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Controlled generalized fusion frame in the tensor product of Hilbert spaces

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Abstract. We present controlled by operators generalized fusion frame in the tensor product of Hilbert spaces and discuss some of its properties. We also describe the frame operator for a pair of controlled g-fusion Bessel sequences in the tensor product of Hilbert spaces.

Key Words: Frame, fusion frame, g-frame, g-fusion frame, frame operator, tensor product of Hilbert spaces, tensor product of frames
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1 Introduction

Frame for a Hilbert space was first introduced by Duffin and Schaeffer [4] in 1952 to study some fundamental problems in non-harmonic Fourier series. In abstract Hilbert spaces, Daubechies et al. [3] gave the formal definition of frame in 1986. Frame theory has been widely used in signal and image processing, filter bank theory, coding and communications, etc. Several generalizations of frames, namely, g-frames [18], fusion frames [1], g-fusion frames [17], etc., have been introduced in recent times. A generalized fusion frame is used to generalize the theory of fusion frame and g-frame. Frame operator for the pair of g-fusion Bessel sequences was studied by the authors in [8], who also presented the stability of dual g-fusion frames in Hilbert spaces in [7].

The basic concept of the tensor product of Hilbert spaces was presented by Robinson [16]. Frames and bases in the tensor product of Hilbert spaces were introduced by Khosravi and Asgari [12]. Reddy et al. [19] studied the frame in the tensor product of Hilbert spaces and presented the tensor frame operator on the tensor product of Hilbert spaces. The concepts of fusion frames and g-frames in the tensor product of Hilbert spaces were introduced by Khosravi and Mirzaee Azandaryani [13]. Fusion frame and its alternative dual in the tensor product of Hilbert spaces were studied by the authors [9]. Also, we considered a generalized fusion frame in the tensor product of Hilbert spaces [10].

In this paper, we present a controlled generalized fusion frame in the tensor product of Hilbert spaces and establish some of its properties. The relation between the frame operators for the pair of controlled g-fusion Bessel sequences in Hilbert spaces and the tensor product of Hilbert spaces is also obtained.

Throughout this paper, H and K are considered to be separable Hilbert spaces with associated inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. By I_H and I_K we denote the identity operators on H and K, and $\mathcal{B}(H, K)$ is the collection of all bounded linear operators from H to K. In particular, $\mathcal{B}(H)$ stands for the space of all bounded linear operators on H. By P_V we denote the orthogonal projection onto the closed subspace $V \subset H$. Everywhere below, I and J stand for subsets of $\mathbb{Z}$. Let $\{V_i\}_{i \in I}$ and $\{W_j\}_{j \in J}$ be collections of closed subspaces of H and K, $\{H_i\}_{i \in I}$ and $\{K_j\}_{j \in J}$ be collections of Hilbert spaces, $\{\Lambda_i \in \mathcal{B}(H, H_i)\}_{i \in I}$ and $\{\Gamma_j \in \mathcal{B}(K, K_j)\}_{j \in J}$ be sequences of operators. By $\mathcal{GB}(H)$ we denote the set of all bounded linear operators which have bounded inverse. If $S, R \in \mathcal{GB}(H)$, then R^*, R^{-1} and SR also belong to $\mathcal{GB}(H)$. Let $\mathcal{GB}^+(H)$ be the set of all positive operators in $\mathcal{GB}(H)$ and T, U be invertible operators in $\mathcal{GB}(H)$.

For any $I \subset \mathbb{Z}$, define

$$l^2\left(\{H_i\}_{i \in I}\right) = \left\{ \{f_i\}_{i \in I} : f_i \in H_i, \sum_{i \in I} \|f_i\|^2 < \infty \right\}$$

with the inner product given by

$$\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle_{H_i}.$$

Clearly (see also [17]), $l^2\left(\{H_i\}_{i \in I}\right)$ is a Hilbert space with respect to the above inner product. We can similarly define the space $l^2\left(\{K_j\}_{j \in J}\right)$.

2 Preliminaries

In this section, we recall some basic definitions and theorems.

Theorem 1 *[6] Let $V \subset H$ be a closed subspace and $T \in \mathcal{B}(H)$. Then $P_V T^* = P_V T^* P_{\overline{TV}}$. If T is a unitary operator, i.e., $T^* T = I_H$, then $P_{\overline{TV}} T = T P_V$.*

Theorem 2 *[2] The set $\mathcal{S}(H)$ of all self-adjoint operators on H is a partially ordered set with respect to the partial order $\leq$: for $T, S \in \mathcal{S}(H)$, $T \leq S$ if and only if $\langle Tf, f \rangle_1 \leq \langle Sf, f \rangle_1$ for all $f \in H$.*

A sequence $\{f_i\}_{i\in I}$ of elements in H is called a frame for H (see [2]) if there exist constants $A, B > 0$ such that

$$A\|f\|_1^2 \leq \sum_{i\in I} |\langle f, f_i\rangle_1|^2 \leq B\|f\|_1^2$$

for all $f \in H$. The constants A and B are called frame bounds.

Let $\{v_i\}_{i\in I}$ be a collection of positive weights. The family $\Lambda = \{(V_i, \Lambda_i, v_i)\}_{i\in I}$ is called a g-fusion frame for H with respect to $\{H_i\}_{i\in I}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\,\|f\|_1^2 \leq \sum_{i\in I} v_i^2 \,\|\Lambda_i P_{V_i}(f)\|_1^2 \leq B\,\|f\|_1^2, \qquad f \in H. \tag{1}$$

The constants A and B are called the lower and the upper bounds of g-fusion frame, respectively. If $A = B$, then Λ is called tight g-fusion frame, and if $A = B = 1$, we say that Λ is a Parseval g-fusion frame. If Λ satisfies only the right inequality in (1), it is called a g-fusion Bessel sequence with the bound B in H (see [17]).

Let $\Lambda = \{(V_i, \Lambda_i, v_i)\}_{i\in I}$ and $\Lambda' = \{(V_i', \Lambda_i', v_i')\}_{i\in I}$ be two g-fusion Bessel sequences in H with bounds D_1 and D_2, respectively. The operator $S_{\Lambda\Lambda'} : H \to H$ defined by

$$S_{\Lambda\Lambda'}(f) = \sum_{i\in I} v_i v_i' P_{V_i} \Lambda_i^* \Lambda_i' P_{V_i'}(f), \qquad f \in H,$$

is called the frame operator for the pair of g-fusion Bessel sequences Λ and Λ' (see [8]).

Now we recall the notion of a controlled g-fusion frame and related concepts (see [14]). Let $\{W_j\}_{j\in J}$ be a collection of closed subspaces of H and $\{v_j\}_{j\in J}$ be a collection of positive weights. Let $\{H_j\}_{j\in J}$ be a sequence of Hilbert spaces, $T, U \in \mathcal{GB}(H)$ and $\Lambda_j \in \mathcal{B}(H, H_j)$, $j \in J$. The family $\Lambda_{TU} = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ is called a (T, U)-controlled g-fusion frame for H if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|_1^2 \leq \sum_{j\in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f\right\rangle_1 \leq B\|f\|_1^2, \qquad f \in H. \tag{2}$$

If $A = B$, then Λ_{TU} is called (T, U)-controlled tight g-fusion frame and if $A = B = 1$, we say that Λ_{TU} is a (T, U)-controlled Parseval g-fusion frame. If Λ_{TU} satisfies only the right inequality in (2), it is called a (T, U)-controlled g-fusion Bessel sequence in H.

Let Λ_{TU} be a (T, U)-controlled g-fusion Bessel sequence in H with a bound B. The synthesis operator $T_C : \mathcal{K}_{\Lambda_j} \to H$ is defined by

$$T_C\left(\left\{v_j\left(T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U\right)^{1/2} f\right\}_{j\in J}\right) = \sum_{j\in J} v_j^2 T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f,$$

and the analysis operator $T_C^* : H \to \mathcal{K}_{\Lambda_j}$ is given by

$$T_C^* f = \left\{ v_j \left(T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U \right)^{1/2} f \right\}_{j \in J}, \qquad f \in H,$$

where

$$\mathcal{K}_{\Lambda_j} = \left\{ \left\{ v_j \left(T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U \right)^{1/2} f \right\}_{j \in J} : f \in H \right\} \subset l^2 \left(\{H_j\}_{j \in J} \right),$$

$J \subset \mathbb{Z}$. The frame operator $S_C : H \to H$ is defined as follows

$$S_C f = T_C T_C^* f = \sum_{j \in J} v_j^2 T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f, \qquad f \in H,$$

and it is easy to verify that for any $f \in H$,

$$\langle S_C f, f \rangle_1 = \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle_1.$$

Furthermore, if Λ_{TU} is a (T, U)-controlled g-fusion frame with bounds A and B, then $A I_H \leq S_C \leq B I_H$. Hence, S_C is bounded, invertible, self-adjoint and positive linear operator. It is easy to verify that $B^{-1} I_H \leq S_C^{-1} \leq A^{-1} I_H$ (see [14]).

Let $K \in \mathcal{B}(H)$, $\{W_j\}_{j \in J}$ be a collection of closed subspaces of H and $\{v_j\}_{j \in J}$ be a collection of positive weights. Let $\{H_j\}_{j \in J}$ be a sequence of Hilbert spaces, $T, U \in \mathcal{GB}(H)$ and $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$. The family $\Lambda_{TU} = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is called a (T, U)-controlled K-g-fusion frame for H (see [15]) if there exist constants $0 < A \leq B < \infty$ such that

$$A \|K^* f\|_1^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle_1 \leq B \|f\|_1^2, \qquad f \in H.$$

There are several ways to introduce the tensor product of Hilbert spaces. The tensor product of Hilbert spaces H and K is a certain linear space of operators which was presented by Folland [5] and independently by Kadison and Ringrose [11]. Below we give the definition according to [19].

The tensor product of Hilbert spaces H and K is denoted by $H \otimes K$ and is defined to be an inner product space associated with the inner product

$$\langle f \otimes g, f' \otimes g' \rangle = \langle f, f' \rangle_1 \langle g, g' \rangle_2 \tag{3}$$

for all $f, f' \in H$ and $g, g' \in K$. The norm on $H \otimes K$ is given by

$$\|f \otimes g\| = \|f\|_1 \|g\|_2, \qquad f \in H, g \in K. \tag{4}$$

The space $H \otimes K$ is complete with respect to the above inner product. Therefore, the space $H \otimes K$ is a Hilbert space.

For $Q \in \mathcal{B}(H)$ and $T \in \mathcal{B}(K)$, the tensor product of operators Q and T is denoted by $Q \otimes T$ and is defined as

$$(Q \otimes T)\, A = Q A T^{*}, \qquad A \in H \otimes K.$$

It can be easily verified that $Q \otimes T \in \mathcal{B}(H \otimes K)$ (see, for example, [5]).

Theorem 3 *[5] Suppose $Q, Q' \in \mathcal{B}(H)$ and $T, T' \in \mathcal{B}(K)$. Then*

(i) $Q \otimes T \in \mathcal{B}(H \otimes K)$ and $\|Q \otimes T\| = \|Q\|\|T\|$;

(ii) $(Q \otimes T)\,(f \otimes g) = Q(f) \otimes T(g)$ for all $f \in H, g \in K$;

(iii) $(Q \otimes T)\,(Q' \otimes T') = (QQ') \otimes (TT')$;

(iv) $Q \otimes T$ is invertible if and only if Q and T are invertible, in which case $(Q \otimes T)^{-1} = (Q^{-1} \otimes T^{-1})$;

(v) $(Q \otimes T)^{*} = (Q^{*} \otimes T^{*})$;

(vi) $I_H \otimes I_K = I_{H \otimes K}$, where $I_{H \otimes K}$ is the identity operator on $H \otimes K$.

3 Controlled g-fusion frame in $H \otimes K$

In this section, we present the notion of a controlled g-fusion frame in tensor product of Hilbert spaces. This concept is a generalization of the notion of g-fusion frame in $H \otimes K$.

Let $\{v_i\}_{i \in I}$ and $\{w_j\}_{j \in J}$ be two families of positive weights. Suppose $T, U \in \mathcal{GB}(H)$, $T_1, U_1 \in \mathcal{GB}(K)$ and $\Lambda_i \in \mathcal{B}(H, H_i)$, $\Gamma_j \in \mathcal{B}(K, K_j)$ for each $i \in I$, $j \in J$. The family $\Delta = \{(V_i \otimes W_j, \Lambda_i \otimes \Gamma_j, v_i w_j)\}_{i \in I, j \in J}$ is called a generalized fusion frame controlled by the operators $(T \otimes T_1, U \otimes U_1)$ (or $(T \otimes T_1, U \otimes U_1)$-controlled g-fusion frame) for $H \otimes K$ with respect to $\{H_i \otimes K_j\}_{i \in I, j \in J}$ if there exist constants $0 < A \le B < \infty$ such that

$$\begin{aligned} & A\|h\|^2 \\ & \le \sum_{i,j} v_i^2 w_j^2 \left\langle \Delta_{ij} P_{V_i \otimes W_j} \left(U \otimes U_1\right) h, \Delta_{ij} P_{V_i \otimes W_j} \left(T \otimes T_1\right) h \right\rangle \\ & \le B\|h\|^2, \end{aligned} \tag{5}$$

$h = f \otimes g \in H \otimes K$, where $P_{V_i \otimes W_j}$ is the orthogonal projection of $H \otimes K$ onto $V_i \otimes W_j$ and $\Delta_{ij} = \Lambda_i \otimes \Gamma_j$. The constants A and B are called the frame bounds. If $A = B$, then it is called a $(T \otimes T_1, U \otimes U_1)$-controlled tight g-fusion frame.

Let us make several remarks.
(i) If the family Δ satisfies only the right inequality in (5), it is called

a $(T \otimes T_1, U \otimes U_1)$-controlled g-fusion Bessel sequence in $H \otimes K$ with the bound B.
(ii) If $T = I_H$ and $T_1 = I_K$, then Δ is called a $(I_{H \otimes K}, U \otimes U_1)$-controlled g-fusion frame for $H \otimes K$.
(iii) If $T = U = I_H$ and $T_1 = U_1 = I_K$, then Δ is called a $(I_{H \otimes K}, I_{H \otimes K})$-controlled g-fusion frame for $H \otimes K$. In this case, inequalities (5) can be written as follows

$$A \|f \otimes g\|^2 \leq \sum_{i,j} v_i^2 w_j^2 \left\|(\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j}(f \otimes g)\right\|^2 \leq B \|f \otimes g\|^2,$$

for all $f \otimes g \in H \otimes K$. Hence, Δ is a g-fusion frame for $H \otimes K$. For more details on g-fusion frame in the tensor product of Hilbert spaces, see [10].

For $i \in I$ and $j \in J$, define the space

$$l^2(\{H_i \otimes K_j\}) = \left\{\{f_i \otimes g_j\} : f_i \otimes g_j \in H_i \otimes K_j, \sum_{i,j} \|f_i \otimes g_j\|^2 < \infty\right\}$$

with the inner product

$$\begin{aligned}
&\left\langle \{f_i \otimes g_j\}, \{f_i' \otimes g_j'\} \right\rangle_{l^2} = \sum_{i,j} \left\langle f_i \otimes g_j, f_i' \otimes g_j' \right\rangle \\
&= \sum_{i,j} \langle f_i, f_i' \rangle_{H_i} \left\langle g_j, g_j' \right\rangle_{K_j} = \left(\sum_{i \in I} \langle f_i, f_i' \rangle_{H_i}\right) \left(\sum_{j \in J} \left\langle g_j, g_j' \right\rangle_{K_j}\right) \\
&= \left\langle \{f_i\}_{i \in I}, \{f_i'\}_{i \in I} \right\rangle_{l^2\left(\{H_i\}_{i \in I}\right)} \left\langle \{g_j\}_{j \in J}, \{g_j'\}_{j \in J} \right\rangle_{l^2\left(\{K_j\}_{j \in J}\right)}.
\end{aligned}$$

The space $l^2(\{H_i \otimes K_j\})$ is complete with the above inner product. Therefore, it is a Hilbert space with respect to the above inner product.

Since $\{V_i\}_{i \in I}$, $\{W_j\}_{j \in J}$ and $\{V_i \otimes W_j\}_{i,j}$ are the families of closed subspaces of the Hilbert spaces H, K and $H \otimes K$, respectively, it is easy to verify that $P_{V_i \otimes W_j} = P_{V_i} \otimes P_{W_j}$.

For the remaining part of this paper, we denote $\Lambda_i \otimes \Gamma_j$ by Δ_{ij}, and the families $\{(V_i, \Lambda_i, v_i)\}_{i \in I}$ and $\{(W_j, \Gamma_j, w_j)\}_{j \in J}$ by Λ_{TU} and $\Gamma_{T_1 U_1}$, respectively.

Theorem 4 *The family* $\Delta = \{(V_i \otimes W_j, \Lambda_i \otimes \Gamma_j, v_i w_j)\}_{i \in I, j \in J}$ *is a* $(T \otimes T_1, U \otimes U_1)$*-controlled* g*-fusion frame for* $H \otimes K$ *with respect to* $\{H_i \otimes K_j\}_{i \in I, j \in J}$ *if and only if* Λ_{TU} *is a* (T, U)*-controlled* g*-fusion frame for* H *with respect to* $\{H_i\}_{i \in I}$ *and* $\Gamma_{T_1 U_1}$ *is a* (T_1, U_1)*-controlled* g*-fusion frame for* K *with respect to* $\{K_j\}_{j \in J}$.

Proof. First, suppose that Δ is a $(T \otimes T_1, U \otimes U_1)$-controlled g-fusion frame for $H \otimes K$ with respect to $\{H_i \otimes K_j\}_{i \in I, j \in J}$ having bounds A and B. Then for each $h = f \otimes g \in H \otimes K - \{\theta \otimes \theta\}$, using (3) and (4), we have

$$A \|h\|^2 \leq \sum_{i,j} v_i^2 w_j^2 \left\langle \Delta_{ij} P_{V_i \otimes W_j} (U \otimes U_1) h, \Delta_{ij} P_{V_i \otimes W_j} (T \otimes T_1) h \right\rangle \leq B \|h\|^2 ,$$

and hence,

$$A \|h\|^2 \leq \sum_{i,j} v_i^2 w_j^2 \left\langle \Lambda_i P_{V_i} U f \otimes \Gamma_j P_{W_j} U_1 g, \Lambda_i P_{V_i} T f \otimes \Gamma_j P_{W_j} T_1 g \right\rangle \leq B \|h\|^2 .$$

Further,

$$\begin{aligned} A\|f\|_1^2\|g\|_2^2 &\leq \sum_{i,j} v_i^2 w_j^2 \left\langle \Lambda_i P_{V_i} U f, \Lambda_i P_{V_i} T f \right\rangle_1 \left\langle \Gamma_j P_{W_j} U_1 g, \Gamma_j P_{W_j} T_1 g \right\rangle_2 \\ &\leq B\|f\|_1^2\|g\|_2^2, \end{aligned}$$

and thus,

$$\begin{aligned} A\|f\|_1^2\|g\|_2^2 &\leq \sum_{i \in I} v_i^2 \left\langle \Lambda_i P_{V_i} U f, \Lambda_i P_{V_i} T f \right\rangle_1 \sum_{j \in J} w_j^2 \left\langle \Gamma_j P_{W_j} U_1 g, \Gamma_j P_{W_j} T_1 g \right\rangle_2 \\ &\leq B\|f\|_1^2\|g\|_2^2. \end{aligned}$$

Since $f \otimes g$ is non-zero vector, f and g are also non-zero vectors, and therefore,

$$\sum_{i \in I} v_i^2 \left\langle \Lambda_i P_{V_i} U f, \Lambda_i P_{V_i} T f \right\rangle_1 , \qquad \sum_{j \in J} w_j^2 \left\langle \Gamma_j P_{W_j} U_1 g, \Gamma_j P_{W_j} T_1 g \right\rangle_2$$

are non-zero. Thus, from the above inequality, we get

$$\begin{aligned} \frac{A \|g\|_2^2 \|f\|_1^2}{\sum_{j \in J} w_j^2 \left\langle \Gamma_j P_{W_j} U_1 g, \Gamma_j P_{W_j} T_1 g \right\rangle_2} &\leq \sum_{i \in I} v_i^2 \left\langle \Lambda_i P_{V_i} U f, \Lambda_i P_{V_i} T f \right\rangle_1 \\ &\leq \frac{B \|g\|_2^2 \|f\|_1^2}{\sum_{j \in J} w_j^2 \left\langle \Gamma_j P_{W_j} U_1 g, \Gamma_j P_{W_j} T_1 g \right\rangle_2}. \end{aligned}$$

Then

$$A_1 \|f\|_1^2 \leq \sum_{i \in I} v_i^2 \left\langle \Lambda_i P_{V_i} U f, \Lambda_i P_{V_i} T f \right\rangle_1 \leq B_1 \|f\|_1^2 , \qquad f \in H,$$

where

$$A_1 = \inf_{g \in K} \frac{A \|g\|_2^2}{\sum_{j \in J} w_j^2 \left\langle \Gamma_j P_{W_j} U_1 g, \Gamma_j P_{W_j} T_1 g \right\rangle_2}$$

and

$$B_1 = \sup_{g \in K} \frac{B \|g\|_2^2}{\sum_{j \in J} w_j^2 \left\langle \Gamma_j P_{W_j} U_1 g, \Gamma_j P_{W_j} T_1 g \right\rangle_2}.$$

This shows that Λ_{TU} is a (T, U)-controlled g-fusion frame for H with respect to $\{H_i\}_{i \in I}$. Similarly, it can be shown that $\Gamma_{T_1 U_1}$ is (T_1, U_1)-controlled g-fusion frame for K with respect to $\{K_j\}_{j \in J}$.

Conversely, suppose that Λ_{TU} is a (T, U)-controlled g-fusion frame for H with respect to $\{H_i\}_{i \in I}$ and $\Gamma_{T_1 U_1}$ is a (T_1, U_1)-controlled g-fusion frame for K with respect to $\{K_j\}_{j \in J}$ having bounds A, B and C, D, respectively. Then

$$A \|f\|_1^2 \leq \sum_{i \in I} v_i^2 \left\langle \Lambda_i P_{V_i} U f, \Lambda_i P_{V_i} T f \right\rangle_1 \leq B \|f\|_1^2, \qquad f \in H, \tag{6}$$

$$C \|g\|_2^2 \leq \sum_{j \in J} w_j^2 \left\langle \Gamma_j P_{W_j} U_1 g, \Gamma_j P_{W_j} T_1 g \right\rangle_2 \leq D \|g\|_2^2, \qquad g \in K. \tag{7}$$

Multiplying (6) with (7) and using (3) and (4), we get

$$\begin{aligned} AC \|f \otimes g\|^2 &\leq \sum_{i,j} v_i^2 w_j^2 \left\langle \Lambda_i P_{V_i} U f, \Lambda_i P_{V_i} T f \right\rangle_1 \left\langle \Gamma_j P_{W_j} U_1 g, \Gamma_j P_{W_j} T_1 g \right\rangle_2 \\ &\leq BD \|f \otimes g\|^2, \end{aligned}$$

$$\begin{aligned} A \|f \otimes g\|^2 &\leq \sum_{i,j} v_i^2 w_j^2 \left\langle \Lambda_i P_{V_i} U f \otimes \Gamma_j P_{W_j} U_1 g, \Lambda_i P_{V_i} T f \otimes \Gamma_j P_{W_j} T_1 g \right\rangle \\ &\leq B \|f \otimes g\|^2, \end{aligned}$$

$$\begin{aligned} AC \|h\|^2 &\leq \sum_{i,j} v_i^2 w_j^2 \left\langle \Delta_{ij} P_{V_i \otimes W_j} (U \otimes U_1) h, \Delta_{ij} P_{V_i \otimes W_j} (T \otimes T_1) h \right\rangle \\ &\leq BD \|h\|^2 \end{aligned}$$

for all $h = f \otimes g \in H \otimes K$. Hence, Δ is a $(T \otimes T_1, U \otimes U_1)$-controlled g-fusion frame for $H \otimes K$ with respect to $\{H_i \otimes K_j\}_{i \in I, j \in J}$ with bounds AC and BD. This completes the proof. □

Note, that Theorem 3.4 in [10] can be obtained as a corollary of Theorem 4 by substituting $T = U = I_H$ and $T_1 = U_1 = I_K$.

Now, we validate this theorem by considering the following example.

Example. Let $H = \mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ be an orthonormal basis for H. Suppose that $V_1 = \overline{span}\{e_2, e_3\}$, $V_2 = \overline{span}\{e_1, e_3\}$ and $V_3 = \overline{span}\{e_1, e_2\}$

with $V_i = H_i$, $i = 1, 2, 3$. Define $\Lambda_1 f = \langle f, e_2\rangle e_3$, $\Lambda_2 f = \langle f, e_3\rangle e_1$ and $\Lambda_3 f = \langle f, e_1\rangle e_2$. Consider the following two operators on H:

$$T(f_1, f_2, f_3) = (2f_1, 3f_2, 5f_3), \qquad U(f_1, f_2, f_3) = \left(\frac{f_1}{2}, \frac{f_2}{3}, \frac{f_3}{4}\right).$$

It is easy to verify that $T, U \in \mathcal{GB}^+(H)$, $TU = UT$. Now, for any $f = (f_1, f_2, f_3) \in H$,

$$\sum_{i=1}^{3} \langle \Lambda_i P_{V_i} U f, \Lambda_i P_{V_i} T f\rangle = f_1^2 + f_2^2 + \frac{5}{4} f_3^2.$$

Thus,

$$\|f\|^2 \leq \sum_{i=1}^{3} \langle \Lambda_i P_{V_i} U f, \Lambda_i P_{V_i} T f\rangle \leq \frac{5}{4}\|f\|^2, \qquad f \in H.$$

Hence, $\{(V_i, \Lambda_i, 1)\}_{i=1}^{3}$ is a (T, U)-controlled g-fusion frame for H with bounds 1 and $5/4$.

Next, we consider the Hilbert space $K = \mathbb{R}^2$ with an orthonormal basis $\{e_1, e_2\}$. Let $W_j = \overline{span}\{e_j\}$, $j = 1, 2$, with $W_j = K_j$, $j = 1, 2$. Define $\Gamma_1 g = \langle g, e_1\rangle e_1$, $\Gamma_2 g = \langle g, e_2\rangle e_2$. Consider two operators on H defined by

$$T_1(g_1, g_2) = (5g_1, 4g_2), \qquad U_1(g_1, g_2) = \left(\frac{g_1}{6}, \frac{g_2}{3}\right).$$

It is easy to verify that $T, U \in \mathcal{GB}^+(H)$ and $TU = UT$. Then, for any $g = (g_1, g_2) \in H$, we have

$$\sum_{j=1}^{2} \left\langle \Gamma_j P_{W_j} U_1 g, \Gamma_j P_{W_j} T_1 g\right\rangle = \frac{5}{6} g_1^2 + \frac{4}{3} g_2^2.$$

Thus, $\{(W_j, \Gamma_j, 1)\}_{j=1}^{2}$ is a (T_1, U_1)-controlled g-fusion frame for K with bounds $5/6$ and $4/3$. Therefore, according to Theorem 4, the family $\{(V_i \otimes W_j, \Lambda_i \otimes \Gamma_j, 1)\}_{i\in I, j\in J}$ is a $(T \otimes T_1, U \otimes U_1)$-controlled g-fusion frame for $H \otimes K = \mathbb{R}^6$ with bounds $5/6$ and $5/3$.

Remark 1 *Let Δ be a $(T \otimes T_1, U \otimes U_1)$-controlled g-fusion frame for the Hilbert space $H \otimes K$. According to its definition, the frame operator $S_\Delta : H \otimes K \to H \otimes K$ is described by*

$$S_\Delta h = \sum_{i,j} v_i^2 w_j^2 (U \otimes U_1)^* P_{V_i \otimes W_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} (T \otimes T_1) h$$

for all $h = f \otimes g \in H \otimes K$.

Theorem 5 *If S_{TU}, $S_{T_1U_1}$ and S_Δ are the corresponding frame operators for the controlled g-fusion frames Λ_{TU}, $\Gamma_{T_1U_1}$ and Δ, respectively, then $S_\Delta = S_{TU} \otimes S_{T_1U_1}$ and $S_\Delta^{-1} = S_{TU}^{-1} \otimes S_{T_1U_1}^{-1}$.*

Proof. For each $h = f \otimes g \in H \otimes K$, we have

$$\begin{aligned}
S_\Delta h &= \sum_{i,j} v_i^2 w_j^2 \left(U \otimes U_1\right)^* P_{V_i \otimes W_j} \left(\Lambda_i \otimes \Gamma_j\right)^* \left(\Lambda_i \otimes \Gamma_j\right) P_{V_i \otimes W_j} \left(T \otimes T_1\right) h \\
&= \sum_{i,j} v_i^2 w_j^2 \left(U^* \otimes U_1^*\right) \left(P_{V_i} \otimes P_{W_j}\right) \left(\Lambda_i^* \otimes \Gamma_j^*\right) \left(\Lambda_i \otimes \Gamma_j\right) \left(P_{V_i} T f \otimes P_{W_j} T_1 g\right) \\
&= \sum_{i,j} v_i^2 w_j^2 \left(U^* P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} T f \otimes U_1^* P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} T_1 g\right) \\
&= \left(\sum_{i \in I} v_i^2 U^* P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} T f\right) \otimes \left(\sum_{j \in J} w_j^2 U_1^* P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} T_1 g\right) \\
&= S_{TU} f \otimes S_{T_1U_1} g = \left(S_{TU} \otimes S_{T_1U_1}\right)(f \otimes g).
\end{aligned}$$

This shows that $S_\Delta = S_{TU} \otimes S_{T_1U_1}$. Since S_{TU} and $S_{T_1U_1}$ are invertible, by (iv) of Theorem 3, it follows that

$$S_\Delta^{-1} = \left(S_{TU} \otimes S_{T_1U_1}\right)^{-1} = S_{TU}^{-1} \otimes S_{T_1U_1}^{-1}.$$

This completes the proof. □

Theorem 6 *If S_{TU}, $S_{T_1U_1}$ and S_Δ are the corresponding frame operators for the controlled g-fusion frames Λ_{TU}, $\Gamma_{T_1U_1}$ and Δ, respectively. Then*

$$AC I_{H \otimes K} \leq S_\Delta \leq BD I_{H \otimes K},$$

where (A, B) and (C, D) are frame bounds of Λ_{TU} and $\Gamma_{T_1U_1}$, respectively, and $I_{H \otimes K}$ is the identity operator on $H \otimes K$.

Proof. Suppose Λ_{TU} is a (T, U)-controlled g-fusion frame for H with respect to $\{H_i\}_{i \in I}$ and $\Gamma_{T_1U_1}$ is a (T_1, U_1)-controlled g-fusion frame for K with respect to $\{K_j\}_{j \in J}$ having bounds A, B and C, D, respectively. Then

$$A I_H \leq S_{TU} \leq B I_H, \qquad C I_K \leq S_{T_1U_1} \leq D I_K.$$

Taking tensor product of the above two inequalities, we get

$$AC\left(I_H \otimes I_K\right) \leq S_{TU} \otimes S_{T_1U_1} \leq BD\left(I_H \otimes I_K\right).$$

Hence,

$$AC I_{H \otimes K} \leq S_\Delta \leq BD I_{H \otimes K}.$$

□

The frame decomposition formula is the most important result in the frame theory. It shows that if Λ_{TU} and $\Gamma_{T_1U_1}$ are controlled g-fusion frames in H and K, respectively, then every element in $H \otimes K$ has a representation as a superposition of the frame elements. Now, we present the frame decomposition formula in $H \otimes K$.

Theorem 7 *Let Λ_{TU} be a (T, U)-controlled g-fusion frame for H and $\Gamma_{T_1U_1}$ be a (T_1, U_1)-controlled g-fusion frames for K with the corresponding frame operators S_{TU} and $S_{T_1U_1}$, respectively. Then for each $f \otimes g \in H \otimes K$, we have*

$$f \otimes g = \sum_{i,j} v_i^2 w_j^2 S_\Delta^{-1} (U \otimes U_1)^* P_{V_i \otimes W_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} h$$

and

$$f \otimes g = \sum_{i,j} v_i^2 w_j^2 (U \otimes U_1)^* P_{V_i \otimes W_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} S_\Delta^{-1} h,$$

where $h = (T \otimes T_1)(f \otimes g)$.

Proof. Since S_{TU} and $S_{T_1U_1}$ are the corresponding frame operators for Λ_{TU} and $\Gamma_{T_1U_1}$, respectively, for each $f \in H$ and $g \in K$, we have

$$\begin{aligned} f &= S_{TU} S_{TU}^{-1} f = \sum_{i \in I} v_i^2 U^* P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} T S_{TU}^{-1} f, \\ g &= \sum_{j \in J} w_j^2 U_1^* P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} T_1 S_{T_1U_1}^{-1} g. \end{aligned}$$

Then for each $f \otimes g \in H \otimes K$, we can write

$$\begin{aligned} f \otimes g &= \left(\sum_{i \in I} v_i^2 U^* P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} T S_{TU}^{-1} f \right) \otimes \\ &\quad \left(\sum_{j \in J} w_j^2 U_1^* P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} T_1 S_{T_1T_1}^{-1} g \right) \\ &= \sum_{i,j} v_i^2 w_j^2 \left(U^* P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} T S_{TU}^{-1} f \otimes U_1^* P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} T_1 S_{T_1U_1}^{-1} g \right) \\ &= \sum_{i,j} v_i^2 w_j^2 (U \otimes U_1)^* P_{V_i \otimes W_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} S_\Delta^{-1} h, \end{aligned}$$

where $h = (T \otimes T_1)(f \otimes g)$. On the other hand, for each $f \otimes g \in H \otimes K$,

$$\begin{aligned} f \otimes g &= S_{TU}^{-1} S_{TU} f \otimes S_{T_1U_1}^{-1} S_{T_1U_1}^{-1} g \\ &= \sum_{i,j} v_i^2 w_j^2 S_\Delta^{-1} (U \otimes U_1)^* P_{V_i \otimes W_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} h. \end{aligned}$$

This completes the proof. □

To get the frame decomposition in $H \otimes K$, we need to find the operator S_{Δ}^{-1}. However, from the practical point of view, this is difficult. To avoid such a problem, we consider controlled tight g-fusion frame.

Corollary 1 *Let Λ_{TU} be a (T, U)-controlled tight g-fusion frame for H and let $\Gamma_{T_1 U_1}$ be a (T_1, U_1)-controlled tight g-fusion frame for K with bounds A_1 and A_2, respectively. Then for each $f \otimes g \in H \otimes K$,*

$$f \otimes g = \frac{1}{A_1 A_2} \sum_{i,j} v_i^2 w_j^2 \left(U \otimes U_1\right)^* P_{V_i \otimes W_j} \left(\Lambda_i \otimes \Gamma_j\right)^* \left(\Lambda_i \otimes \Gamma_j\right) P_{V_i \otimes W_j} h,$$

where $h = \left(T \otimes T_1\right)\left(f \otimes g\right)$.

In the next theorem, using controlled K-g-fusion frames and some elements of $\mathcal{B}(H)$ and $\mathcal{B}(K)$, we construct a new controlled g-fusion frame in $H \otimes K$.

Theorem 8 *Let $K_1 \in \mathcal{B}(H)$ and $K_2 \in \mathcal{B}(K)$. Let Λ_{TU} be (T, U)-controlled K_1-g-fusion frame for H and let $\Gamma_{T_1 U_1}$ be (T_1, U_1)-controlled K_2-g-fusion frame for K with bounds A, B and C, D and frame operators S_{TU} and $S_{T_1 U_1}$, respectively. Let $C_1 \in \mathcal{B}(H)$ and $C_2 \in \mathcal{B}(K)$ be invertible operators such that C_1^* commutes with T and U, while C_2^* commutes with T_1 and U_1, respectively. Then the family*

$$\Theta = \left\{\left(C_1 \otimes C_2\right)\left(V_i \otimes W_j\right), \left(\Lambda_i \otimes \Gamma_j\right) P_{V_i \otimes W_j} \left(C_1 \otimes C_2\right)^*, v_i w_j\right\}_{i \in I, j \in J}$$

is a $(T \otimes T_1, U \otimes U_1)$-controlled $\left(C_1 \otimes C_2\right)\left(K_1 \otimes K_2\right)\left(C_1 \otimes C_2\right)^*$-fusion frame for $H \otimes K$. The corresponding frame operator for Θ is

$$\left(C_1 \otimes C_2\right) S_{\Delta} \left(C_1 \otimes C_2\right)^*.$$

Proof. Take $\Theta_{ij} = \left(\Lambda_i \otimes \Gamma_j\right) P_{V_i \otimes W_j} \left(C_1 \otimes C_2\right)^* P_{\left(C_1 \otimes C_2\right)\left(V_i \otimes W_j\right)}$. Then by Theorems 1 and 3, we get

$$\begin{aligned}\Theta_{ij} &= \left(\Lambda_i \otimes \Gamma_j\right)\left(P_{V_i} \otimes P_{W_j}\right)\left(C_1^* \otimes C_2^*\right)\left(P_{C_1 V_i} \otimes P_{C_2 W_j}\right) \\ &= \Lambda_i P_{V_i} C_1^* P_{C_1 V_i} \otimes \Gamma_j P_{W_j} C_2^* P_{C_2 W_j} = \Lambda_i P_{V_i} C_1^* \otimes \Gamma_j P_{W_j} C_2^*.\end{aligned}$$

Using the obtained relation, for each $f \otimes g \in H \otimes K$, we can write

$$\begin{aligned}&\sum_{i,j} v_i^2 w_j^2 \left\langle \Theta_{ij}\left(U \otimes U_1\right)\left(f \otimes g\right), \Theta_{ij}\left(T \otimes T_1\right)\left(f \otimes g\right)\right\rangle \\ &= \sum_{i,j} v_i^2 w_j^2 \left\langle \Lambda_i P_{V_i} C_1^* U f \otimes \Gamma_j P_{W_j} C_2^* U_1 g, \Lambda_i P_{V_i} C_1^* T f \otimes \Gamma_j P_{W_j} C_2^* T_1 g\right\rangle \\ &= \sum_{i,j} v_i^2 w_j^2 \left\langle \Lambda_i P_{V_i} C_1^* U f, \Lambda_i P_{V_i} C_1^* T f\right\rangle_1 \left\langle \Gamma_j P_{W_j} C_2^* U_1 g, \Gamma_j P_{W_j} C_2^* T_1 g\right\rangle_2\end{aligned}$$

$$\begin{aligned}
&= \sum_{i\in I} v_i^2 \left\langle \Lambda_i P_{V_i} U C_1^* f, \Lambda_i P_{V_i} T C_1^* f \right\rangle_1 \times \\
&\qquad\qquad \sum_{j\in J} w_j^2 \left\langle \Gamma_j P_{W_j} U_1 C_2^* g, \Gamma_j P_{W_j} T_1 C_2^* g \right\rangle_2 \\
&\leq B \left\| C_1^* f \right\|_1^2 D \left\| C_2^* g \right\|_2^2 \leq BD \left\| C_1 \right\|^2 \left\| f \right\|_1^2 \left\| C_2 \right\|^2 \left\| g \right\|_2 \\
&= BD \left\| C_1 \otimes C_2 \right\|^2 \left\| f \otimes g \right\|^2.
\end{aligned}$$

On the other hand, for each $f \otimes g \in H \otimes K$, we have

$$\begin{aligned}
&\frac{AC}{\left\| C_1 \otimes C_2 \right\|^2} \left\| \left[(C_1 \otimes C_2)(K_1 \otimes K_2)(C_1 \otimes C_2)^* \right]^* (f \otimes g) \right\|^2 \\
&= \frac{AC}{\left\| C_1 \otimes C_2 \right\|^2} \left\| (C_1 \otimes C_2)(K_1^* \otimes K_2^*)(C_1^* \otimes C_2^*)(f \otimes g) \right\|^2 \\
&= \frac{AC}{\left\| C_1 \otimes C_2 \right\|^2} \left\| (C_1 K_1^* C_1^* \otimes C_2 K_2^* C_2^*)(f \otimes g) \right\|^2 \\
&= \frac{AC}{\left\| C_1 \right\|^2 \left\| C_2 \right\|^2} \left\| C_1 K_1^* C_1^* f \right\|_1^2 \left\| C_2 K_2^* C_2^* g \right\|_2^2 \\
&\leq A \left\| K_1^* C_1^* f \right\|_1^2 C \left\| K_2^* C_2^* g \right\|_2^2 \\
&\leq \sum_{i\in I} v_i^2 \left\langle \Lambda_i P_{V_i} U C_1^* f, \Lambda_i P_{V_i} T C_1^* f \right\rangle_1 \times \\
&\qquad\qquad \sum_{j\in J} w_j^2 \left\langle \Gamma_j P_{W_j} U_1 C_2^* g, \Gamma_j P_{W_j} T_1 C_2^* g \right\rangle_2 \\
&= \sum_{i,j} v_i^2 w_j^2 \left\langle \Theta_{ij} (U \otimes U_1)(f \otimes g), \Theta_{ij} (T \otimes T_1)(f \otimes g) \right\rangle.
\end{aligned}$$

Hence, Θ is a $(T \otimes T_1, U \otimes U_1)$-controlled $(C_1 \otimes C_2)(K_1 \otimes K_2)(C_1 \otimes C_2)^*$-fusion frame for $H \otimes K$.

Further, take $\Delta = (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} (C_1 \otimes C_2)^*$. Then, applying Theorem 3, we obtain

$$\begin{aligned}
\Delta^* \Delta &= \left((\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} (C_1 \otimes C_2)^* \right)^* (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} (C_1 \otimes C_2)^* \\
&= (C_1 \otimes C_2) \left(P_{V_i} \otimes P_{W_j} \right) \left(\Lambda_i^* \otimes \Gamma_j^* \right) (\Lambda_i \otimes \Gamma_j) \left(P_{V_i} \otimes P_{W_j} \right) (C_1^* \otimes C_2^*) \\
&= C_1 P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} C_1^* \otimes C_2 P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} C_2^*.
\end{aligned}$$

Now,

$$\begin{aligned}
&P_{(C_1 \otimes C_2)(V_i \otimes W_j)} \Delta^* \Delta P_{(C_1 \otimes C_2)(V_i \otimes W_j)} \\
&= \left(P_{C_1 V_i} \otimes P_{C_2 W_j} \right) \Delta^* \Delta \left(P_{C_1 V_i} \otimes P_{C_2 W_j} \right) \\
&= P_{C_1 V_i} C_1 P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} C_1^* P_{C_1 V_i} \otimes P_{C_2 W_j} C_2 P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} C_2^* P_{C_2 W_j} \\
&= (P_{V_i} C_1^*)^* \Lambda_i^* \Lambda_i P_{V_i} C_1^* \otimes \left(P_{W_j} C_2^* \right)^* \Gamma_j^* \Gamma_j P_{W_j} C_1^* \\
&= C_1 P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} C_1^* \otimes C_2 P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} C_2^*.
\end{aligned}$$

Therefore, for each $h = f \otimes g \in H \otimes K$, we have

$$\begin{aligned}
&\sum_{i,j} v_i^2 w_j^2 \left(U \otimes U_1\right)^* P_{(C_1 \otimes C_2)(V_i \otimes W_j)} \Delta^* \Delta P_{(C_1 \otimes C_2)(V_i \otimes W_j)} \left(T \otimes T_1\right) h \\
&= \sum_{i,j} v_i^2 w_j^2 \left(U^* \otimes U_1^*\right) \left(C_1 P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} C_1^* \otimes C_2 P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} C_2^*\right) \left(T f \otimes T_1 g\right) \\
&= \sum_{i,j} v_i^2 w_j^2 \left(U^* C_1 P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} C_1^* T f \otimes U_1^* C_2 P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} C_2^* T_1 g\right) \\
&= \left(\sum_{i \in I} v_i^2 C_1 U^* P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} T C_1^* f\right) \otimes \sum_{j \in J} w_j^2 C_2 U_1^* P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} T_1 C_2^* g \\
&= C_1 S_{TU} C_1^* f \otimes C_2 S_{T_1 U_1} C_2^* g \\
&= \left(C_1 \otimes C_2\right) \left(S_{TU} \otimes S_{T_1 U_1}\right) \left(C_1 \otimes C_2\right)^* \left(f \otimes g\right) \\
&= \left(C_1 \otimes C_2\right) S_\Delta \left(C_1 \otimes C_2\right)^* \left(f \otimes g\right).
\end{aligned}$$

Thus, the corresponding frame operator for Θ is $\left(C_1 \otimes C_2\right) S_\Delta \left(C_1 \otimes C_2\right)^*$. □

In the following theorem, we show that controlled g-fusion Bessel sequences in H and K become controlled g-fusion frames in $H \otimes K$.

Theorem 9 *Let $\Lambda_{TU} = \{(V_i, \Lambda_i, v_i)\}_{i \in I}$, $\Lambda'_{TU} = \{(V'_i, \Lambda'_i, v'_i)\}_{i \in I}$ be (T, U)-controlled g-fusion Bessel sequences with bounds B, D in H and $\Gamma_{T_1 U_1} = \{(W_j, \Gamma_j, w_j)\}_{j \in J}$, $\Gamma'_{T_1 U_1} = \left\{\left(W'_j, \Gamma'_j, w'_j\right)\right\}_{j \in J}$ be (T_1, U_1)-controlled g-fusion Bessel sequences with bounds E, F in K, respectively. Suppose $(T_\Lambda, T_{\Lambda'})$ and $(T_\Gamma, T_{\Gamma'})$ are their synthesis operators such that $T_{\Lambda'} T_\Lambda^* = I_H$ and $T_{\Gamma'} T_\Gamma^* = I_K$. Then*

$$\Delta = \{(V_i \otimes W_j, \Lambda_i \otimes \Gamma_j, v_i w_j)\}_{i \in I, j \in J}$$

and

$$\Delta' = \left\{\left(V'_i \otimes W'_j, \Lambda'_i \otimes \Gamma'_j, v'_i w'_j\right)\right\}_{i \in I, j \in J}$$

are $(T \otimes T_1, U \otimes U_1)$-controlled g-fusion frames for $H \otimes K$.

Proof. By Theorem 4, Δ and Δ' are $(T \otimes T_1, U \otimes U_1)$-controlled g-fusion Bessel sequences in $H \otimes K$ with bounds BE and DF, respectively. For each $f \otimes g \in H \otimes K$, we can write

$$\begin{aligned}
\|f \otimes g\|^4 &= |\langle f \otimes g, f \otimes g\rangle|^2 = |\langle f, f\rangle_1|^2 \, |\langle g, g\rangle_2|^2 \\
&= |\langle T_\Lambda^* f, T_{\Lambda'}^* f\rangle_1|^2 \, |\langle T_\Gamma^* g, T_{\Gamma'}^* g\rangle_2|^2 \\
&\leq \|T_\Lambda^* f\|_1^2 \, \|T_{\Lambda'}^* f\|_1^2 \, \|T_\Gamma^* g\|_2^2 \, \|T_{\Gamma'}^* g\|_2^2
\end{aligned}$$

$$= \sum_{i\in I}(v_i')^2 \left\langle \Lambda_i' P_{V_i'} U f, \Lambda_i' P_{V_i'} T f\right\rangle_1 \times \sum_{j\in J}(w_j')^2 \left\langle \Gamma_j' P_{W_j'} U_1 g, \Gamma_j' P_{W_j'} T_1 g\right\rangle_2$$
$$\times \sum_{i\in I} v_i^2 \left\langle \Lambda_i P_{V_i} U f, \Lambda_i P_{V_i} T f\right\rangle_1 \times \sum_{j\in J} w_j^2 \left\langle \Gamma_j P_{W_j} U_1 g, \Gamma_j P_{W_j} T_1 g\right\rangle_2$$
$$\leq DF \|f\|_1^2 \|g\|_2^2 \left(\sum_{i\in I} v_i^2 \left\langle \Lambda_i P_{V_i} U f, \Lambda_i P_{V_i} T f\right\rangle_1\right) \times$$
$$\left(\sum_{j\in J} w_j^2 \left\langle \Gamma_j P_{W_j} U_1 g, \Gamma_j P_{W_j} T_1 g\right\rangle_2\right)$$
$$= DF\|f\otimes g\|^2 \times \sum_{i,j} v_i^2 w_j^2 \left\langle \Lambda_i P_{V_i} U f \otimes \Gamma_j P_{W_j} U_1 g, \Lambda_i P_{V_i} T f \otimes \Gamma_j P_{W_j} T_1 g\right\rangle.$$

Thus, for $h = f\otimes g \in H\otimes K$, we can write

$$\frac{1}{DF}\|h\|^2 \leq \sum_{i,j} v_i^2 w_j^2 \left\langle \Delta_{ij} P_{V_i\otimes W_j}\left(U\otimes U_1\right)h, \Delta_{ij} P_{V_i\otimes W_j}\left(T\otimes T_1\right)h\right\rangle.$$

Hence, Δ is a $(T\otimes T_1, U\otimes U_1)$-controlled g-fusion frame for $H\otimes K$. Similarly, it can be shown that Δ' is also a $(T\otimes T_1, U\otimes U_1)$-controlled g-fusion frame for $H\otimes K$. □

Now, we present the frame operator for a pair of g-fusion Bessel sequences in $H\otimes K$.

Let $\Lambda_{TU} = \{(V_i, \Lambda_i, v_i)\}_{i\in I}$ and $\Lambda'_{TU} = \{(V_i', \Lambda_i', v_i')\}_{i\in I}$ be (T, U)-controlled g-fusion Bessel sequences in H. The operator $S_{\Lambda\Lambda'} : H \to H$ defined by

$$S_{\Lambda\Lambda'}(f) = \sum_{j\in J} v_j w_j U^* P_{W_j}\Lambda_j^*\Gamma_j P_{V_j} T f, \qquad f\in H,$$

is called the frame operator for the pair of controlled g-fusion Bessel sequences Λ_{TU} and Λ'_{TU}.

Let the families

$$\Delta = \{(V_i\otimes W_j, \Lambda_i\otimes\Gamma_j, v_i w_j)\}_{i\in I, j\in J}$$

and

$$\Delta' = \left\{\left(V_i'\otimes W_j', \Lambda_i'\otimes\Gamma_j', v_i' w_j'\right)\right\}_{i\in I, j\in J}$$

be two $(T\otimes T_1, U\otimes U_1)$-controlled g-fusion Bessel sequences in $H\otimes K$. The operator $S : H\otimes K \to H\otimes K$ defined by

$$Sh = \sum_{i,j} v_i w_j v_i' w_j' \left(U\otimes U_1\right)^* P_{V_i\otimes W_j}\left(\Lambda_i\otimes\Gamma_j\right)^*\left(\Lambda_i'\otimes\Gamma_j'\right) P_{V_i'\otimes W_j'} h,$$

where $h = \left(T\otimes T_1\right)\left(f\otimes g\right)$, $f\otimes g\in H\otimes K$, is called the frame operator for the pair of $(T\otimes T_1, U\otimes U_1)$-controlled g-fusion Bessel sequences Δ and Δ'.

If we put $T = U = I_H$ and $T_1 = U_1 = I_K$ in the above definition, the operator S becomes the frame operator for the pair of g-fusion Bessel sequences in $H \otimes K$. Thus, this concept is a generalization of the concept given in [10].

Now, let Δ and Δ' be two $(T \otimes T_1, U \otimes U_1)$-controlled g-fusion frames for $H \otimes K$. If we take $v_i = v_i'$, $V_i = V_i'$, $\Lambda_i = \Lambda_i'$ and $w_j = w_j'$, $W_j = W_j'$, $\Gamma_j = \Gamma_j'$ for each $i \in I$ and $j \in J$, we obtain $\Delta = \Delta'$. According to Remark 1, it follows that $S = S_\Delta$.

In the next theorem, we establish a relationship between the frame operator for a pair of controlled g-fusion Bessel sequences in $H \otimes K$ with those of H and K.

Theorem 10 *Let $S_{\Lambda\Lambda'}$ be the frame operator for the pair of (T, U)-controlled g-fusion Bessel sequences $\Lambda_{TU} = \{(V_i, \Lambda_i, v_i)\}_{i \in I}$ and $\Lambda'_{TU} = \{(V_i', \Lambda_i', v_i')\}_{i \in I}$ in H, and let $S_{\Gamma\Gamma'}$ be the frame operator for the pair of (T_1, U_1)-controlled g-fusion Bessel sequences*

$$\Gamma_{T_1U_1} = \{(W_j, \Gamma_j, w_j)\}_{j \in J} \qquad \text{and} \qquad \Gamma'_{T_1U_1} = \left\{\left(W_j', \Gamma_j', w_j'\right)\right\}_{j \in J}$$

in K. Then $S = S_{\Lambda\Lambda'} \otimes S_{\Gamma\Gamma'}$.

Proof. Noting that S is the associated frame operator for the pair of $(T \otimes T_1, U \otimes U_1)$-controlled g-fusion Bessel sequences Δ and Δ', for all $h = f \otimes g \in H \otimes K$, we can write

$$\begin{aligned}
Sh &= \sum_{i,j} v_i w_j v_i' w_j' \,(U \otimes U_1)^* \,P_{V_i \otimes W_j}\,(\Lambda_i \otimes \Gamma_j)^* \left(\Lambda_i' \otimes \Gamma_j'\right) P_{V_i' \otimes W_j'}\,(T \otimes T_1)\,h \\
&= \sum_{i,j} v_i w_j v_i' w_j' \left(U^* P_{V_i} \Lambda_i^* \Lambda_i' P_{V_i'} T f \otimes U_1^* P_{W_j} \Gamma_j^* \Gamma_j' P_{W_j'} T_1 g\right) \\
&= \left(\sum_{i \in I} v_i v_i' U^* P_{V_i} \Lambda_i^* \Lambda_i' P_{V_i'} T f\right) \otimes \left(\sum_{j \in J} w_j w_j' U_1^* P_{W_j} \Gamma_j^* \Gamma_j' P_{W_j'} T_1 g\right) \\
&= S_{\Lambda\Lambda'}(f) \otimes S_{\Gamma\Gamma'}(g) = \left(S_{\Lambda\Lambda'} \otimes S_{\Gamma\Gamma'}\right)(f \otimes g).
\end{aligned}$$

This shows that $S = S_{\Lambda\Lambda'} \otimes S_{\Gamma\Gamma'}$. □

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