some cases,* $E = \varnothing$.

Proof. The sufficiency follows from Theorem 1.

Let us prove the necessity. Suppose that $A \subseteq \mathbb{R}$ is $\diamond$-connected in $(\mathbb{R}, \mathcal{U}, \mathcal{I}_n)$.

(a) Let us initially consider the case in which $\overset{0}{A} \neq \varnothing$. Since A is $\diamond$-connected, there is only one maximal interval E with more than one element contained in A. We have that $A \backslash E \in \mathcal{I}_n$. Indeed, suppose that there is a $u \in \overset{0}{\overline{A \backslash E}}$. Thus, there exists $\varepsilon > 0$ such that $(u - \varepsilon, u + \varepsilon) \subseteq \overline{A \backslash E}$.

(i) Suppose there exists $\inf(E)$ and $\sup(E)$. Since $\overset{0}{\overline{A \backslash E}} \subseteq \overset{0}{\overline{\mathbb{R} \backslash E}} = (-\infty, \inf(E)) \cup (\sup(E), \infty)$, we have $u < \inf(E)$ or $\sup(E) < u$. If $u < \inf(E)$, we can assume that $\varepsilon < \inf(E) - u$. By the maximality of E, it is clear that $\{\inf(E), \sup(E)\} \subseteq E$ and $(u + \varepsilon, \inf(E)) \nsubseteq A$.

Choose $v \in (u + \varepsilon, \inf(E)) \backslash A$. If $r \in (u - \varepsilon, u + \varepsilon)$, then there exists a sequence $\{a_n\}$ in $A \backslash E$ such that $a_n \to r$. We can assume that $a_n \in (u - \varepsilon, u + \varepsilon)$ for each $n \geq 1$. Hence, $r \in \overline{A \cap (-\infty, v)}$. Thus, $(u - \varepsilon, u + \varepsilon) \subseteq \overline{A \cap (-\infty, v)}$. Furthermore, $E \subseteq A \cap (v, \infty)$. This allows us to conclude that $\{A \cap (-\infty, v), A \cap (v, \infty)\} \cap \mathcal{I}_n = \varnothing$. But since $A = [A \cap (-\infty, v)] \sqcup [A \cap (v, \infty)]$ and A is $\diamond$-connected, we reached a contradiction. Analogously we obtain a contradiction by supposing that $\sup(E) < u$.

(ii) If there exists $\inf(E)$ but not $\sup(E)$ or if there exists $\sup(E)$ but not $\inf(E)$, we can proceed as in the case (i) to arrive to a contradiction.

Hence, there is a $J \in \mathcal{I}_n$ with $A \backslash E = J$, and thus $A = E \cup J$.

(b) Suppose now that $\overset{0}{A} = \varnothing$. We are going to show that $A \in \mathcal{I}_n$. If $A \notin \mathcal{I}_n$, then there is an interval $(a, b) \subseteq \overline{A}$. Given that $\overset{0}{A} = \varnothing$, we can choose a $z \in (a, b) \backslash A$. In this case, we have that $(a, z) \subseteq \overline{A \cap (-\infty, z)}$ and $(z, b) \subseteq \overline{A \cap (z, \infty)}$, and hence $\{A \cap (-\infty, z), A \cap (z, \infty)\} \cap \mathcal{I}_n = \varnothing$, which is not possible since $A = [A \cap (-\infty, z)] \sqcup [A \cap (z, \infty)]$ and A is $\diamond$-connected. $\square$

Note also that, as it is easy to see, in the space $(\mathbb{R}, \mathcal{L}, \mathcal{I}_n)$, a set A is $\diamond$-connected if and only if $A \in \mathcal{I}_n$.

Theorem 2 *The space $(X, \tau, \mathcal{I})$ is $\diamond$-connected if and only if for each continuous function $f : (X, \tau) \to (\{0, 1\}, \mathcal{P}(\{0, 1\}))$, it holds $f^{-1}(\{0\}) \in \mathcal{I}$ or $f^{-1}(\{1\}) \in \mathcal{I}$.*

Proof. Suppose that $f : (X, \tau) \to (\{0, 1\}, \mathcal{P}(\{0, 1\}))$ is continuous. Since $X = f^{-1}(\{0\}) \sqcup f^{-1}(\{1\})$ then, $f^{-1}(\{0\}) \in \mathcal{I}$ or $f^{-1}(\{1\}) \in \mathcal{I}$.

For the converse, let us assume that $\{U, V\} \subseteq \tau \backslash \{\varnothing\}$ and $X = U \sqcup V$. The function $f : (X, \tau) \to (\{0, 1\}, \mathcal{P}(\{0, 1\}))$ defined by $f(x) = 1$ if $x \in U$ and $f(x) = 0$ if $x \in V$ is continuous. Thus, $V = f^{-1}(\{0\}) \in \mathcal{I}$ or $U = f^{-1}(\{1\}) \in \mathcal{I}$. $\square$

For ideal topological space $(X,\tau,\mathcal{I})$, denote by $\overline{\mathcal{I}}$ the ideal of all $A\subseteq X$ such that there is a $J\in\mathcal{I}$ with $A\subseteq\overline{J}$ (see [5] for details). Note that $I\in\overline{\mathcal{J}}$ if and only if $\overline{I}\in\overline{\mathcal{J}}$.

Theorem 3 *If A and B are subsets of the space $(X,\tau,\mathcal{I})$ and A is $\diamond$-connected, then:*
1) If $A\subseteq B\subseteq\overline{A}$, then B is $\diamond$-connected in the space $\left(X,\tau,\overline{\mathcal{I}}\right)$.
2) If $A\subseteq B\subseteq A\cup A^{\star}$, then B is $\diamond$-connected in the space $(X,\tau,\mathcal{I})$.

Proof. 1) Suppose that there exists $\{U,V\}\subseteq\tau$ such that $B=(B\cap U)\sqcup(B\cap V)$. Hence, $A=(A\cap U)\sqcup(A\cap V)$, and thus $A\cap U\in\mathcal{I}$ or $A\cap V\in\mathcal{I}$. If $A\cap U=I\in\mathcal{I}$, then $A\subseteq I\cup(X\backslash U)$, and therefore, $B\subseteq\overline{A}\subseteq\overline{I}\cup(X\backslash U)$. Thus, $B\cap U\subseteq\overline{I}$ and $B\cap U\in\overline{\mathcal{I}}$. Similarly, if $A\cap V\in\mathcal{I}$, then $B\cap V\in\overline{\mathcal{I}}$. This implies that B is $\diamond$-connected in $\left(X,\tau,\overline{\mathcal{I}}\right)$.

2) This can be easily verified. □

The following example shows that if a set A is $\diamond$-connected with respect to an ideal $\mathcal{I}$, it is not necessarily that $\overline{A}$ is $\diamond$-connected with respect to $\mathcal{I}$.

Example 3 *Let $X=\{0,1,2\}$, $\tau=\{\varnothing,X,\{0\},\{1,2\}\}$, $\mathcal{I}=\{\varnothing,\{1\}\}$ and $A=\{0,1\}$. Then A is $\diamond$-connected in $(X,\tau,\mathcal{I})$. However, $\overline{A}=X$ is not $\diamond$-connected in $(X,\tau,\mathcal{I})$ because $X=\{0\}\cup\{1,2\}$ but $\{\{0\},\{1,2\}\}\cap\mathcal{I}=\varnothing$.*

The three theorems that follow show us how to build $\diamond$-connected sets from some known ones. If $\mathcal{I}$ is an ideal on X, then the ideal $\mathcal{P}\left(\bigcup\limits_{I\in\mathcal{I}}I\right)$ is denoted by $\mathcal{I}^{\circledast}$. Note that $\mathcal{I}\subseteq\mathcal{I}^{\circledast}$.

Theorem 4 1) *If $\{A_\alpha\}_{\alpha\in\Delta}$ is a collection of $\diamond$-connected subsets of an ideal topological space $(X,\tau,\mathcal{I})$ and there exists $a\in\bigcap\limits_{\alpha\in\Delta}A_\alpha$ such that $\{a\}\notin\mathcal{I}$, then the set $A=\bigcup\limits_{\alpha\in\Delta}A_\alpha$ is $\diamond$-connected in the space $(X,\tau,\mathcal{I}^{\circledast})$.*
2) *If A and B are $\diamond$-connected subsets of $(X,\tau,\mathcal{I})$ such that $A\circledast B\neq\varnothing$, then $A\cup B$ is $\diamond$-connected in $(X,\tau,\mathcal{I})$. Here $A\circledast B=\overline{A\backslash\bigcup\limits_{I\in\mathcal{I}}I}\cap\left(B\backslash\bigcup\limits_{I\in\mathcal{I}}I\right)$. In the particular case $\mathcal{I}=\{\varnothing\}$, if A and B are connected sets in the space (X,τ) and $\overline{A}\cap B\neq\varnothing$, then $A\cup B$is connected.*

Proof. 1) Suppose that there is $\{U,V\}\subseteq\tau$ such that $A=(A\cap U)\sqcup(A\cap V)$. Then, for each $\alpha\in\Delta$, $A_\alpha=(A_\alpha\cap U)\sqcup(A_\alpha\cap V)$, and hence $A_\alpha\cap U\in\mathcal{I}$ or $A_\alpha\cap V\in\mathcal{I}$. If $a\in U$, then $\{a\}\subseteq U\cap A_\alpha$, and thus $U\cap A_\alpha\notin\mathcal{I}$ for all $\alpha\in\Delta$. This implies that $V\cap A_\alpha\in\mathcal{I}$ for all $\alpha\in\Delta$, and then $V\cap A\in\mathcal{I}^{\circledast}$. Similarly, if $a\in V$, we obtain that $U\cap A\in\mathcal{I}^{\circledast}$.

2) Suppose that $z\in A\circledast B$. Let $\{U,V\}\subseteq\tau$ and $A\cup B=[(A\cup B)\cap U]\sqcup[(A\cup B)\cap V]$. This implies that $B=(B\cap U)\sqcup(B\cap V)$ and $A=(A\cap U)\sqcup$

$(A \cap V)$. Without loss of generality, we can assume $z \in U$. Since $\{z\} \notin \mathcal{I}$, we have that $B \cap U \notin \mathcal{I}$, and thus $B \cap V \in \mathcal{I}$ because B is $\diamond$-connected. Now, there is an element $a \in U \cap \left(A \backslash \bigcup_{I \in \mathcal{I}} I \right)$. Since $\{a\} \notin \mathcal{I}$, we have $U \cap A \notin \mathcal{I}$, and in this case $A \cap V \in \mathcal{I}$, because A is $\diamond$-connected. Under these conditions, $(A \cup B) \cap V \in \mathcal{I}$. Thus, $A \cup B$ is $\diamond$-connected. □

Corollary 2 *If $(X, \tau, \mathcal{I})$ is an ideal space such that $\mathcal{I}$ is closed for arbitrary unions, then for each $a \in X \backslash \bigcup_{I \in \mathcal{I}} I$, there is a maximal $\diamond$-connected set $\mathcal{P}(a)$ such that $a \in \mathcal{P}(a)$. Furthermore, sets $\mathcal{P}(a) \backslash \bigcup_{I \in \mathcal{I}} I$ determine a partition of $X \backslash \bigcup_{I \in \mathcal{I}} I$.*

Proof. It is clear that in this case, $\mathcal{I}^{\circledast} = \mathcal{I}$. If $a \in X \backslash \bigcup_{I \in \mathcal{I}} I$, then $\{a\} \notin \mathcal{I}$. If $\mathcal{H} = \{A \subseteq X : a \in A \text{ and } A \text{ is } \diamond\text{-connected}\}$, then $\{a\} \in \mathcal{H}$. The set $\mathcal{P}(a) = \bigcup_{H \in \mathcal{H}} H$ is a maximal $\diamond$-connected. Now, if $\{a, b\} \subseteq X \backslash \bigcup_{I \in \mathcal{I}} I$ and there is $c \in \left(\mathcal{P}(a) \backslash \bigcup_{I \in \mathcal{I}} I \right) \cap \left(\mathcal{P}(b) \backslash \bigcup_{I \in \mathcal{I}} I \right)$, then $\mathcal{P}(a) \cup \mathcal{P}(b)$ is $\diamond$-connected given that $\{c\} \notin \mathcal{I}$. The maximality of $\mathcal{P}(a)$ and $\mathcal{P}(b)$ forces that $\mathcal{P}(a) = \mathcal{P}(a) \cup \mathcal{P}(b) = \mathcal{P}(b)$. □

Note that, according to Corollary 1, $\bigcup_{I \in \mathcal{I}} I \subseteq \mathcal{P}(a)$ for all $a \in X \backslash \bigcup_{I \in \mathcal{I}} I$. The sets $\mathcal{P}(a)$ are what one might call the *$\diamond$-connected components of* $(X, \tau, \mathcal{I})$.

If A and B are $\diamond$-connected and $A \cap B \neq \varnothing$, then it may happen that $A \cup B$ is not $\diamond$-connected, which the following example shows.

Example 4 *If $(X, \tau, \mathcal{I})$ and A are as in Example 3 and $B = \{1, 2\}$, then A and B are $\diamond$-connected sets but $A \cup B = X$ is not $\diamond$-connected.*

Theorem 5 *If $\{A_n\}_{n \in \mathbb{Z}^+}$ is a collection of $\diamond$-connected subsets of an ideal space $(X, \tau, \mathcal{I})$ such that for each $n \in \mathbb{Z}^+$, there exists $z_n \in A_n \cap A_{n+1}$ with $\{z_n\} \notin \mathcal{I}$, then the set $A = \bigcup_{n \in \mathbb{Z}^+} A_n$ is $\diamond$-connected in the space $(X, \tau, \mathcal{I}^{\circledast})$.*

Proof. For each $n \geq 1$, define $B_n = A_1 \cup A_2 \cup \cdots A_n$. It is clear that $\bigcup_{n \geq 1} B_n = \bigcup_{n \geq 1} A_n$, $\bigcap_{n \geq 1} B_n = A_1$ and B_1 is $\diamond$-connected in $(X, \tau, \mathcal{I})$. Suppose that for some $k \in \mathbb{Z}^+$, B_k is $\diamond$-connected in $(X, \tau, \mathcal{I})$. Since $B_{k+1} = B_k \cup A_{k+1}$, $z_k \in A_k \cap A_{k+1} \subseteq B_k \cap A_{k+1}$ and $\{z_k\} \notin \mathcal{I}$, then it is easy to see that B_{k+1} is $\diamond$-connected in $(X, \tau, \mathcal{I})$. Now, given that $z_1 \in A_1 = \bigcap_{n \geq 1} B_n$ and $\{z_1\} \notin \mathcal{I}$, we have that $\bigcup_{n \geq 1} B_n$ is $\diamond$-connected in $(X, \tau, \mathcal{I}^{\circledast})$ by Theorem 4. □

Without the hypothesis that for each $n \geq 1$, there is $z_n \in A_n \cap A_{n+1}$ with $\{z_n\} \notin \mathcal{I}$, it is possible that the conclusion of the theorem turns out to be false. On the other hand, it is also possible that under the hypothesis of Theorem 5, it is not true that the set $A = \bigcup_{n \in \mathbb{Z}^+} A_n$ is $\diamond$-connected in $(X, \tau, \mathcal{I})$. The following example illustrates these two situations.

Example 5 1) *If $2\mathbb{Z}$ is the set of even integers then in the space $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mathcal{P}_f(2\mathbb{Z}))$, we have that the set $A_n = \{n, n+1\}$ is $\diamond$-connected for each $n \in \mathbb{Z}$ but the set $\mathbb{Z} = \bigcup_{n \in \mathbb{Z}} A_n$ is not $\diamond$-connected in $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mathcal{P}(2\mathbb{Z}))$ (see Example 1). Note that $(\mathcal{P}_f(2\mathbb{Z}))^{\circledast} = \mathcal{P}(2\mathbb{Z})$.*

2) *If A is the set of all positive odd integers and $\mathcal{P}_f(A)$ is the collection of all finite $B \subseteq A$, then in the space $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mathcal{P}_f(A))$, we have that the set $A_n = \{2k+1 : 0 \leq k \leq n-1\} \cup \{2\}$ is $\diamond$-connected for each $n \geq 1$. Furthermore, $2 \in A_n \cap A_{n+1}$ for all $n \geq 1$ and $\{2\} \notin \mathcal{P}_f(A)$. However, the set $D = \bigcup_{n \geq 1} A_n$ is not $\diamond$-connected in $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mathcal{P}_f(A))$.*

Theorem 6 1) *If in the space $(X, \tau, \mathcal{I})$ there is a singleton $\{a\} \notin \mathcal{I}$ such that for each $x \in X$, there exists a $\diamond$-connected set E_x satisfying $\{a, x\} \subseteq E_x$, then $(X, \tau, \mathcal{I}^{\circledast})$ is $\diamond$-connected.*
2) *If in the space $(X, \tau, \mathcal{I})$, for each $\{x, y\} \subseteq X$, there exists a $\diamond$-connected set $E_{x,y}$ with $\{x, y\} \subseteq E_{x,y}$, then $(X, \tau, \mathcal{I}^{\circledast})$ is $\diamond$-connected.*

Proof. 1) This is a consequence of Theorem 4 given that $X = \bigcup_{x \in X} E_x$, $a \in \bigcap_{a \in X} E_x$ and $\{a\} \notin \mathcal{I}$.

2) If $X = \bigcup_{I \in \mathcal{I}} I$, then the statement is obvious. If $X \neq \bigcup_{I \in \mathcal{I}} I$, one can choose $a \in X \setminus \bigcup_{I \in \mathcal{I}} I$ and then apply 1). $\square$

Note that under any of the assumptions of Theorem 6, $(X, \tau, \mathcal{I})$ does not have to be $\diamond$-connected. In the following example we show this.

Example 6 *If $\mathcal{I} = \mathcal{P}_f(\mathbb{R} \setminus \{0\})$, then in the space $(\mathbb{R}, \mathcal{L}, \mathcal{I})$, we have that $\{0\} \notin \mathcal{I}$ and $E_r = \{0, r\}$ is $\diamond$-connected for each $r \in \mathbb{R}$. Despite this, $(\mathbb{R}, \mathcal{L}, \mathcal{I})$ is not $\diamond$-connected given that $\mathbb{R} = (-\infty, 1) \cup [1, \infty)$ but $\{(-\infty, 1), [1, \infty)\} \cap \mathcal{I} = \varnothing$. Further, note that $E_{r,s} = \{r, s\}$ is $\diamond$-connected for each $\{r, s\} \subseteq \mathbb{R}$.*

In the theorem that follows, we characterize $\diamond$-connectedness in terms of separated sets.

Theorem 7 *The set A is $\diamond$-connected in the space $(X, \tau, \mathcal{I})$ if and only if for each pair of separated sets H and K, from $A = H \cup K$ it follows that $H \in \mathcal{I}$ or $K \in \mathcal{I}$.*

Proof. If $A = H \cup K$ and H and K are separated sets, then H and K are closed sets in A. Given that A is $\diamond$-connected, we have that $H \in \mathcal{I}$ or $K \in \mathcal{I}$.

Conversely, if $A = H \sqcup K$ where H and K are closed sets in A, then H and K are separated sets. The hypothesis implies that $H \in \mathcal{I}$ or $K \in \mathcal{I}$. Hence, A is $\diamond$-connected. □

Corollary 3 *If the set A is $\diamond$-connected in the space $(X, \tau, \mathcal{I})$ and H and K are separated sets such that $A \subseteq H \cup K$, then $A \cap H \in \mathcal{I}$ or $A \cap K \in \mathcal{I}$.*

Proof. Since $A = (A \cap H) \sqcup (A \cap K)$ and $A \cap H$, $A \cap K$ are separated sets, we have that $A \cap H \in \mathcal{I}$ or $A \cap K \in \mathcal{I}$ by Theorem 7. □

Theorem 8 *If $(X, \tau, \mathcal{I})$ is an ideal space, $\{A, B\} \subseteq \tau$, $A \cap B$ is connected and $A \cup B$ is $\diamond$-connected, then A and B are $\diamond$-connected sets.*

Proof. Suppose that $\{U, V\} \subseteq \tau$ and $A = U \sqcup V$. Then $A \cap B = (U \cap B) \sqcup (V \cap B)$ and since $A \cap B$ is connected, we have that $U \cap B = \varnothing$ or $V \cap B = \varnothing$. Suppose that $U \cap B = \varnothing$. Given that $A \cup B = U \sqcup (V \cup B)$ and $A \cup B$ is $\diamond$-connected, we have that $U \in \mathcal{I}$ or $V \cup B \in \mathcal{I}$, and thus $U \in \mathcal{I}$ or $V \in \mathcal{I}$. Similarly, if $V \cap B = \varnothing$, we obtain that $U \in \mathcal{I}$ or $V \in \mathcal{I}$. Hence, A is $\diamond$-connected. The $\diamond$-connectedness of B can be shown in the similar way. □

Now we present some functional properties of $\diamond$-connectedness.

If $f : X \to Y$ is a function and $\mathcal{I}$ is an ideal on X, we will denote by $\mathcal{J}_f^{\mathcal{I}}$ the set $\{B \subseteq Y : f(f^{-1}(B)) \in f(\mathcal{I})\}$. It is clear that $\mathcal{J}_f^{\mathcal{I}}$ is an ideal on Y and that $f(\mathcal{I}) \subseteq \mathcal{J}_f^{\mathcal{I}}$. Moreover, if f is surjective, then $f(\mathcal{I}) = \mathcal{J}_f^{\mathcal{I}}$.

Example 7 *If $X = \{0, 1\}$, $Y = \{0, 1, 2\}$, $\mathcal{I} = \{\varnothing, \{0\}\}$ and $f : X \to Y$ is defined by $f(0) = 0$ and $f(1) = 1$, then $f(\mathcal{I}) = \{\varnothing, \{0\}\}$ and $\{2\} \in \mathcal{J}_f^{\mathcal{I}}$. Hence, $f(\mathcal{I}) \neq \mathcal{J}_f^{\mathcal{I}}$.*

Theorem 9 1) *If $f : (X, \tau) \to (Y, \beta)$ is a continuous function and $\mathcal{I}$ is an ideal on X such that $(X, \tau, \mathcal{I})$ is $\diamond$-connected, then $f(X)$ is $\diamond$-connected in $\left(Y, \beta, \mathcal{J}_f^{\mathcal{I}}\right)$.*
2) *If $f : (X, \tau) \to (Y, \beta)$ is a bijective open function and $(Y, \beta, \mathcal{J})$ is $\diamond$-connected, then $(X, \tau, f^{-1}(\mathcal{J}))$ is $\diamond$-connected.*

Proof. 1) Suppose that H and K are separated sets in Y such that $f(X) = H \cup K$. Given that $f^{-1}(H)$ and $f^{-1}(K)$ are separated sets in X and $X = f^{-1}(H) \cup f^{-1}(K)$, we have that $f^{-1}(H) \in \mathcal{I}$ or $f^{-1}(K) \in \mathcal{I}$, and thus $f\left(f^{-1}(H)\right) \in f(\mathcal{I})$ or $f\left(f^{-1}(K)\right) \in f(\mathcal{I})$. Hence, $H \in \mathcal{J}_f^{\mathcal{I}}$ or $K \in \mathcal{J}_f^{\mathcal{I}}$.

2) This statement is verified without difficulty. □

Corollary 4 *If $f : (X, \tau) \to (Y, \beta)$ is a continuous surjective function and $\mathcal{I}$ is an ideal on X such that $(X, \tau, \mathcal{I})$ is $\diamond$-connected, then $(Y, \beta, f(\mathcal{I}))$ is $\diamond$-connected.*

Corollary 5 *If $f : (X, \tau) \to (Y, \beta)$ is continuous and the space $(X, \tau, \mathcal{I})$ is $\diamond$-connected, then the set $Gr(f) = \{(x, f(x)) : x \in X\}$ is $\diamond$-connected in the space $(X \times Y, \tau \times \beta, \mathcal{J}(f, \mathcal{I}))$ where $\mathcal{J}(f, \mathcal{I})$ is the ideal $\{J \subseteq X \times Y : p_1 (J \cap Gr(f)) \in \mathcal{I}\}$. Here $p_1 : X \times Y \to X$ is the first projection.*

Proof. It is not difficult to verify that $\mathcal{J}(f, \mathcal{I})$ is an ideal on $X \times Y$. Since the function $g : X \to X \times Y$ defined by $g(x) = (x, f(x))$ for all $x \in X$ is continuous, Theorem 9 implies that $Gr(f)$ is $\diamond$-connected in $\left(X \times Y, \tau \times \beta, \mathcal{J}_g^{\mathcal{I}}\right)$. Now, given that g is injective, we have that for each $J \subseteq X \times Y$, $g\left(g^{-1}(J)\right) \in g(\mathcal{I})$ if and only if $g^{-1}(J) \in \mathcal{I}$. Furthermore, for all $J \subseteq X \times Y$, it is true that $g^{-1}(J) = p_1(J \cap Gr(f))$. Hence, $\mathcal{J}_g^{\mathcal{I}} = \mathcal{J}(f, \mathcal{I})$. □

Corollary 6 *If (X^*, d^*) is the completion of a metric space (X, d) and $(X, d, \mathcal{I})$ is a $\diamond$-connected space, then $\left(X^*, d^*, \overline{\mathcal{J}_j^{\mathcal{I}}}\right)$ is $\diamond$-connected, where $j : X \to X^*$ is the inclusion funtion. Note that $\mathcal{J}_j^{\mathcal{I}} = \{I \cup A : I \in \mathcal{I}$ and $A \subseteq X^* \backslash X\}$.*

A surjective function $f : (X, \tau) \to (Y, \beta)$ is a *quotient function* if for each $B \subseteq Y$, we have $f^{-1}(B) \in \tau$ if and only if $B \in \beta$.

Theorem 10 *Let $f : (X, \tau) \to (Y, \beta)$ be a quotient funtion with $f^{-1}(\{y\})$ connected for each $y \in Y$. Let $(Y, \beta, \mathcal{J})$ be a $\diamond$-connected space. Then $(X, \tau, \mathcal{I}_{f,\mathcal{J}})$ is $\diamond$-connected.*

Proof. Suppose that there is $\{U, V\} \subseteq \tau$ such that $X = U \sqcup V$. If $U = \varnothing$ or $V = \varnothing$, then $\{U, V\} \cap \mathcal{I}_{f,\mathcal{J}} \neq \varnothing$. Thus, we can assume that $U \neq \varnothing$ and $V \neq \varnothing$. There is no $y_0 \in Y$ such that $f^{-1}(\{y_0\}) \cap U \neq \varnothing$ and $f^{-1}(\{y_0\}) \cap V \neq \varnothing$ since the set $f^{-1}(\{y_0\})$ is connected. Hence, there is $\{Y_1, Y_2\} \subseteq \mathcal{P}(Y)$ with $Y_1 \cap Y_2 = \varnothing$, $U = f^{-1}(Y_1)$ and $V = f^{-1}(Y_2)$. Note that $\{Y_1, Y_2\} \subseteq \beta$ since f is a quotient function. But we have that $Y = f(U) \sqcup f(V) = Y_1 \sqcup Y_2$, and since Y is $\diamond$-connected, we conclude that $Y_1 \in \mathcal{J}$ or $Y_2 \in \mathcal{J}$. This allows us to affirm that $U \in \mathcal{I}_{f,\mathcal{J}}$ or $V \in \mathcal{I}_{f,\mathcal{J}}$, and thus $(X, \tau, \mathcal{I}_{f,\mathcal{J}})$ is $\diamond$-connected. □

We recall that a topological space (X,τ) is said to be *completely Hausdorff* if for each $\{a,b\} \subseteq X$ with $a \neq b$, there exists a continuous funtion $f : X \to [0,1]$ such that $f(a) = 0$ and $f(b) = 1$.

Theorem 11 *If $(X,\tau,\mathcal{I})$ is a $\diamond$-connected space having more than one point such that $\mathcal{I}$ is codense and (X,τ) is completely Hausdorff, then X is not countable.*

Proof. Let $\{a,b\} \subseteq X$, $a \neq b$, and suppose that $f : X \to [0,1]$ is a continuous function such that $f(a) = 0$ and $f(b) = 1$. If $r \in (0,1)$, $U = f^{-1}([0,r))$ and $V = f^{-1}((r,1])$, then U and V are disjoint and nonempty open sets in (X,τ) and, since $(X,\tau,\mathcal{I})$ is $\diamond$-connected and $\mathcal{I}$ is codense, we have that $X \neq U \cup V$. Thus, there is $x_r \in X$ such that $f(x_r) = r$. Additionally, it is clear that if $\{r_1,r_2\} \subseteq (0,1)$ and $r_1 \neq r_2$, then $x_{r_1} \neq x_{r_2}$. Therefore, X is not countable. □

Corollary 7 *If (X,τ) is a T_4 space with more than one point and $\mathcal{I}$ is a codense ideal in X such that $(X,\tau,\mathcal{I})$ is $\diamond$-connected, then X is not countable.*

Proof. By Urysohn's Lemma, each T_4 space is a completely Hausdorff space. □

The following property is related to the intersection of $\diamond$-connected sets.

Theorem 12 *Suppose that:*

1) (X,τ) *is a compact and Hausdorff space;*

2) $\{A_\alpha\}_{\alpha\in\Delta}$ *is a collection of closed and $\diamond$-connected subsets of $(X,\tau,\mathcal{I})$;*

3) *There exists $a \in \bigcap_{\alpha\in\Delta} A_\alpha$ such that $\{a\} \notin \mathcal{I}$;*

4) *For each pair $\alpha \neq \beta$ in Δ, there exists $\theta \in \Delta$ such that $A_\theta \subseteq A_\alpha$ and $A_\theta \subseteq A_\beta$.*

Then the set $A = \bigcap_{\alpha\in\Delta} A_\alpha$ is $\diamond$-connected.

Proof. . Suppose that A is not $\diamond$-connected. Then there exists $\{U,V\} \subseteq \tau$ such that $A = (A \cap U) \sqcup (A \cap V)$ and $\{A \cap U, A \cap V\} \cap \mathcal{I} = \varnothing$. Note that $A \cap U \neq \varnothing$ and $A \cap V \neq \varnothing$. Since A is closed and the sets $A \cap U$ and $A \cap V$ are closed in A, these sets are closed in X, and thus are compact. But (X,τ) is Hausdorff, and hence there are disjoint open sets T and R such that $A \cap U \subseteq T$ and $A \cap V \subseteq R$. It is clear that $A \cap U \subseteq A_\alpha \cap T$, $A \cap V \subseteq A_\alpha \cap R$, and that $A_\alpha \backslash (T \cup R)$ is closed for each $\alpha \in \Delta$. If there is a $\lambda \in \Delta$ such that $A_\lambda \backslash (T \cup R) = \varnothing$, then $A_\lambda = (A_\lambda \cap T) \sqcup (A_\lambda \cap R)$. Given that A_λ is

$\diamond$-connected, we have that $\{A_\lambda \cap T, A_\lambda \cap R\} \cap \mathcal{I} \neq \varnothing$, and thus $\{a\} \in \mathcal{I}$, which is impossible. Hence $A_\alpha \backslash (T \cup R) \neq \varnothing$ for each $\alpha \in \Delta$. By hypothesis 4, the collection $\{A_\alpha \backslash (T \cup R) : \alpha \in \Delta\}$ has the finite intersection property. By the compactness of X, we can conclude that $\bigcap_{\alpha\in\Delta} [A_\alpha \backslash (T \cup R)] \neq \varnothing$ or, equivalently, $A \backslash (T \cup R) \neq \varnothing$. This is not possible, since $A \subseteq T \cup R$. □

Theorem 13 *Suppose that $(X, \tau, \mathcal{I})$ is $\diamond$-connected, $Y \subseteq X$ is connected and that $X \backslash Y = A \cup B$, where A and B are separated sets in X. Then $Y \cup A$ and $Y \cup B$ are $\diamond$-connected.*

Proof. We will proceed with $Y \cup A$. Suppose that H and K are separated sets in X such that $Y \cup A = H \cup K$. Since Y is connected, we have that $Y \subseteq H$ or $Y \subseteq K$. Without loss of generality, suppose that $Y \subseteq H$. Thus, $K \subseteq A$, and hence $\overline{K} \cap B = \varnothing = K \cap \overline{B}$. Now, $X = Y \cup (X \backslash Y) = Y \cup (A \cup B) = (Y \cup A) \cup B = (H \cup K) \cup B = (B \cup H) \cup K$. Also, $\overline{B \cup H} \cap K = (\overline{B} \cap K) \cup (\overline{H} \cap K) = \varnothing$ and $(B \cup H) \cap \overline{K} = (B \cap \overline{K}) \cup (H \cap \overline{K}) = \varnothing$. Given that X is $\diamond$-connected, it is true that $B \cup H \in \mathcal{I}$ or $K \in \mathcal{I}$ by Theorem 7 In this case $H \in \mathcal{I}$ or $K \in \mathcal{I}$, and therefore, $Y \cup A$ is $\diamond$-connected. □

In the next theorem we establish a relationship between the $\diamond$-connectedness of a Tychonoff space and the $\diamond$-connectedness of its Stone-Cĕch compactification.

Theorem 14 *Let $(\beta(X), \beta(\tau))$ be the Stone-Cĕch compactification of the Tychonoff space (X, τ).*

1) *If $(X, \tau, \mathcal{I})$ is $\diamond$-connected, then $(\beta(X), \beta(\tau), \beta(\mathcal{I}))$ is $\diamond$-connected where $\beta(\mathcal{I}) = \{B \subseteq \beta(X) :$ there is $I \in \mathcal{I}$ with $B \subseteq adh_{\beta(\tau)}(I)\}$.*

2) *If $(\beta(X), \beta(\tau), \mathcal{J})$ is $\diamond$-connected, then $(X, \tau, \mathcal{J}_X)$ is $\diamond$-connected. In particular, if $\mathcal{I}$ is an ideal on X and $(\beta(X), \beta(\tau), \mathcal{I})$ is $\diamond$-connected, then $(X, \tau, \mathcal{I})$ is $\diamond$-connected.*

Proof. 1) Note that $\mathcal{I}$ is an ideal on $\beta(X)$. Since $\beta(\tau)_X = \tau$, we have that X is $\diamond$-connected in $(\beta(X), \beta(\tau), \mathcal{I})$. Theorem 3 implies that the set $adh_{\beta(\tau)}(X) = \beta(X)$ is $\diamond$-connected on $(\beta(X), \beta(\tau), \beta(\mathcal{I}))$.

2) Suppose that U and V are disjoint and nonempty open sets such that $X = U \cup V$. The characteristic function $\chi_U : X \to \{0, 1\}$ is continuous, and thus, it can be extended to a continuous function $F : \beta(X) \to \{0, 1\}$. Hence, $\beta(X) = F^{-1}(\{0\}) \sqcup F^{-1}(\{1\})$ with $U \subseteq F^{-1}(\{1\})$ and $V \subseteq F^{-1}(\{0\})$. Given that $(\beta(X), \beta(\tau), \mathcal{J})$ is $\diamond$-connected, we have that $F^{-1}(\{1\}) \in \mathcal{J}$ or $F^{-1}(\{0\}) \in \mathcal{J}$, and thus $U \in \mathcal{J}_X$ or $V \in \mathcal{J}_X$. □

Let $\mathcal{C}$ be a collection of open sets in a topological space (X, τ). If a and b are points in X, then a finite sequence $U_1, U_2, ..., U_m$ of sets in $\mathcal{C}$ is said to be a *simple chain* connecting a with b modulo $\mathcal{C}$ if any of the following conditions are true:

(i) $m = 1$ and $\{a, b\} \subseteq U_1$;

(ii) $m > 1$, $a \in U_1$ only, $b \in U_m$ only, and $U_i \cap U_j \neq \varnothing$ iff $|i - j| \leq 1$.

In this case, we will write $a \rightsquigarrow_{\mathcal{C}} b$

Theorem 15 *Suppose that* $(X, \tau, \mathcal{I})$ *is* $\diamond$*-connected and* $\{a\} \notin \mathcal{I}$ *for some* $a \in X$. *If* $\mathcal{C}$ *is an open cover of* X *and if* $F = \{x \in X : a \rightsquigarrow_{\mathcal{C}} x\}$, *then* $X \backslash F \in \mathcal{I}$.

Proof. It is well known that F is an open and closed set (see, for example, [10], section 26). Given that $(X, \tau, \mathcal{I})$ is $\diamond$-connected, we have that $F \in \mathcal{I}$ or $X \backslash F \in \mathcal{I}$. But since $a \in F$ and $\{a\} \notin \mathcal{I}$, we conclude that $X \backslash F \in \mathcal{I}$. $\square$

Our last statement establishes a relationship between $\diamond$-connectedness and simple extensions of topologies.

We recall that if (X, τ) is a topological space and $A \subseteq X$, then the *simple extension* of τ over A is the topology $\tau(A) = \{U \cup (V \cap A) : \{U, V\} \subseteq \tau\}$.

Theorem 16 *If* $(X, \tau, \mathcal{I})$ *is an ideal space and the set* $F \in \mathcal{I}$ *is closed, then* $(X, \tau(F), \mathcal{I})$ is $\diamond$ *-connected if and only if* $\left(X \backslash F, \tau_{X \backslash F}, \mathcal{I}_{X \backslash F}\right)$ *is* $\diamond$*-connected.*

Proof. Suppose that $\{U, V\} \subseteq \tau$ and $X \backslash F = U \sqcup V$. Then $X = U \sqcup (V \cup F)$ with $\{U, V \cup F\} \subseteq \tau(F)$. The hypothesis implies that $U \in \mathcal{I}$ or $V \cup F \in \mathcal{I}$, and thus $U \in \mathcal{I}$ or $V \in \mathcal{I}$. Since $U \subseteq X \backslash F$ and $V \subseteq X \backslash F$, we have that $U \in \mathcal{I}_{X \backslash F}$ or $V \in \mathcal{I}_{X \backslash F}$.

Conversely, if $\{U_1, U_2, V_1, V_2\} \subseteq \tau$ and $X = [U_1 \cup (V_1 \cap F)] \sqcup [U_2 \cup (V_2 \cap F)]$, then $X \backslash F = (U_1 \backslash F) \sqcup (U_2 \backslash F)$, and hence $\{U_1 \backslash F, U_2 \backslash F\} \cap \mathcal{I}_{X \backslash F} \neq \varnothing$. Since $\mathcal{I}_{X \backslash F} \cup \{F\} \subseteq \mathcal{I}$, we obtain $\{U_1, U_2\} \cap \mathcal{I} \neq \varnothing$, which allows us to conclude that $\{U_1 \cup (V_1 \cap F), U_2 \cup (V_2 \cap F)\} \cap \mathcal{I} \neq \varnothing$. $\square$

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