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Annular Region Containing all the Zeros of Lacunary-Type Polynomials

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Abstract. In this paper, we find the annular region containing all the zeros of lacunary-type polynomials, whose coefficients are subjected to certain restrictions.

Key Words: Lacunary-type polynomial, Eneström–Kakeya theorem
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Introduction

Let $P(z) = \sum_{j=0}^{n} a_j z^j$, $a_n \neq 0$ be a polynomial of degree n. Then, according to the well-known result of Cauchy [2], all the zeros of polynomial $P(z)$ lie in the disk

$$|z| \leq 1 + \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right|.$$

Locating zeros of polynomials with specific conditions on the coefficients, in general, and, in particular, finding the number of zeros of complex polynomials in a disk when their coefficients are restricted to certain conditions has applications in many areas of applied mathematics, including linear control systems, electrical networks, root approximation and signal processing. For this reason, there is always a need for better estimates for the region containing some or all the zeros of a polynomial.

A review on the location of zeros of polynomials can be found in [4, 9, 13, 16]. One of the most elegant results on the bounds of zeros of a polynomial with restrictions on its coefficients, known as Eneström–Kakeya theorem (for reference see section 8.3 of [18]), states that if $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients, such that $a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 > 0$, then $P(z)$ has all its zeros in $|z| \leq 1$. In the literature, there exist various extensions and generalizations of Eneström–Kakeya theorem [3–18]. In 1996, Aziz and Zargar [1] proved the following results for the regions containing the zeros of the lacunary-type polynomials.

Theorem 1 *If $P(z) = a_n z^n + a_p z^p + \ldots + a_1 z + a_0$, $0 \le p \le n-1$ is a polynomial of degree n and $M = \max|a_j/a_n|$, $j = 0, 1, \ldots, p$, then all the zeros of $P(z)$ lie in $|z| < K$, where K is a unique positive root of the trinomial equation*

$$x^{n-p} - x^{n-p-1} - M = 0.$$

Theorem 2 *If $P(z) = a_n z^n + a_p z^p + \ldots + a_1 z + a_0$, $0 \le p \le n-1$ is a polynomial of degree n and $A = \max|a_j|$, $j = 0, 1, \ldots, p$, then all the zeros of $P(z)$ lie in the ring shaped region*

$$\frac{|a_0|}{2(1+A)^{n-1}\{1+(p+1)A\}} \le |z| < 1 + \alpha_0 A,$$

where α_0 is a unique positive root of the equation

$$x = 1 - \frac{1}{(1+Ax)^{p+1}}$$

in the interval $(0, 1)$.

1 Main Results

In this paper, we obtain the annular region containing the zeros of the polynomials, whose coefficients are subjected to Eneström–Kakeya type restrictions. In fact, we prove the following result:

Theorem 3 *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu < n$, $a_0 \ne 0$ be a polynomial of degree n, where for some $t > 0$ and some $\mu \le k \le n$,*

$$t^{\mu}|a_{\mu}| \ge ... \ge t^{k-1}|a_{k-1}| \ge t^k|a_k| \le t^{k+1}|a_{k+1}| \le ... \le t^{n-1}|a_{n-1}| \le t^n|a_n|$$

and $|arg\ a_j - \beta| \le \alpha \le \pi/2$ for $\mu \le j \le n$ and some real α and β. Then all the zeros of $P(z)$ lie in the region defined by the following inequalities

$$\min(r_2, t) \le |z| \le \max\left(r_1, t^{-1}\right),$$

where

$$r_1 = \frac{2M^2}{t^2|ta_n - a_{n-1}|(|a_n| - M) + (t^4|ta_n - a_{n-1}|^2(|a_n| - M)^2 + 4|a_n|t^2M^3)^{1/2}},$$

$$r_2 = \frac{1}{2M_1^2}\left[(t^2|a_0| - M_1)t^2|a_0| + \left([(t^2|a_0| - M_1)t^2|a_0|]^2 + 4M_1^3|a_0|t^3\right)^{1/2}\right],$$

$$\begin{aligned} M = {} & t^{n-k}(1+t^2)\big(|a_0|t^k - |a_k|\cos\alpha\big) + t^{n-\mu}|a_\mu|(t^2+\cos\alpha+\sin\alpha) \\ & + t^2|a_n|(\cos\alpha+\sin\alpha) + (1-t^2)\cos\alpha\sum_{j=\mu+1}^{k-1}|a_j|t^{n-j} \\ & + (t^2-1)\cos\alpha\sum_{j=k+1}^{n-1}|a_j|t^{n-j} + (1+t^2)\sin\alpha\sum_{j=\mu+1}^{n-1}|a_j|t^{n-j}, \end{aligned}$$

$$\begin{aligned} M_1 = {} & |a_0|t + (1+\sin\alpha+\cos\alpha)\big(|a_\mu|t^{\mu+1} + |a_n|t^{n+1}\big) - 2\cos\alpha|a_k|t^{k+1} \\ & + 2\sin\alpha\sum_{j=\mu+1}^{n-1}|a_j|t^{j+1}. \end{aligned}$$

For the proof of the theorem, we need the following lemmas:

Lemma 1 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients such that for some real* α *and* β, $|arg\ a_j - \beta| \leq \alpha \leq \pi/2$, $0 \leq j \leq n$ *and* $|a_j| \geq |a_{j-1}|$, $0 \leq j \leq n$. *Then*

$$\big|a_j - a_{j-1}\big| \leq (|a_j| - |a_{j-1}|)\cos\alpha + (|a_j| + |a_{j-1}|)\sin\alpha$$

The above lemma is due to Govil and Rahman [9].

Lemma 2 *If* $P(z)$ *is analytic in* $|z| \leq R$, $P(0) = 0$, $P'(0) = b$, *then for* $|z| \leq R$,

$$|P(z)| \leq \frac{M|z|(M|z| + R^2|b|)}{R^2(M + |z||b|)},$$

where $M = \max_{|z|=R}|P(z)|$.

The above lemma can be deduced from the result due to Govil et al. [8].

Proof of Theorem 3. Consider the polynomial

$$\begin{aligned} F(z) &= (t-z)P(z) \\ &= a_0(t-z) + ta_\mu z^\mu + \sum_{j=\mu+1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1}. \qquad (1) \end{aligned}$$

Let us also consider another polynomial defined by

$$\begin{aligned} G(z) &= z^{n+1}F(1/z) \\ &= a_0 t z^{n+1} - a_0 z^n + ta_\mu z^{n-\mu+1} + \sum_{j=\mu+1}^{n}(ta_j - a_{j-1})z^{n-j+1} - a_n \end{aligned}$$

such that

$$|G(z)| \geq |a_n| - |H(z)|, \tag{2}$$

where

$$H(z) = a_0 t z^{n+1} - a_0 z^n + t a_\mu z^{n-\mu+1} + \sum_{j=\mu+1}^{n} (t a_j - a_{j-1}) z^{n-j+1}.$$

For $|z| = t$, we have

$$\begin{aligned} |H(z)| &= \left| a_0 t z^{n+1} - a_0 z^n + t a_\mu z^{n-\mu+1} + \sum_{j=\mu+1}^{n} (t a_j - a_{j-1}) z^{n-j+1} \right| \\ &\leq |a_0| t^{n+2} + |a_0| t^n + |a_\mu| t^{n-\mu+2} + \sum_{j=\mu+1}^{k} |t a_j - a_{j-1}| t^{n-j+1} \\ &\quad + \sum_{j=k+1}^{n} |t a_j - a_{j-1}| t^{n-j+1}. \end{aligned}$$

Further, using Lemma 1, we obtain

$$\begin{aligned} |H(z)| &\leq |a_0| t^{n+2} + |a_0| t^n + |a_\mu| t^{n-\mu+2} \\ &\quad + \sum_{j=\mu+1}^{k} [(|a_{j-1}| - t|a_j|)\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha] t^{n-j+1} \\ &\quad + \sum_{j=k+1}^{n} [(t|a_j| - |a_{j-1}|)\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha] t^{n-j+1} \\ &= t^{n-k}(1+t^2)[|a_0| t^k - |a_k|\cos\alpha] + t^{n-\mu}|a_\mu|(t^2 + \cos\alpha + \sin\alpha) \\ &\quad + t^2|a_n|(\cos\alpha + \sin\alpha) + (1-t^2)\cos\alpha \sum_{j=\mu+1}^{k-1} |a_j| t^{n-j} \\ &\quad + (t^2-1)\cos\alpha \sum_{j=k+1}^{n-1} |a_j| t^{n-j} + (1+t^2)\sin\alpha \sum_{j=\mu+1}^{n-1} |a_j| t^{n-j} \\ &= M. \end{aligned}$$

Clearly, $H(0) = 0$, $H'(0) = t a_n - a_{n-1}$ and $|H(z)| \leq M$ for $|z| = t$. Therefore, it follows by Lemma 2, that for $|z| \leq t$,

$$|H(z)| \leq \frac{M|z|}{t^2} \frac{M|z| + t^2|t a_n - a_{n-1}|}{M + |t a_n - a_{n-1}||z|}.$$

Using this in (2), we obtain for $|z| \leq t$,

$$\begin{aligned}|G(z)| \geq & |a_n| - \frac{M|z|}{t^2}\frac{M|z| + t^2|ta_n - a_{n-1}|}{M + |ta_n - a_{n-1}||z|} \\ = & \frac{-M^2|z|^2 + t^2|ta_n - a_{n-1}|(|a_n| - M)|z| + |a_n|t^2M}{t^2(M + |ta_n - a_{n-1}||z|)}.\end{aligned}$$

From the equality $G(z) = 0$, we find

$$|z| = \frac{t^2|ta_n - a_{n-1}|(|a_n| - M) \pm \left(t^4|ta_n - a_{n-1}|^2(|a_n| - M)^2 + 4M^3|a_n|t^2\right)^{1/2}}{2M^2}.$$

That is, $G(z) > 0$ if

$$\begin{aligned}&\frac{t^2|ta_n - a_{n-1}|(|a_n| - M) - \left(t^4|ta_n - a_{n-1}|^2(|a_n| - M)^2 + 4M^3|a_n|t^2\right)^{1/2}}{2M^2} < |z| \\ &< \frac{t^2|ta_n - a_{n-1}|(|a_n| - M) + \left(t^4|ta_n - a_{n-1}|^2(|a_n| - M)^2 + 4M^3|a_n|t^2\right)^{1/2}}{2M^2}.\end{aligned}$$

In the obtained inequality, the left-hand term is clearly negative, and thus, $G(z) > 0$ if

$$\begin{aligned}|z| &< \frac{t^2|ta_n - a_{n-1}|(|a_n| - M) + \left(t^4|ta_n - a_{n-1}|^2(|a_n| - M)^2 + 4M^3|a_n|t^2\right)^{1/2}}{2M^2} \\ &= r_1^{-1}.\end{aligned}$$

This shows that all the zeros of $F(z)$ lie in the region defined by

$$|z| \leq \max\left(r_1, t^{-1}\right). \tag{3}$$

From (1), we have

$$|F(z)| \geq |a_0|t - T(z) \tag{4}$$

where

$$T(z) = -a_0 z + ta_\mu z^\mu + \sum_{j=\mu+1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1}.$$

Now, for $|z| = t$, we can write

$$
\begin{aligned}
|T(z)| &\leq |a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{n} |ta_j - a_{j-1}|t^j + |a_n|t^{n+1} \\
&= |a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} |ta_j - a_{j-1}|t^j + \sum_{j=k+1}^{n} |ta_j - a_{j-1}|t^j + |a_n|t^{n+1} \\
&= |a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} \left((|a_{j-1}| - t|a_j|)\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha\right) t^j \\
&\quad + \sum_{j=k+1}^{n} \left((t|a_j| - |a_{j-1}|)\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha\right) t^j + |a_n|t^{n+1} \\
&= |a_0|t + (1 + \sin\alpha + \cos\alpha)(|a_\mu|t^{\mu+1} + |a_n|t^{n+1}) - 2\cos\alpha|a_k|t^{k+1} \\
&\quad + 2\sin\alpha \sum_{j=\mu+1}^{n-1} |a_j|t^{j+1} \\
&= M_1.
\end{aligned}
$$

Clearly $T(0) = 0$, $T'(0) = -a_0$, and $T(z) \leq M_1$ for $|z| = t$. Therefore, it follows by Lemma 2,

$$|T(z)| \leq \frac{M_1|z|(M_1|z| + t^2|a_0|)}{t^2(M_1 + t|a_0||z|)}.$$

Using this in (4), we get

$$
\begin{aligned}
|F(z)| &\geq |a_0|t - \frac{M_1|z|(M_1|z| + t^2|a_0|)}{t^2(M_1 + t|a_0||z|)} \\
&= \frac{-1}{t^2(M_1 + t|a_0||z|)} \left(M_1^2|z|^2 - (t^2|a_0| - M_1)t^2|a_0||z| - M_1|a_0|t^3\right).
\end{aligned}
$$

Since $F(z) = 0$ for

$$|z| = \frac{(|a_0|t^2 - M_1)t^2|a_0| \pm \left((|a_0|t^2 - M_1)^2t^4|a_0|^2 + 4|a_0|M_1^3t^3\right)^{1/2}}{2M_1^2},$$

we conclude that $F(z) > 0$ if

$$
\begin{aligned}
&\frac{(|a_0|t^2 - M_1)t^2|a_0| - \left((|a_0|t^2 - M_1)^2t^4|a_0|^2 + 4|a_0|M_1^3t^3\right)^{1/2}}{2M_1^2} < |z| \\
&\quad < \frac{(|a_0|t^2 - M_1)t^2|a_0| + \left((|a_0|t^2 - M_1)^2t^4|a_0|^2 + 4|a_0|M_1^3t^3\right)^{1/2}}{2M_1^2},
\end{aligned}
$$

or, equivalently, if

$$\begin{aligned} |z| &< \frac{(|a_0|t^2 - M_1)t^2|a_0| + \{(|a_0|t^2 - M_1)^2t^4|a_0|^2 + 4|a_0|M_1^3t^3\}^{1/2}}{2M_1^2} \\ &= r_2. \end{aligned}$$

This shows that all the zeros of $F(z)$, and hence, of the polynomial $P(z)$ lie in the region defined by

$$|z| \geq \min(r_2, t). \tag{5}$$

Combining (3) and (5), the desired result follows. □

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