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On k-deficit Banach Frames

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Abstract

k-deficit Banach frames have been defined and studied. It has been proved that if a Banach space has a Banach frame having k-deficit ($k \geq 0$), then its second conjugate space has a retro Banach frame. Also, we prove results regarding existence of k-deficit Banach frames in subspaces and super spaces of a Banach space and deduce that ℓ^∞ does not have a k-deficit Banach frame for any k. Finally, we prove the equivalence of two statements regarding Banach frames.

Key Words: Banach Frame, retro-Banach frames, deficit Banach frames.
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1 Introduction

The *excess* of the frame is the greatest integer n such that n elements can be deleted from the frame and still leave a complete set, or ∞ if there is no upper bound to the number of elements that can be removed. In the former case, it can be shown that the frame is simply a Riesz basis to which finitely many elements have been adjoined [6]. Such frames are called "near Riesz bases" and behave in many respects like Riesz bases. A frame with infinite excess need not contain a Riesz basis as a subset; an example was constructed in [2].

R. Balan et al. [1] studied deficits and excesses of frames in Hilbert spaces and proved that if $\mathcal{F}$ is any complete sequence in a Banach space which has infinite excess, then there exists a countably infinite subset $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{F} \setminus \mathcal{G}$ is complete. They also related the concepts of deficit and excess of a Bessel sequence to the dimension of the kernels of the analysis operator and the synthesis operator associated with the Bessel sequence. They proved several other results and applied them to the specific cases of Weyl-Heisenberg and wavelet system.

The concept of *excess in Banach spaces was defined and studied in [12].

In the present paper, we define and study k-deficit Banach frames and prove that if E has a Banach frame having k-deficit, then E^{**} has a retro Banach frame. Also, if a Banach space E has a Banach frame having k-deficit ($k \geq 0$), then every closed subspace of E also has a Banach frame having k_1-deficit, for some $k_1 \leq k$. Further, it has been proved that if a Banach space E has a k-deficit Banach frame, then every superspace of E with finite co-dimension also has a k-deficit Banach frame. Finally, we prove the equivalence of two statements regarding Banach frames.

2 Preliminaries

Throughout this paper E will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* the conjugate space of E, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E, $[\widetilde{f_n}]$ the closed linear span of $\{f_n\}$ in the $\sigma(E^*, E)$-topology, E_d the associated Banach space of the scalars-valued sequences indexed by $\mathbb{N}$.

A sequence $\{f_n\}$ in E^* is said to be total over E if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$.

The following result which is refered in this paper is listed in the form of a lemma.

Lemma 1 ([9])**.** *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$.*

Definition 2 ([5])**.** Let E be a Banach space and E_d an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. Then the pair $(\{f_n\}, S)$ is called a *Banach frame* for E with respect to E_d if

(i) $\{f_n(x)\} \in E_d$, for each $x \in E$

(ii) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E \tag{1}$$

(iii) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \quad x \in E.$$

The positive constants A and B, respectively, are called lower and upper frame bounds of the Banach frame $(\{f_n\}, S)$. The operator $S : E_d \to E$ is called the reconstruction operator (or the pre-frame operator). The inequality (2.1) is called the frame inequality. It is easy to observe that frame bounds need not be unique. Further, if $T : E \to E_d$ is the coefficient map given by $T(x) = \{f_n(x)\}, x \in E$, then $(\|S\|)^{-1}$ and $\|T\|$ satisfying $A \leq \|S\|^{-1} \leq \|T\| \leq B$, are also frame bounds for the Banach frame $(\{f_n\}, S)$.

The Banach frame $(\{f_n\}, S)$ is called tight if $A = B$ and normalized tight if $A = B = 1$. Also, the Banach frame $(\{f_n\}, S)$ is said to be *exact* if there exists no reconstruction operator S_0 such that $(\{f_n\}_{n\neq i}, S_0)$ $(i \in \mathbb{N})$ is a Banach frame for E.

For more results and concepts regarding frames in Banach spaces, one may refer to [3, 4, 10, 11].

Finally, in this section, we give the following definition of a retro Banach frame introduced in [7].

Definition 3. Let E be a Banach space and E^* be its conjugate space. Let $(E^*)_d$ be a Banach space of scaler valued sequences associated with E^* indexed by $\mathbb{N}$. Let $\{x_n\} \subset E$ and $T : (E^*)_d \to E^*$ be given. The pair $(\{x_n\}, T)$ is called a *retro Banach frame* (RBF) for E^* with respect to $(E^*)_d$ if

(i) $\{f(x_n)\} \in (E^*)_d$, for each $f \in E^*$,

(ii) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B\|f\|_{E^*}, \quad f \in E^* \tag{2}$$

(iii) T is a bounded linear operator such that

$$T(\{f(x_n)\}) = f, \quad f \in E^*.$$

The positive constant A and B, respectively, are called the lower and the upper frame bounds of the retro Banach frame $(\{x_n\}, T)$. The operator $T : (E^*)_d \to E^*$ is called the reconstruction operator (or, the pre-frame operator). The inequality (2.2) is called the retro frame inequality. As in case of Banach frames, the retro Banach frame is called *tight* if $A = B$ and is said to be *exact* if there exists no reconstruction operator T_0 such that $(\{x_n\}_{i\neq i}, T_0)$ $(i \in \mathbb{N})$ is a retro Banach frame for E^*. Further, in view of Lemma 3.1 in [7], if $(\{x_n\}, T)$ is an exact retro Banach frame for E^*, then there exists a sequence $\{f_n\} \subset E^*$ such that $f_i(x_j) = \delta_{ij}$ (Kronecker delta), for all $i, j \in \mathbb{N}$. The seqeuence $\{f_n\} \subset E^*$ is called an *admissible sequence* to the retro Banach frame $(\{x_n\}, T)$.

3 Main Results

Definition 4. A Banach frame $(\{f_n\}, S)$ $(\{f_n\} \subset E, S : E_d \to E)$ is said to have *k-deficit* if there exists an integer $k \geq 0$ such that every $(k + 1)$ dimensional subspace A_{k+1} of E^* meets $[f_n]$ non-trivially and there exists a k-dimensional subspace B_k of E^* such that $B_k \cap [f_n] = \{0\}$.

Regarding existence of k-deficit Banach frames, we have

Example 5. (a) Let $E = c_0$ and let $\{e_n\}$ be the sequence of unit vectors in E and $\{g_n\}$ be the sequence of unit vectors in E^*. Then, by Lemma 1, there exists an associated Banach space $E_d = \{\{g_n(x)\} : x \in E\}$ and a reconstruction operator $T : E_d \to E$ such that $(\{g_n\}, T)$ is a Banach frame for E with respect to E_d. Also, $g_i(e_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Further, since $[g_n] = E^*$, $(\{g_n\}, T)$ is a 0-deficit Banach frame.

(b) Let $E = \ell^1$ and let $\{e_n\}$ be the sequence of unit vectors in E and $\{f_n\}$ be the sequence of unit vectors in E^*. Then, as in Example 5(a), there exists an associated Banach space E_d and a reconstruction operator $S : E_d \to E$ such that $(\{f_n\}, S)$ is a Banach frame for E with respect to E_d. Also, $f_i(e_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. But $[f_n] \neq E^*$. So, $(\{f_n\}, S)$ is not a 0-deficit Banach frame for E.

(c) Let J be the James space [13]. Let $\{e_n\}$ be the sequence of unit vectors in J. Define $\{y_n\} \subset J$ and $\{g_n\} \subset J^*$ by

$$y_n = \sum_{i=1}^{n} e_i\,, \qquad n \in \mathbb{N}$$
$$g_n = e_n - e_{n+1}\,, \qquad n \in \mathbb{N}$$

Then $g_i(y_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Also, $\{g_n\}$ is total over J. So, by Lemma 1, there exists an associated Banach space $J_d = \{\{g_n(x)\} : x \in J\}$ with norm $\|\{g_n(x)\}\|_{J_d} = \|x\|_J$, $x \in J$ and a reconstruction operator $T : J_d \to J$ such that $(\{g_n\}, T)$ is a Banach frame for J with respect to J_d. Further, since $e_1 \notin [g_n]$, $e_n \in [g_n]$ for all $n = 2, 3, \ldots$ and $[e_n] = J^*$, it follows that $\dim J^* - \dim[g_n] = 1$. Hence $(\{g_n\}, T)$ is a 1-deficit Banach frame for J.

Theorem 6. *If E has a k-deficit Banach frame $(k > 0)$, then E^{**} has a retro Banach frame with respect to some associated Banach space.*

Proof. Let $(\{f_n\}, S)$ be a Banach frame for E having k-deficit. Let $\{g_i\}_{i=1}^k \subset E^*$ be a linearly independent set such that $E^* = [f_n] \oplus [g_i]_{i=1}^k$. Define a sequence $\{h_n\} \subset E^*$ by

$$\begin{cases} h_1 = g_k \\ h_n = f_{n-1}, \quad n = 2, 3, \ldots \end{cases}$$

Then $\{h_n\}$ is a finitely linearly independent sequence in E^*. If $h_n(x) = 0$, for all $n \in \mathbb{N}$, then, by the frame inequality for the Banach frame $(\{f_n\}, S), x = 0$. Therefore, by Lemma 1, there exists an associated Banach space $E_{d_0} = \{\{h_n(x)\} : x \in E\}$ with norm given by $\|\{h_n(x)\}\|_{E_{d_0}} = \|x\|_E$, $x \in E$ together with a reconstruction operator $S_0 : E_{d_0} \to E$ given by $S_0(\{h_n(x)\}) = x, x \in E$ such that $(\{h_n\}, S_0)$ is a Banach frame for E with respect to E_{d_0}. Since $(\{f_n\}, S)$ have k-deficit, $\dim(E^* \backslash [f_n]) = k$. So $\dim(E^* \backslash [h_n]) = k-1$. Then, there exists a $(k-1)$-dimensional subspace B of E^* such that $B \cap [h_n] = \{0\}$. If A is any k-dimensional subspace of E^*, then $A \cap [h_n] \neq \{0\}$ because otherwise $\dim(E^* \setminus [h_n]) = k$. Hence $(\{h_n\}, S_0)$ is a $(k-1)$-deficit Banach frame for E. Repeating the above process $(k-1)$-times, we obtain a sequence $\{\phi_n\} \subset E^*$ and a bounded linear operator $T : \{\{\phi_n(x)\} : x \in E\} \to E$ such that $(\{\phi_n\}, T)$ is a 0-deficit Banach frame for E. This gives $[\phi_n] = E^*$. Hence, by Lemma 1, there exits an associated Banach space $(E^{**})_d = \{\{\phi(g_n)\} : \phi \in E^{**}\}$ with norm given by $\|\{\phi(g_n)\}\|_{(E^{**})_d} = \|\phi\|_{E^{**}}$, $\phi \in E^{**}$ together with a reconstruction operator $\cup : (E^{**})_d \to E^{**}$ given by $\cup(\{\phi(g_n)\}) = \phi$, $\phi \in E^{**}$ such that $(\{g_n\}, \cup)$ is a retro Banach frame for E^{**} with respect to $(E^{**})_d$. □

Next, we prove that if a Banach space has a Banach frame having k-deficit ($k \geq 0$), then every closed subspace of it also has a Banach frame having k_1-deficit ($k_1 \leq k$).

Theorem 7. *If E has a Banach frame having k-deficit ($k \geq 0$), then every closed subspace G of E has a Banach frame having k_1-deficit, for some $k_1 \leq k$.*

Proof. Let $(\{f_n\}, S)(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E having k-deficit ($k \geq 0$). Then, there exists a k-dimensional subspace B_k of E^* such that $E^* = [f_n] \oplus B_k$. Let G be a closed subspace of E. Let $g_n = f_n|_G$, $n \in \mathbb{N}$. Then $\{g_n\} \subset G^*$. If $x \in G$ is such that $g_n(x) = 0$ for all $n \in \mathbb{N}$, then $f_n(x) = 0$, $n \in \mathbb{N}$. So, by the frame inequality for the Banach frame $(\{f_n\}, S)$, $x = 0$. Therefore, by Lemma 1, there exists an associated Banach space $G_d = \{\{g_n(x)\} : x \in G\}$ with norm $\|\{g_n(x)\|_{G_d} = \|x\|_G$, $x \in G$, together with a reconstruction operator $T : G_d \to G$ such that $(\{g_n\}, T)$ is a Banach frame for G. Further, we have $G^* = [g_n] \oplus A_{k_1}$, where A_{k_1} is a k_1-dimensional subspace of G^* with $k_1 = \dim D \leq k$, where $D = \{f \in G^* : f \in \phi|_G, \forall\ \phi \in B_k\}$. □

Corollary 8. *The Banach space ℓ^∞ does not have a k-deficit Banach frame for any k.*

Note. The Banach space ℓ^∞ does have a Banach frame [8].

The next result is regarding the existence of a Banach frame having k-deficit Banach frame in every superspace with finite co-dimension, of a Banach space having k-deficit Banach frame.

Theorem 9. *If E has a k-deficit Banach frame, then every superspace X of E with finite co-dimension also has a k-deficit Banach frame.*

Proof. Let $(\{f_n\}, S)(\{f_n\} \subset E^*, S : E_d \to E)$ be a k-deficit Banach frame for E. Let $\dim X/E = m$ and let F be an m-dimensional subspace of X such that $X = E \oplus F$. Let $(\{y_i\}_{i=1}^m, U)$ be an exact retro Banach frame for F^* with admissible sequence $\{g_i\}_{i=1}^m \subset F^*$. For each $n \in \mathbb{N}$, let ϕ_n be the extension of f_n to X such that $\phi_n(y) = 0$, $n \in \mathbb{N}$; $y \in F$ and for each $j = 1, 2, \ldots, m$, let ψ_j be the extension of g_j to X such that $\psi_j(z) = 0$, for all

$j = 1, 2, \ldots, m$; $z \in E$. Define $\{h_n\} \subset E^*$ by

$$h_i = \begin{cases} \psi_i, & i = 1, 2, \ldots, m \\ \phi_{i-m}, & i \geq m+1 \end{cases}$$

Let $x \in X$ be such that $h_n(x) = 0$, for all $n \in \mathbb{N}$. Since $X = E \oplus F$, there exists an $x_0 \in E$ and $y_0 \in F$ such that $x = x_0 + y_0$. Then

$$\begin{cases} \psi_i(x_0 + y_0) = 0, & i = 1, 2, \ldots, m \\ \phi_{i-m}(x_0 + y_0) = 0, & i \geq m+1 \end{cases}$$

So

$$\begin{cases} g_i(y_0) = 0, & i = 1, 2, \ldots, m \\ f_j(x_0) = 0, & j = 1, 2, \ldots \end{cases}$$

This gives $x = 0$. Therefore, by Lemma 1, there exists an associated Banach space $X_d = \{\{h_n(x)\} : x \in X\}$ with norm $\|\{h_n(x)\}\|_{X_d} = \|x\|_X, x \in X$ together with a reconstruction operator $S_1 : X_d \to X$ given by $S_1(\{h_n(x)\}) = x$, $x \in X$ such that $(\{h_n\}, S_1)$ is a Banach frame for X with respect to X_d. Further, since $(\{f_n\}, S)$ is a k-deficit Banach frame for E and $\dim F^* = m$, it follows that $(\{h_n\}, S_1)$ is a k-deficit Banach frame for X. □

The following result gives a characterization of Banach frames having zero deficit.

Theorem 10. *For a sequence $\{f_n\} \subset E^*$, there exists an associated Banach space E_d and a reconstruction operator $S : E_d \to E$ such that $(\{f_n\}, S)$ is a Banach frame for E having zero deficit if and only if there exists an associated Banach space $(E^{**})_d$ together with a reconstruction operator $U : (E^{**})_d \to E^{**}$ such that $(\{f_n\}, U)$ is a retro Banach frame for E^{**}.*

Proof. Suppose first that $(\{f_n\}, U)$ is a retro Banach frame for E^{**}. Then, by Theorem 3.1 in [7], $[f_n] = E^*$. Therefore, by Lemma 1, there exists an associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$ and a reconstruction operator $S : E_d \to E$ such that $(\{f_n\}, S)$ is a Banach frame for E with respect to E_d.

The other part of the result follows in view of the arguments used in Theorem 6. □

Finally, we prove the equivalence of two statements regarding Banach frames.

Theorem 11. *Let $(\{f_n\}, T)$ $(\{f_n\} \subset E^*, T : E_d \to E)$ be a Banach frame for E with respect to E_d. Let u be the canonical mapping of E into $[f_n]^*$. Then, for an integer $k \geq 0$, $\dim([f_n]^* \backslash u(E)) = k$ if and only if there exists a linearly independent set $\{g_i\}_{i=1}^k \subset [f_n]^* \backslash u(E)$ such that, for every bounded $\sigma(E, [f_n])$-Cauchy sequence $\{z_j\} \subset E$, there exists a unique $x \in E$ and a uniquely determined finite set $\{\alpha_i\}_{i=1}^k$ of scalars satisfying*

$$\lim_{j \to \infty} f_n(z_j) = f_n(x) + \sum_{i=1}^{k} \alpha_i g_i(f_n), \quad n \in \mathbb{N}$$

Proof. Since $(\{f_n\}, T)$ is a Banach frame for E with respect to E_d, there exists constants A, B with $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|^2, \quad x \in E \tag{3}$$

Suppose first that $\dim([f_n]^* \setminus u(E)) = k$. Let $\{g_i\}_{i=1}^k$ be a linearly independent set in $[f_n]^* \setminus u(E)$ such that

$$[f_n]^* = [g_i]_{i=1}^k \oplus u(E). \tag{4}$$

Let $\{z_j\}$ be a bounded $\sigma(E, [f_n])$-Cauchy sequence in E. Then $\sup\limits_{1 \leq j < \infty} \|u(z_j)\| < \infty$. Since $[f_n]$ is separable, there exists a subsequence $\{u(z_{j_k})\}$ of $\{u(z_j)\}$ and a $g \in [f_n]^*$ such that

$$g = \sigma([f_n]^*, [f_n]) - \lim_{k \to \infty} u(z_{j_k})\,.$$

This gives

$$\begin{aligned} g(f) &= \lim_{k \to \infty} u(z_{j_k})(f) \\ &= \lim_{k \to \infty} f(z_k), \quad \text{for all} \quad f \in [f_n]. \end{aligned}$$

Therefore, by (3.1) and (3.2), there exists a unique $x \in E$ and unique scalars $\{\alpha_i\}_{i=1}^k \subset \mathbb{K}$ such that

$$\lim_{k \to \infty} f(z_k) = g(f) = f(x) + \sum_{i=1}^k \alpha_i g_i(f), \quad f \in [f_n].$$

Conversely, let $\phi \in [f_n]^* \setminus u(E)$ be any element such that $\|\phi\| = 1$. Let $\phi_0 \in S_{E^{**}}$ such that $\phi = \phi_0|_{[f_n]}$. Since S_E is $\sigma([f_n]^*, [f_n])$-dense in $S_{[f_n]^*}$, there exists a sequence $\{z_j\}$ in S_E such that$\{u(z_j)\}$ is $\sigma([f_n]^*, [f_n])$-conergent to ϕ. Then $\phi(f_n) = \lim\limits_{j \to \infty} f_n(z_j)$,$n \in \mathbb{N}$. So $\{z_j\}$ is a bounded $\sigma(E, [f_n])$-Cauchy sequence in E. Therefore, there exists an $x \in E$ and unique scalers $\{\alpha_i\}_{i=1}^k \subset \mathbb{K}$ such that

$$\phi(f_n) = f_n(x) + \sum_{i=1}^k \alpha_i g_i(f_n), \quad n \in \mathbb{N}$$

This gives $\phi = u(x) + \sum_{i=1}^k \alpha_i g_i$. So, we have

$$\phi + u(E) = \sum_{i=1}^k \alpha_i (g_i + u(E)).$$

This can be done for each $\phi \in [f_n]^* \setminus u(E)$. Hence $\dim([f_n]^* \setminus u(E)) = k$.

□

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References

[1] R. Balan, P.G. Casazza, C. Heil and Z. Landau, Deficits and excesses of frames, *Adv. Comp. Math.*, **18**, 2003, pp. 93–116.

[2] P.G. Casazza and O. Christensen, Frames containing a Riesz basis and preservation of this property under perturbations, *SIAM J. Math. Anal.*, **29**, 1998, pp. 266–278.

[3] O. Christensen and C. Heil, Perturbation of Banach frames and atomic decompositions, *Math. Nachr*, **185**, 1997, pp. 33–47.

[4] H.G. Feichinger and K. Gröchenig, A unified approach to atomic decompositions via integrable group representations, In: Proc. Conf. "Function Spaces and Applications", *Lecture Notes in Math.* **1302**, Berlin — Heidelberg — New York, Springer 1988, pp. 52–73.

[5] K. Gröchenig, Describing functions: Atomic decomposiions versus frames, *Monatsh. Math.*, **112**, 1991, pp. 1–41.

[6] J. R. Holub, Pre-frame operators, Besselian frames, and near-Riesz bases in Hilbert space, *Proc. Amer. Math. Soc.*, **122**, 1994, pp. 779–785.

[7] P.K. Jain, S.K. Kaushik and L.K. Vashisht, Banach frames for conjugate Banach spaces, *Zeitschrift für Analysis und ihre Anwendungen*, **23** (4), 2004, pp. 713–720.

[8] P.K. Jain, S.K. Kaushik and L.K. Vashisht, On Banach frames, *Indian J. Pure Appl. Math.*, **37** (5), 2006, pp. 265–272.

[9] P.K. Jain, S.K. Kaushik and L.K. Vashisht, On perturbation of Banach frames, *International Journal of Wavelet Analysis, Multiresolution and Image Processing* (Hong Kong), **4** (3), 2006, pp. 559–565.

[10] P.K. Jain, S.K. Kaushik and Nisha Gupta, On frame systems in Banach spaces, *International Jour. of Wavelet Analysis, Multiresolution and Image processing*, **7** (1), 2009, pp. 1–7.

[11] P.K. Jain, S.K. Kaushik and Nisha Gupta, On near exact Banach frames in Banach spaces, *Bull. Aust. Math. Soc.*, 78, 2008, pp. 335–342.

[12] P.K. Jain, S.K. Kaushik and Nisha Gupta, A note on bounded Banach frames, *International Journal of Pure and Applied Mathematics*, **41** (2), 2007, pp. 257–266.

[13] R.C. James, Bases and reflexivity of Banach spaces, *Ann. of Math.* **52**, 1950, pp. 518–527.