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Hypergeometric connections between balancing polynomials and Chebyshev polynomials of first and second kinds

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Abstract. In the present study, we find several connections between balancing polynomials and the Chebyshev polynomials of the first and second kinds. The Chebyshev polynomials of the first and second kinds are expressed as the sum of two terms of balancing polynomials with hypergeometric coefficients. As an inversion, the balancing polynomials are also expressed as the sum of two terms of the Chebyshev polynomials of the first kind and the Chebyshev polynomials of the second kind with hypergeometric coefficients.

Key Words: Balancing polynomials, Chebyshev polynomials of the first kind, Chebyshev polynomials of the second kind, Hypergeometric functions
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Introduction

The connection between various classical orthogonal polynomials play an important role in many problems of pure and applied mathematics. The study of the problems connecting number and polynomial sequences with different orthogonal polynomials is of old and recent interest for many number theorists (see, e.g., [6, 11–14, 16]). For instance, Abd-Elhameed and Youssri [1] and Abd-Elhameed et al. [2] studied the connection formulas between Fibonacci and Lucas polynomials and Chebychev polynomials of the first and second kinds in terms of hypergeometric functions of the type ${}_2F_1$. The hypergeometric function ${}_2F_1(a,b;c;z)$ is a solution to a second-order linear ordinary differential equation and is defined by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, $$

where $(a)_n, (b)_n$ and $(c)_n$ are Pochhammer symbols, defined by

$$(d)_n = \begin{cases} 1, & n = 0; \\ d(d+1)\cdots(d+n-1), & n > 0. \end{cases}$$

Behera and Panda [4] introduced the concept of balancing numbers, which are the solutions of the simple Diophantine equation $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$, where r is called as balancer corresponding to the balancing number n. Balancing numbers are generated by the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}, \ n \geq 1 \tag{1}$$

with initial terms $B_0 = 0$ and $B_1 = 1$. Equivalently, the nth balancing numbers can also be obtained by using Binet's formula

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2},$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$ are roots of the characteristic equation $\lambda^2 - 6\lambda + 1 = 0$. To know more about this number sequence, the reader can refer to [7, 8].

Balancing polynomials are one of the generalizations of balancing numbers which are generated by using the recurrence relation

$$B_{n+1}(x) = 6xB_n(x) - B_{n-1}(x), \ n \geq 1 \tag{2}$$

with initial terms $B_o(x) = 0$ and $B_1(x) = 1$, see [15]. Frontczak [9] showed the relationships between balancing and Lucas-balancing polynomials and Chebyshev polynomials of the first and second kinds as $B_n(x) = U_{n-1}(3x)$ and $C_n(x) = T_n(3x)$. One can give the expression of balancing polynomials in hypergeometric function as $B_{n+1}(x) = (n+1)_2F_1\big(-n, n+2, 3/2; (1-3x)/2\big)$, using the result (3.5) of [5]. Supposedly, till now, the hypergeometric connection formulas between balancing polynomials and various orthogonal polynomials are traceless in literature. This gives us the inspiration to explore such problems.

The aim of this work is to find some hypergeometric connection formulas between balancing polynomials and the Chebyshev polynomials of the first and second kinds. The complete study is motivated by the works of Abd-Elhameed and Youssri [1].

1 Preliminaries

In this section, we discuss some results relating to balancing polynomials, Chebyshev polynomials of the first kind and Chebyshev polynomials of the second kind.

The recurrence relation of balancing polynomials in (2) can be written in the following combinatorial form (see [15]):

$$B_j(x) = \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} (-1)^m \binom{j-m-1}{m} (6x)^{j-2m-1}, \ j \geq 1.$$

We recall that the Binet's formula of balancing numbers can be written as

$$B_j = \frac{(3+\sqrt{8})^j - (3-\sqrt{8})^j}{2\sqrt{8}} = \frac{3^j}{2\sqrt{8}} \left\{ \left(1 + \frac{\sqrt{8}}{3}\right)^j - \left(1 - \frac{\sqrt{8}}{3}\right)^j \right\},$$

which concedes the identity

$${}_2F_1\left(\begin{matrix} a, \ \frac{1}{2}+a \\ \frac{3}{2} \end{matrix}; z^2 \right) = \frac{1}{2} z^{-1} (1-2a)^{-1} \left[(1+z)^{1-2a} - (1-z)^{1-2a} \right]$$

(see (15.1.10), [3]). Assuming $a = (1-j)/2$ and $z = \sqrt{8}/3$, we get

$$B_j = j \cdot 3^{j-1} {}_2F_1\left(\begin{matrix} \frac{1-j}{2}, \ \frac{2-j}{2} \\ \frac{3}{2} \end{matrix}; \frac{8}{9} \right). \tag{3}$$

The linear transformation formula of hypergeometric function is

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a, \ b \\ c \end{matrix}; z \right) =& \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1\left(\begin{matrix} a, \ b \\ a+b-c+1 \end{matrix}; 1-z \right) \\ &+ (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} {}_2F_1\left(\begin{matrix} c-a, \ c-b \\ c-a-b+1 \end{matrix}; 1-z \right). \end{aligned} \tag{4}$$

(see [3], p. 559). Since in (3), we have $a+b-c = -j$ (where $a = (1-j)/2$, $b = (2-j)/2$ and $c = 3/2$), it is seen that one of the gamma terms in the numerator of (4) is not defined. Thus, we use formula (15.3.11) in [3], p. 559 to get the following identity

$${}_2F_1\left(\begin{matrix} a, \ b \\ a+b+m \end{matrix}; z \right) = \frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)} {}_2F_1\left(\begin{matrix} a, \ b \\ 1-m \end{matrix}; 1-z \right),$$

which is indeed a special case of (15.3.11) where a or b is a negative integer and $m \in \mathbf{Z}^+ \cup \{\mathbf{0}\}$. Now, we have

$${}_2F_1\left(\begin{matrix} \frac{1-j}{2}, \ \frac{2-j}{2} \\ \frac{3}{2} \end{matrix}; \frac{8}{9} \right) = \frac{\Gamma(j)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1+j}{2}\right)\Gamma\left(\frac{2+j}{2}\right)} {}_2F_1\left(\begin{matrix} \frac{1-j}{2}, \ \frac{2-j}{2} \\ 1-j \end{matrix}; \frac{1}{9} \right).$$

Using the duplication formula (see, e.g., [3], p. 256) for $\Gamma(z)$, we obtain

$$\begin{aligned} {}_2F_1\left(\begin{matrix} \frac{1-j}{2}, \ \frac{2-j}{2} \\ \frac{3}{2} \end{matrix}; \frac{8}{9} \right) =& \frac{\frac{1}{\sqrt{2\pi}} \cdot 2^{j-\frac{1}{2}} \cdot \Gamma\left(\frac{j}{2}\right) \cdot \Gamma\left(\frac{j+1}{2}\right) \cdot \frac{1}{2} \cdot \sqrt{\pi}}{\Gamma\left(\frac{j+1}{2}\right) \cdot \frac{j}{2} \cdot \Gamma\left(\frac{j}{2}\right)} {}_2F_1\left(\begin{matrix} \frac{1-j}{2}, \ \frac{2-j}{2} \\ 1-j \end{matrix}; \frac{1}{9} \right) \\ =& \frac{2^{j-1}}{j} {}_2F_1\left(\begin{matrix} \frac{1-j}{2}, \ \frac{2-j}{2} \\ 1-j \end{matrix}; \frac{1}{9} \right). \end{aligned}$$

Putting the above value in (3), we get the following useful formula for balancing numbers

$$B_j = 6^{j-1} {}_2F_1\left(\begin{matrix} \frac{1-j}{2}, \frac{2-j}{2} \\ 1-j \end{matrix}; \frac{1}{9}\right), \ j \geq 1. \tag{5}$$

The Chebyshev polynomials of the first kind $\{T_j(x)\}_{j\in\mathbf{N}}$ and the second kind $\{U_j(x)\}_{j\in\mathbf{N}}$ are generated by means of the recurrence relations $T_j(x) = 2xT_{j-1}(x) - T_{j-2}(x),\ j > 1$ with initialization $T_0(x) = 1$ and $T_1(x) = x$ and $U_j(x) = 2xU_{j-1}(x) - U_{j-2}(x),\ j > 1$ with initialization $U_0(x) = 1$ and $U_1(x) = 2x$, respectively. The Chebyshev polynomials of the first and second kinds have the following explicit formulas

$$T_j(x) = \frac{j}{2}\sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^m}{j-m}\binom{j-m}{m}(2x)^{j-2m}, \ j \geq 1, \tag{6}$$

$$U_j(x) = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor}(-1)^m\binom{j-m}{m}(2x)^{j-2m}, \ j \geq 0. \tag{7}$$

The following identities hold by putting $x = 1$ and $x = \cos\theta$ in (6) and (7), respectively,

$$T_j(x) = 1, \qquad U_j(x) = j+1,$$

$$T_j(x) = \cos(j\theta), \qquad U_j(x) = \frac{\sin(j+1)\theta}{\sin\theta}. \tag{8}$$

The q-th derivative, $q \geq 1$ of $T_j(x)$ and $U_j(x)$ at $x = 1$ are defined as

$$D^qT_j(x)|_{x=1} = \prod_{m=0}^{q-1}\frac{(j-m)(j+m)}{2m+1} = \frac{j(j+q-1)!\sqrt{\pi}}{(j-q)!2^q\Gamma\left(q+\frac{1}{2}\right)} \tag{9}$$

and

$$D^qU_j(x)|_{x=1} = (j+1)\prod_{m=0}^{q-1}\frac{(j-m)(j+m+2)}{2m+3} = \frac{(j+q+1)!\sqrt{\pi}}{(j-q)!2^{q+1}\Gamma\left(q+\frac{3}{2}\right)}. \tag{10}$$

The following two results are found in [10].

Lemma 1 *Let j, r be non-negative integers, and let $\epsilon_j = 2 - \delta_{j,0}$. Then we have the following representations:*

$$\sum_{i_1+i_2+\cdots+i_{r+1}=j} B_{i_1+1}(x)B_{i_2+1}(x)\dots B_{i_{r+1}+1}(x)$$

$$= \frac{1}{r!}\sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \epsilon_{j-2m}\binom{r+m}{r}(j-m+r)_r T_{j-2m}(3x) \tag{11}$$

$$= \frac{1}{r!}\sum_{m=0}^{\lfloor \frac{j}{2} \rfloor}(j-2m+1)\binom{r+m-1}{r}(j-m+r)_{r-1}B_{j-2m+1}(x). \tag{12}$$

Lemma 2 *Let j, r be non-negative integers. Then we have the following expressions:*

$$\sum_{i_1+i_2+\cdots+i_{r+1}=j} B_{i_1+1}(x)B_{i_2+1}(x)\dots B_{i_{r+1}+1}(x)$$

$$= \frac{3^j(j+r)}{r!}\sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{\epsilon_{j-2m}}{m!(j-m)!}\,{}_2F_1\left(\begin{matrix}-m,\ m-j\\ -j-r\end{matrix};\frac{1}{9}\right)T_{j-2m}(x) \tag{13}$$

$$= \frac{3^j(j+r)}{r!}\sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(j-2m+1)}{m!(j-m+1)!}\,{}_2F_1\left(\begin{matrix}-m,\ m-j-1\\ -j-r\end{matrix};\frac{1}{9}\right)U_{j-2m}(x). \tag{14}$$

2 Connection formulas between balancing polynomials and Chebyshev polynomials of the first and second kinds

In this section, we give some useful results that are used subsequently in order to prove our main results Theorems 1–4. Basically, the results in Theorems 3 and 4 are an inversion of the results in Theorems 1 and 2.

Lemma 3 *For any two positive integers j and m, if we assume $A_{j,m} = {}_2F_1\left(\begin{matrix}-m,\ j-m\\ j-2m+2\end{matrix};\frac{1}{9}\right)$, then the following recurrence relation holds:*

$$\binom{j-m+1}{m}A_{j,m} = -\frac{1}{9}\binom{j-m+1}{m-1}A_{j-1,m-1} + \binom{j-m}{m-1}A_{j-2,m-1} + \binom{j-m}{m}A_{j-1,m}.$$

Proof. In order to prove the relation, we proceed as follows

$$\begin{aligned}
&-\frac{1}{9}\binom{j-m+1}{m-1}\sum_{k=0}^{m-1}\frac{(j-m)_k(-m+1)_k}{(j-2m+3)_k k!}\left(\frac{1}{9}\right)^k \\
&+\binom{j-m}{m-1}\sum_{k=0}^{m-1}\frac{(j-m-1)_k(-m+1)_k}{(j-2m+2)_k k!}\left(\frac{1}{9}\right)^k \\
&+\binom{j-m}{m}\sum_{k=0}^{m}\frac{(j-m-1)_k(-m)_k}{(j-2m+1)_k k!}\left(\frac{1}{9}\right)^k \\
= &-\frac{1}{9}\frac{(j-m+1)!}{(m-1)!(j-2m+2)!} \\
&\times\sum_{k=0}^{m-1}\frac{(j-m+k-1)!(-m+k)!(j-2m+2)!}{(j-m-1)!(-m)!(j-2m+k+2)!k!}\left(\frac{1}{9}\right)^k \\
&+\frac{(j-m)!}{(m-1)!(j-2m+1)!} \\
&\times\sum_{k=0}^{m-1}\frac{(j-m+k-2)!(-m+k)!(j-2m+1)!}{(j-m-2)!(-m)!(j-2m+k+1)!k!}\left(\frac{1}{9}\right)^k \\
&+\frac{(j-m)!}{m!(j-2m)!}\sum_{k=0}^{m}\frac{(j-m+k-2)!(-m+k-1)!(j-2m)!}{(j-m-2)!(-m-1)!(j-2m+k)!k!}\left(\frac{1}{9}\right)^k \\
= &-\frac{1}{9}\frac{(j-m+1)(j-m)}{(m-1)!(-m)!}\sum_{k=0}^{m-1}\frac{(j-m+k-1)!(m-k)!}{(j-2m+k+2)!k!}\left(\frac{1}{9}\right)^k \\
&+\frac{(j-m)(j-m-1)}{(m-1)!(-m)!}\sum_{k=0}^{m-1}\frac{(j-m+k-2)!(-m+k)!}{(j-2m+k+1)!k!}\left(\frac{1}{9}\right)^k \\
&+\frac{(j-m)(j-m-1)}{m!(-m-1)!}\sum_{k=0}^{m}\frac{(j-m+k-2)!(-m+k-1)!}{(j-2m+k)!k!}\left(\frac{1}{9}\right)^k.
\end{aligned}$$

The common denominator gives the following simplification:

$$\frac{(j-m+1)(j-m)}{(m-1)!(-m)!}\sum_{k=0}^{m-1}\frac{(j-m+k-1)!(m-k)!}{(j-2m+k+2)!k!}\left(\frac{1}{9}\right)^k,$$

which proves the result. □

The following results can be obtained in the same way as Lemma 3, and thus, their proofs will be omitted.

Lemma 4 *For any two positive integers* j *and* m*, if we assume* $N_{j,m} = {}_2F_1\left(\begin{matrix}-m,\ j-m-1\\ j-2m+1\end{matrix};\frac{1}{9}\right)$*, then the following recurrence relation holds:*

$$\binom{j-m+1}{m}N_{j+1,m} = -\frac{1}{9}\binom{j-m+1}{m-1}N_{j,m-1} + \binom{j-m}{m-1}N_{j-1,m-1} + \binom{j-m}{m}N_{j,m}.$$

Lemma 5 *For any two positive integers* j *and* m*, if we assume* $C_{j,m} = {}_2F_1\left(\begin{matrix}-m,\ -j+m-1\\ -j\end{matrix};9\right)$*, then the following recurrence relation holds:*

$$\binom{j}{m}(j-m)(j-2m+1)C_{j,m} = -9\binom{j-2}{m-1}(j-2m+1)(j-m+1)C_{j-2,m-1} + \binom{j-1}{m-1}(j-2m+2)(j-m)C_{j-1,m-1} + \binom{j-1}{m}(j-2m)(j-m+1)C_{j-1,m}.$$

Lemma 6 *For any two positive integers* j *and* m*, if we assume* $D_{j,m} = {}_2F_1\left(\begin{matrix}-m,\ -j+m\\ -j+1\end{matrix};9\right)$*, then the following recurrence relation holds:*

$$\binom{j}{m}(j-m)(j-2m+1)D_{j+1,m} = \binom{j-1}{m-1}(j-2m+2)(j-m)D_{j,m-1} + \binom{j-1}{m}(j-2m)(j-m+1)D_{j,m} - 9\binom{j-2}{m-1}(j-2m+1)(j-m+1)D_{j-1,m-1}.$$

Lemma 7 *For any two positive integers* j *and* m*, if we assume* $E_{j,m} = {}_2F_1\left(\begin{matrix}-m,\ j-m\\ j-2m\end{matrix};9\right)$*, then the following recurrence relation holds:*

$$\binom{j-m+1}{m}E_{j+2,m} = -9\binom{j-m+1}{m-1}E_{j+1,m-1} + \binom{j-m}{m-1}E_{j,m-1} + \binom{j-m}{m}E_{j+1,m}.$$

Lemma 8 *For any two positive integers* j *and* m*, if we assume* $F_{j,m} = {}_2F_1\left(\begin{matrix}-m,\ j-m-1\\ j-2m-1\end{matrix};9\right)$*, then the following recurrence relation holds:*

$$\binom{j-m+1}{m}F_{j+3,m} = -9\binom{j-m+1}{m-1}F_{j+2,m-1} + \binom{j-m}{m-1}F_{j+1,m-1} + \binom{j-m}{m}F_{j+2,m}.$$

Lemma 9 *For any two positive integers j and m, if we assume* $G_{j,m} = {}_2F_1\left(\begin{matrix}-m, & -j+m\\ & -j+1\end{matrix};\frac{1}{9}\right)$, *then the following recurrence relation holds:*

$$\binom{j}{m}(j-m)(j-2m+1)G_{j+1,m} = \binom{j-1}{m-1}(j-2m+2)(j-m)G_{j,m-1}$$
$$+\binom{j-1}{m}(j-2m)(j-m+1)G_{j,m}-\frac{1}{9}\binom{j-2}{m-1}(j-2m+1)(j-m+1)G_{j-1,m-1}.$$

Lemma 10 *For any two positive integers j and m, if we assume* $H_{j,m} = {}_2F_1\left(\begin{matrix}-m, & -j+m+1\\ & -j+2\end{matrix};\frac{1}{9}\right)$, *then the following recurrence relation holds:*

$$\binom{j}{m}(j-m)(j-2m+1)H_{j+2,m} = \binom{j-1}{m-1}(j-2m+2)(j-m)H_{j+1,m-1}$$
$$+\binom{j-1}{m}(j-2m)(j-m+1)H_{j+1,m}-\frac{1}{9}\binom{j-2}{m-1}(j-2m+1)(j-m+1)H_{j,m-1}.$$

Theorem 1 *For every non-negative integer j, the recurrence relation for the Chebyshev polynomials of the first kind can be written in terms of balancing polynomials by*

$$\begin{aligned}T_{j+1}(x) =& x\sum_{m=0}^{\lfloor\frac{j}{2}\rfloor}(-1)^m 3^{2m-j} j\binom{j-m}{m}\frac{1}{j-m}{}_2F_1\left(\begin{matrix}-m, \ j-m\\ j-2m+2\end{matrix};\frac{1}{9}\right)B_{j-2m+1}(x)\\ &+\frac{1}{2}\sum_{m=0}^{\lfloor\frac{j-1}{2}\rfloor}(-1)^{m+1}3^{2m-j+1}(j-1)\binom{j-m-1}{m}\frac{1}{j-m-1}\\ &\times {}_2F_1\left(\begin{matrix}-m, \ j-m-1\\ j-2m+1\end{matrix};\frac{1}{9}\right)B_{j-2m}(x). \end{aligned} \tag{15}$$

Proof. The proof of this result is based on induction on j. Clearly, the result is true for $j = 1$; assume that (15) holds for $n < j$. We need to show that

$$T_{j+1}(x) = \sum_{m=0}^{\lfloor\frac{j}{2}\rfloor} g_{j,m}B_{j-2m+1}(x) + \sum_{m=0}^{\lfloor\frac{j-1}{2}\rfloor} h_{j,m}B_{j-2m}(x),$$

where

$$g_{j,m} = x(-1)^m 3^{2m-j} j\binom{j-m}{m}\frac{1}{j-m}{}_2F_1\left(\begin{matrix}-m, \ j-m\\ j-2m+2\end{matrix};\frac{1}{9}\right)$$

and

$$\begin{aligned}h_{j,m} =&\frac{1}{2}(-1)^{m+1}3^{2m-j+1}(j-1)\binom{j-m-1}{m}\frac{1}{j-m-1}\\&\times {}_2F_1\left(\begin{matrix}-m,\ j-m-1\\ j-2m+1\end{matrix};\frac{1}{9}\right).\end{aligned}$$

Using the Chebyshev polynomials of the first kind followed by induction hypothesis, we get

$$\begin{aligned}T_{j+1}(x) =&2x\Big\{\sum_{m=0}^{\lfloor\frac{j-1}{2}\rfloor} g_{j-1,m}B_{j-2m}(x)+\sum_{m=0}^{\lfloor\frac{j}{2}-1\rfloor} h_{j-1,m}B_{j-2m-1}(x)\Big\}\\&-\Big\{\sum_{m=0}^{\lfloor\frac{j}{2}-1\rfloor} g_{j-2,m}B_{j-2m-1}(x)+\sum_{m=0}^{\lfloor\frac{j-3}{2}\rfloor} h_{j-2,m}B_{j-2m-2}(x)\Big\}.\end{aligned}$$

Using (2) in the above expression, we obtain

$$\begin{aligned}T_{j+1}(x) =&\frac{1}{3}\sum_{m=0}^{\lfloor\frac{j-1}{2}\rfloor} g_{j-1,m}B_{j-2m-1}(x)+\frac{1}{3}\sum_{m=0}^{\lfloor\frac{j-1}{2}\rfloor} g_{j-1,m}B_{j-2m+1}(x)\\&-\sum_{m=0}^{\lfloor\frac{j}{2}-1\rfloor} g_{j-2,m}B_{j-2m-1}(x)+\frac{1}{3}\sum_{m=0}^{\lfloor\frac{j}{2}-1\rfloor} h_{j-1,m}B_{j-2m-2}(x)\\&+\frac{1}{3}\sum_{m=0}^{\lfloor\frac{j}{2}-1\rfloor} h_{j-1,m}B_{j-2m}(x)-\sum_{m=0}^{\lfloor\frac{j-3}{2}\rfloor} h_{j-2,m}B_{j-2m-2}(x).\end{aligned}$$

For convenience, we write $T_{j+1}(x)=\sum_1+\sum_2$. These two sums can alternatively be written as

$$\begin{aligned}\sum_1 =&\sum_{m=0}^{\lfloor\frac{j-1}{2}\rfloor}\Big(\frac{1}{3}g_{j-1,m-1}+\frac{1}{3}g_{j-1,m}-g_{j-2,m-1}\Big)B_{j-2m+1}(x)\\&+\frac{1}{3}g_{j-1,\lfloor\frac{j-1}{2}\rfloor}B_{j-2\lfloor\frac{j-1}{2}\rfloor-1}(x)-g_{j-2,\lfloor\frac{j}{2}-1\rfloor}B_{j-2\lfloor\frac{j}{2}-1\rfloor-1}(x)\psi_j\end{aligned}\tag{16}$$

and

$$\begin{aligned}\sum_2 =&\sum_{m=0}^{\lfloor\frac{j}{2}-1\rfloor}\Big(\frac{1}{3}h_{j-1,m-1}+\frac{1}{3}h_{j-1,m}-h_{j-2,m-1}\Big)B_{j-2m}(x)\\&+\frac{1}{3}h_{j-1,\lfloor\frac{j}{2}-1\rfloor}B_{j-2\lfloor\frac{j}{2}\rfloor}(x)-h_{j-2,\lfloor\frac{j-3}{2}\rfloor}B_{j-2\lfloor\frac{j-1}{2}\rfloor}(x)\psi_{j+1},\end{aligned}\tag{17}$$

where

$$\psi_j = \begin{cases} 1, & \text{if } j \text{ is even;} \\ 0, & \text{if } j \text{ is odd.} \end{cases}$$

Using Lemmas 3 and 4, we obtain

$$\frac{1}{3}g_{j-1,m-1} + \frac{1}{3}g_{j-1,m} - g_{j-2,m-1} = g_{j,m} \tag{18}$$

and

$$\frac{1}{3}h_{j-1,m-1} + \frac{1}{3}h_{j-1,m} - h_{j-2,m-1} = h_{j,m}. \tag{19}$$

Applying (18) and (19) in (16) and (17) followed by some algebraic manipulations, we get

$$\sum_1 = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} g_{j,m} B_{j-2m+1}(x) \text{ and } \sum_2 = \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} h_{j,m} B_{j-2m}(x).$$

This completes the proof. □

The proof of the following result is similar to the proof of Theorem 1 using Lemmas 5 and 6.

Theorem 2 *For any non-negative integer j, the recurrence relation for the Chebyshev polynomials of the second kind can be written in terms of balancing polynomials by*

$$\begin{aligned} U_{j+1}(x) =& \frac{2}{3^j} x \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^{2m+1} \binom{j}{m} \frac{-j+2m-1}{j-m+1} {}_2F_1 \left(\begin{matrix} -m, & -j+m-1 \\ & -j \end{matrix} ; 9 \right) \\ & \times B_{j-2m+1}(x) + \frac{1}{3^{j-1}} \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} (-1)^{2m+2} \binom{j-1}{m} \frac{-j+2m}{j-m} \\ & \times {}_2F_1 \left(\begin{matrix} -m, & -j+m \\ & -j+1 \end{matrix} ; 9 \right) B_{j-2m}(x). \end{aligned}$$

Theorem 3 *For every non-negative integer j, the recurrence relation for balancing polynomials can be written in terms of the Chebyshev polynomials of the first kind by*

$$\begin{aligned} B_{j+1}(x) =& 4x \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} (-1)^m \frac{1}{\phi_{j-2m-1}} \binom{j-m-1}{m} 3^{j-2m} {}_2F_1 \left(\begin{matrix} -m, & j-m \\ & j-2m \end{matrix} ; 9 \right) \\ & \times T_{j-2m-1}(x) + 2 \sum_{m=0}^{\lfloor \frac{j-2}{2} \rfloor} (-1)^{m+1} \frac{1}{\phi_{j-2m-2}} \binom{j-m-2}{m} 3^{j-2m-2} \\ & \times {}_2F_1 \left(\begin{matrix} -m, & j-m-1 \\ & j-2m-1 \end{matrix} ; 9 \right) T_{j-2m-2}(x) \end{aligned} \tag{20}$$

where

$$\phi_j = \begin{cases} 2, & \textit{if } j = 0; \\ 1, & \textit{if } j > 0. \end{cases}$$

Proof. We will proceed by induction to prove this theorem. The result is true for $j = 1$; assume that (20) holds for $n < j$. We have to show that

$$B_{j+1}(x) = \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} e_{j,m} T_{j-2m-1}(x) + \sum_{m=0}^{\lfloor \frac{j-2}{2} \rfloor} f_{j,m} T_{j-2m-2}(x),$$

where

$$e_{j,m} = 4x(-1)^m \frac{1}{\phi_{j-2m-1}} \binom{j-m-1}{m} 3^{j-2m} {}_2F_1\left(\begin{matrix} -m, \ j-m \\ j-2m \end{matrix} ; 9 \right)$$

and

$$f_{j,m} = 2(-1)^{m+1} \frac{1}{\phi_{j-2m-2}} \binom{j-m-2}{m} 3^{j-2m-2} {}_2F_1\left(\begin{matrix} -m, \ j-m-1 \\ j-2m-1 \end{matrix} ; 9 \right).$$

Using the balancing polynomials followed by induction hypothesis, we get

$$\begin{aligned} B_{j+1}(x) =& 6x \Big\{ \sum_{m=0}^{\lfloor \frac{j}{2}-1 \rfloor} e_{j-1,m} T_{j-2m-2}(x) + \sum_{m=0}^{\lfloor \frac{j-3}{2} \rfloor} f_{j-1,m} T_{j-2m-3}(x) \Big\} \\ & - \Big\{ \sum_{m=0}^{\lfloor \frac{j-3}{2} \rfloor} e_{j-2,m} T_{j-2m-3}(x) + \sum_{m=0}^{\lfloor \frac{j}{2}-2 \rfloor} f_{j-2,m} T_{j-2m-4}(x) \Big\}. \end{aligned} \tag{21}$$

Substituting $T_j(x) = 2xT_{j-1}(x) - T_{j-2}(x)$, $j > 1$ in (21), we obtain

$$\begin{aligned} B_{j+1}(x) =& 3 \sum_{m=0}^{\lfloor \frac{j}{2}-1 \rfloor} e_{j-1,m} T_{j-2m-1}(x) + 3 \sum_{m=0}^{\lfloor \frac{j}{2}-1 \rfloor} e_{j-1,m} T_{j-2m-3}(x) \\ & - \sum_{m=0}^{\lfloor \frac{j-3}{2} \rfloor} e_{j-2,m} T_{j-2m-3}(x) + 3 \sum_{m=0}^{\lfloor \frac{j-3}{2} \rfloor} f_{j-1,m} T_{j-2m-2}(x) \\ & + 3 \sum_{m=0}^{\lfloor \frac{j-3}{2} \rfloor} f_{j-1,m} T_{j-2m-4}(x) - \sum_{m=0}^{\lfloor \frac{j}{2}-2 \rfloor} f_{j-2,m} T_{j-2m-4}(x). \end{aligned}$$

For convenience, we write $B_{j+1}(x) = \sum_1 + \sum_2$. These two sums can be written in the following alternative forms

$$\begin{aligned} \sum_1 =& \sum_{m=0}^{\lfloor \frac{j}{2}-1 \rfloor} (3e_{j-1,m} + 3e_{j-1,m-1} - e_{j-2,m-1}) T_{j-2m-1}(x) \\ & + 3e_{j-1,\lfloor \frac{j}{2}-1 \rfloor} T_{j-2\lfloor \frac{j}{2}-1 \rfloor-3}(x) - e_{j-2,\lfloor \frac{j-3}{2} \rfloor} T_{j-2\lfloor \frac{j-3}{2} \rfloor-3}(x) \eta_j \end{aligned} \tag{22}$$

and

$$\sum_{2} = \sum_{m=0}^{\lfloor \frac{j-3}{2} \rfloor} (3f_{j-1,m} + 3f_{j-1,m-1} - f_{j-2,m-1}) T_{j-2m-2}(x)$$
$$+ 3f_{j-1,\lfloor \frac{j-3}{2} \rfloor} T_{j-2\lfloor \frac{j-3}{2} \rfloor -4}(x) - f_{j-2,\lfloor \frac{j}{2}-2 \rfloor} T_{j-2\lfloor \frac{j}{2}-2 \rfloor -4}(x) \zeta_j, \tag{23}$$

where

$$\eta_j = \begin{cases} 1, & \text{if } j \text{ is odd;} \\ 0, & \text{if } j \text{ is even;} \end{cases} \quad \text{and} \quad \zeta_j = \begin{cases} 1, & \text{if } j \text{ is even;} \\ 0, & \text{if } j \text{ is odd.} \end{cases}$$

Using Lemmas 7 and 8, we have

$$3e_{j-1,m} + 3e_{j-1,m-1} - e_{j-2,m-1} = e_{j,m} \tag{24}$$

and

$$3f_{j-1,m} + 3f_{j-1,m-1} - f_{j-2,m-1} = f_{j,m}. \tag{25}$$

Applying (24) and (25) in (22) and (23) followed by some algebraic manipulations, we get

$$\sum_{1} = \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} e_{j,m} T_{j-2m-1}(x) \quad \text{and} \quad \sum_{2} = \sum_{m=0}^{\lfloor \frac{j-2}{2} \rfloor} f_{j,m} T_{j-2m-2}(x),$$

which completes the proof. □

The proof of the following result is similar to the proof of Theorem 3 using Lemmas 9 and 10.

Theorem 4 *For any non-negative integer j, the recurrence relation for balancing polynomials can be written in terms of the Chebyshev polynomials of the second kind by*

$$\begin{aligned} B_{j+1}(x) =& 2x \cdot 3^j \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} (-1)^{2m} \binom{j-1}{m} \frac{j-2m}{j-m} {}_2F_1 \left(\begin{matrix} -m, & -j+m \\ & -j+1 \end{matrix} ; \frac{1}{9} \right) \\ & \times U_{j-2m-1}(x) + 3^{j-2} \sum_{m=0}^{\lfloor \frac{j-2}{2} \rfloor} (-1)^{2m+1} \binom{j-2}{m} \frac{j-2m-1}{j-m-1} \\ & \times {}_2F_1 \left(\begin{matrix} -m, & -j+m+1 \\ & -j+2 \end{matrix} ; \frac{1}{9} \right) U_{j-2m-2}(x). \end{aligned}$$

Theorems 3 and 4 are nothing but the inversion relations of Theorems 1 and 2, respectively.

The following results are direct consequences of Theorems 1–4.

Corollary 1 *For every non-negative integer j, the following two recurrence identities hold:*

$$\sum_{m=0}^{\lfloor \frac{j}{2} \rfloor}(-1)^m 3^{2m-j} j\binom{j-m}{m}\frac{1}{j-m}{}_2F_1\left(\begin{matrix}-m,\ j-m\\ j-2m+2\end{matrix};\frac{1}{9}\right)B_{j-2m+1}$$
$$+\frac{1}{2}\sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor}(-1)^{m+1}3^{2m-j+1}(j-1)\binom{j-m-1}{m}\frac{1}{j-m-1}$$
$$\times\,{}_2F_1\left(\begin{matrix}-m,\ j-m-1\\ j-2m+1\end{matrix};\frac{1}{9}\right)B_{j-2m}=1$$

and

$$\frac{2}{3^j}\sum_{m=0}^{\lfloor \frac{j}{2} \rfloor}(-1)^{2m+1}\binom{j}{m}\frac{-j+2m-1}{j-m+1}{}_2F_1\left(\begin{matrix}-m,\quad -j+m-1\\ -j\end{matrix};9\right)B_{j-2m+1}$$
$$+\frac{1}{3^{j-1}}\sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor}(-1)^{2m+2}\binom{j-1}{m}\frac{-j+2m}{j-m}$$
$$\times\,{}_2F_1\left(\begin{matrix}-m,\quad -j+m\\ -j+1\end{matrix};9\right)B_{j-2m}=j+1.$$

Corollary 2 *For every non-negative integer j, the following results for balancing numbers in terms of hypergeometric connections are valid:*

$$B_{j+1}=4\sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor}(-1)^m\frac{1}{\phi_{j-2m-1}}\binom{j-m-1}{m}3^{j-2m}{}_2F_1\left(\begin{matrix}-m,\ j-m\\ j-2m\end{matrix};9\right)$$
$$+2\sum_{m=0}^{\lfloor \frac{j-2}{2} \rfloor}(-1)^{m+1}\frac{1}{\phi_{j-2m-2}}\binom{j-m-2}{m}3^{j-2m-2}{}_2F_1\left(\begin{matrix}-m,\ j-m-1\\ j-2m-1\end{matrix};9\right)$$

and

$$B_{j+1}=2\cdot 3^j\sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor}(-1)^{2m}\binom{j-1}{m}\frac{(j-2m)^2}{j-m}{}_2F_1\left(\begin{matrix}-m,\quad -j+m\\ -j+1\end{matrix};\frac{1}{9}\right)$$
$$+3^{j-2}\sum_{m=0}^{\lfloor \frac{j-2}{2} \rfloor}(-1)^{2m+1}\binom{j-2}{m}\frac{(j-2m-1)^2}{j-m-1}{}_2F_1\left(\begin{matrix}-m,\quad -j+m+1\\ -j+2\end{matrix};\frac{1}{9}\right).$$

It is instructive to note that when we compare the results in Corollary 2 with (1), we find

$$B_j=2\sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor}(-1)^m\frac{1}{\phi_{j-2m-1}}\binom{j-m-1}{m}3^{j-2m-1}{}_2F_1\left(\begin{matrix}-m,\ j-m\\ j-2m\end{matrix};9\right)\qquad(26)$$

and

$$B_j = 3^{j-1}\sum_{m=0}^{\lfloor\frac{j-1}{2}\rfloor}(-1)^{2m}\binom{j-1}{m}\frac{(j-2m)^2}{j-m}\,{}_2F_1\left(\begin{matrix}-m, & -j+m\\ & -j+1\end{matrix};\frac{1}{9}\right). \qquad (27)$$

The following linear transformation formulas are given in [3], p. 559:

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}a, & b\\ & c\end{matrix};z\right) =&(1-z)^{c-a-b}{}_2F_1\left(\begin{matrix}c-a, & c-b\\ & c\end{matrix};z\right),\\
{}_2F_1\left(\begin{matrix}a, & b\\ & c\end{matrix};z\right) =&(1-z)^{-a}{}_2F_1\left(\begin{matrix}a, & c-b\\ & c\end{matrix};\frac{z}{z-1}\right),\\
{}_2F_1\left(\begin{matrix}a, & b\\ & c\end{matrix};z\right) =&(1-z)^{-b}{}_2F_1\left(\begin{matrix}a, & c-a\\ & c\end{matrix};\frac{z}{z-1}\right).
\end{aligned}$$

These three formulas, together with (5), when applied to (26) and (27), yield several identities.

Using (8) in Theorems 3 and 4, we get the following results.

Corollary 3 *For every non-negative integer j, the following two recurrence identities hold:*

$$\begin{aligned}
B_{j+1}(\cos\theta) =&4\cos\theta\sum_{m=0}^{\lfloor\frac{j-1}{2}\rfloor}(-1)^m\frac{1}{\phi_{j-2m-1}}\binom{j-m-1}{m}3^{j-2m}\\
&\times{}_2F_1\left(\begin{matrix}-m, & j-m\\ & j-2m\end{matrix};9\right)\cos(j-2m-1)\theta\\
&+2\sum_{m=0}^{\lfloor\frac{j-2}{2}\rfloor}(-1)^{m+1}\frac{1}{\phi_{j-2m-2}}\binom{j-m-2}{m}3^{j-2m-2}\\
&\times{}_2F_1\left(\begin{matrix}-m, & j-m-1\\ & j-2m-1\end{matrix};9\right)\cos(j-2m-2)\theta
\end{aligned}$$

and

$$\begin{aligned}
B_{j+1}(\cos\theta) =&2\cos\theta\cdot 3^j\sum_{m=0}^{\lfloor\frac{j-1}{2}\rfloor}(-1)^{2m}\binom{j-1}{m}\frac{j-2m}{j-m}\\
&\times{}_2F_1\left(\begin{matrix}-m, & -j+m\\ & -j+1\end{matrix};\frac{1}{9}\right)\frac{\sin(j-2m)\theta}{\sin\theta}\\
&+3^{j-2}\sum_{m=0}^{\lfloor\frac{j-2}{2}\rfloor}(-1)^{2m+1}\binom{j-2}{m}\frac{j-2m-1}{j-m-1}\\
&\times{}_2F_1\left(\begin{matrix}-m, & -j+m+1\\ & -j+2\end{matrix};\frac{1}{9}\right)\frac{\sin(j-2m-1)\theta}{\sin\theta}.
\end{aligned}$$

It is observed that $T_j\Big((x+x^{-1})/2\Big) = (x^n+x^{-n})/2$, which yields the following result:

$$
\begin{aligned}
B_{j+1}\Big(\frac{x+x^{-1}}{2}\Big) =& 2\sum_{m=0}^{\lfloor\frac{j-1}{2}\rfloor}(-1)^m\frac{1}{\phi_{j-2m-1}}\binom{j-m-1}{m}3^{j-2m}\\
&\times {}_2F_1\left(\begin{matrix}-m,\ j-m\\ j-2m\end{matrix};9\right)(x^{j-2m}+x^{-j+2m+2})\\
&+\sum_{m=0}^{\lfloor\frac{j-2}{2}\rfloor}(-1)^{m+1}\frac{1}{\phi_{j-2m-2}}\binom{j-m-2}{m}3^{j-2m-2}\\
&\times {}_2F_1\left(\begin{matrix}-m,\ j-m-1\\ j-2m-1\end{matrix};9\right)(x^{j-2m-2}+x^{-j+2m+2}).
\end{aligned}
$$

The following result holds true by using (9) in Theorems 1 and 2.

Corollary 4 *For every $q \geq 0$, the qth derivative of the sum of two balancing polynomials gives the following values:*

$$
\begin{aligned}
&\sum_{m=0}^{\lfloor\frac{j}{2}\rfloor}(-1)^m 3^{2m-j} j\binom{j-m}{m}\frac{1}{j-m}{}_2F_1\left(\begin{matrix}-m,\ j-m\\ j-2m+2\end{matrix};\frac{1}{9}\right)B^q_{j-2m+1}\\
&+\frac{1}{2}\sum_{m=0}^{\lfloor\frac{j-1}{2}\rfloor}(-1)^{m+1}3^{2m-j+1}(j-1)\binom{j-m-1}{m}\frac{1}{j-m-1}\\
&\times {}_2F_1\left(\begin{matrix}-m,\ j-m-1\\ j-2m+1\end{matrix};\frac{1}{9}\right)B^q_{j-2m} = \frac{(j+1)(j+q)!\sqrt{\pi}}{(j-q+1)!2^q\Gamma\Big(q+\frac{1}{2}\Big)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{2}{3^j}\sum_{m=0}^{\lfloor\frac{j}{2}\rfloor}(-1)^{2m+1}\binom{j}{m}\frac{-j+2m-1}{j-m+1}{}_2F_1\left(\begin{matrix}-m,\ \ -j+m-1\\ -j\end{matrix};9\right)B^q_{j-2m+1}\\
&+\frac{1}{3^{j-1}}\sum_{m=0}^{\lfloor\frac{j-1}{2}\rfloor}(-1)^{2m+2}\binom{j-1}{m}\frac{-j+2m}{j-m}{}_2F_1\left(\begin{matrix}-m,\ \ -j+m\\ -j+1\end{matrix};9\right)B^q_{j-2m}\\
&=\frac{(j+q+2)!\sqrt{\pi}}{(j-q+1)!2^{q+1}\Gamma\Big(q+\frac{3}{2}\Big)}.
\end{aligned}
$$

Using (10) in Theorems 3 and 4, we get the following statements.

Corollary 5 *For every $q \geq 0$, the qth derivative of balancing polynomials gives the following results:*

$$
\begin{aligned}
B_{j+1}^{q} = & \frac{(j-2m-1)(j-2m+q-2)!\sqrt{\pi}}{(j-2m-q-1)!2^{q-2}\Gamma\left(q+\frac{1}{2}\right)} \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} (-1)^{m} \frac{1}{\phi_{j-2m-1}} \binom{j-m-1}{m} \\
& \times 3^{j-2m} {}_2F_1\left(\begin{matrix} -m,\ j-m \\ j-2m \end{matrix}; 9\right) + \frac{(j-2m-2)(j-2m+q-3)!\sqrt{\pi}}{(j-2m-q-2)!2^{q-1}\Gamma\left(q+\frac{1}{2}\right)} \\
& \times \sum_{m=0}^{\lfloor \frac{j-2}{2} \rfloor} (-1)^{m+1} \frac{1}{\phi_{j-2m-2}} \binom{j-m-2}{m} 3^{j-2m-2} {}_2F_1\left(\begin{matrix} -m,\ j-m-1 \\ j-2m-1 \end{matrix}; 9\right)
\end{aligned}
$$

and

$$
\begin{aligned}
B_{j+1}^{q} = & \frac{3^{j}(j-2m+q)!\sqrt{\pi}}{(j-2m-q-1)!2^{q}\Gamma\left(q+\frac{3}{2}\right)} \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} (-1)^{2m} \binom{j-1}{m} \frac{j-2m}{j-m} \\
& \times {}_2F_1\left(\begin{matrix} -m,\ -j+m \\ -j+1 \end{matrix}; \frac{1}{9}\right) + \frac{3^{j-2}(j-2m+q-1)!\sqrt{\pi}}{(j-2m-q-2)!2^{q+1}\Gamma\left(q+\frac{3}{2}\right)} \\
& \times \sum_{m=0}^{\lfloor \frac{j-2}{2} \rfloor} (-1)^{2m+1} \binom{j-2}{m} \frac{j-2m-1}{j-m-1} {}_2F_1\left(\begin{matrix} -m,\ -j+m+1 \\ -j+2 \end{matrix}; \frac{1}{9}\right).
\end{aligned}
$$

Corollary 6 *Let j and r be non-negative integers. Then*

$$
\begin{aligned}
T_{j+1}(x) = & r!x \sum_{i_1+i_2+\cdots+i_{r+1}=j} \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} B_{i_1+1}(x) B_{i_2+1}(x) \cdots B_{i_{r+1}+1}(x) \\
& \times \frac{(-1)^{m} 3^{2m-j} j \binom{j-m}{m} \frac{1}{j-m} {}_2F_1\left(\begin{matrix} -m,\ j-m \\ j-2m+2 \end{matrix}; \frac{1}{9}\right)}{(j-2m+1)\binom{r+m-1}{r-1}(j-m+r)_{r-1}} \\
& + \frac{1}{2} r! \sum_{i_1+i_2+\cdots+i_{r+1}=j-1} \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} B_{i_1+1}(x) B_{i_2+1}(x) \cdots B_{i_{r+1}+1}(x) \\
& \times \frac{(-1)^{m+1} 3^{2m-j+1} (j-1) \binom{j-m-1}{m} \frac{1}{j-m-1} {}_2F_1\left(\begin{matrix} -m,\ j-m-1 \\ j-2m+1 \end{matrix}; \frac{1}{9}\right)}{(j-2m)\binom{r+m-1}{r-1}(j-m+r-1)_{r-1}}
\end{aligned}
$$

and

$$
\begin{aligned}
U_{j+1}(x) =&\frac{2}{3^j}xr! \sum_{i_1+i_2+\cdots+i_{r+1}=j} \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} B_{i_1+1}(x)B_{i_2+1}(x)\cdots B_{i_{r+1}+1}(x) \\
&\times \frac{(-1)^{2m+1}\binom{j}{m}\frac{-j+2m-1}{j-m+1}{}_2F_1\left(\begin{matrix}-m, & -j+m-1 \\ & -j\end{matrix};9\right)}{(j-2m+1)\binom{r+m-1}{r-1}(j-m+r)_{r-1}} \\
&+ \frac{1}{3^{j-1}}r! \sum_{i_1+i_2+\cdots+i_{r+1}=j-1} \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} B_{i_1+1}(x)B_{i_2+1}(x)\cdots B_{i_{r+1}+1}(x) \\
&\times \frac{(-1)^{2m+2}\binom{j-1}{m}\frac{-j+2m}{j-m}{}_2F_1\left(\begin{matrix}-m, & -j+m \\ & -j+1\end{matrix};9\right)}{(j-2m)\binom{r+m-1}{r-1}(j-m+r-1)_{r-1}}.
\end{aligned}
$$

Proof. The proof of the statement is arising from Theorems 1 and 2 using the expression (12). □

Using the expressions (11) and (13) in Theorem 3, we get the following result.

Corollary 7 *Let j and r be non-negative integers, and let $\epsilon_j = 2 - \delta_{j,0}$. Then the following relations hold true:*

$$
\begin{aligned}
B_{j+1}(x) =&4r!x \sum_{i_1+\cdots+i_{r+1}=j-1} \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} B_{i_1+1}\left(\frac{x}{3}\right)\cdots B_{i_{r+1}+1}\left(\frac{x}{3}\right) \\
&\times \frac{(-1)^m 3^{j-2m}\frac{1}{\phi_{j-2m-1}}\binom{j-m-1}{m}{}_2F_1\left(\begin{matrix}-m, & j-m \\ & j-2m\end{matrix};9\right)}{\epsilon_{j-2m-1}\binom{r+m}{r}(j-m+r-1)_r} \\
&+ 2r! \sum_{i_1+\cdots+i_{r+1}=j-2} \sum_{m=0}^{\lfloor \frac{j-2}{2} \rfloor} B_{i_1+1}\left(\frac{x}{3}\right)\cdots B_{i_{r+1}+1}\left(\frac{x}{3}\right) \\
&\times \frac{(-1)^{m+1} 3^{j-2m-2}\frac{1}{\phi_{j-2m-2}}\binom{j-m-2}{m}{}_2F_1\left(\begin{matrix}-m, & j-m-1 \\ & j-2m-1\end{matrix};9\right)}{\epsilon_{j-2m-2}\binom{r+m}{r}(j-m+r-2)_r}
\end{aligned}
$$

and

$$B_{j+1}(x) = \frac{4xr!}{3^{j-1}(j+r-1)!} \sum_{i_1+\cdots+i_{r+1}=j-1} \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} B_{i_1+1}(x)\cdots B_{i_{r+1}+1}(x)$$
$$\times \frac{(-1)^m \frac{1}{\phi_{j-2m-1}} \binom{j-m-1}{m} 3^{j-2m} {}_2F_1\left(\begin{matrix}-m, \; j-m \\ j-2m\end{matrix}; 9\right)}{\frac{\epsilon_{j-2m-1}}{(j-m-1)!m!} {}_2F_1\left(\begin{matrix}-m, \; -j+m+1 \\ -j-r+1\end{matrix}; \frac{1}{9}\right)}$$
$$+ \frac{2r!}{3^{j-2}(j+r-2)!} \sum_{i_1+\cdots+i_{r+1}=j-2} \sum_{m=0}^{\lfloor \frac{j-2}{2} \rfloor} B_{i_1+1}(x)\cdots B_{i_{r+1}+1}(x)$$
$$\times \frac{(-1)^{m+1} \frac{1}{\phi_{j-2m-2}} \binom{j-m-2}{m} 3^{j-2m-2}}{\frac{\epsilon_{j-2m-2}}{(j-m-2)!m!} {}_2F_1\left(\begin{matrix}-m, \; -j+m+2 \\ -j-r+2\end{matrix}; \frac{1}{9}\right)} {}_2F_1\left(\begin{matrix}-m, \; j-m-1 \\ j-2m-1\end{matrix}; 9\right).$$

The next result is obtained by using expression (14) in Theorem 4.

Corollary 8 *Let j and r be non-negative integers. Then the following result holds true:*

$$B_{j+1}(x) = \frac{6xr!}{(j+r-1)!} \sum_{i_1+i_2+\cdots+i_{r+1}=j-1} \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} B_{i_1+1}(x)B_{i_2+1}(x)\cdots B_{i_{r+1}+1}(x)$$
$$\times \frac{(-1)^{2m} \binom{j-1}{m} \frac{j-2m}{j-m} {}_2F_1\left(\begin{matrix}-m, \; -j+m \\ -j+1\end{matrix}; \frac{1}{9}\right)}{\frac{j-2m}{(j-m)!m!} {}_2F_1\left(\begin{matrix}-m, \; -j+m \\ -j-r+1\end{matrix}; \frac{1}{9}\right)}$$
$$+ \frac{r!}{(j+r-2)!} \sum_{i_1+i_2+\cdots+i_{r+1}=j-2} \sum_{m=0}^{\lfloor \frac{j-2}{2} \rfloor} B_{i_1+1}(x)B_{i_2+1}(x)\cdots B_{i_{r+1}+1}(x)$$
$$\times \frac{(-1)^{2m+1} \binom{j-2}{m} \frac{j-2m-1}{j-m-1} {}_2F_1\left(\begin{matrix}-m, \; -j+m+1 \\ -j+2\end{matrix}; \frac{1}{9}\right)}{\frac{j-2m-1}{(j-m-1)!m!} {}_2F_1\left(\begin{matrix}-m, \; -j+m+1 \\ -j-r+2\end{matrix}; \frac{1}{9}\right)}.$$

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