Armenian Journal of Mathematics
Volume 2, Number 3, 2009, 94–101

On Some Formulas for the Index of a Linear Bounded Operator

I. G. Khachatryan*

* *Department of Mathematics, Yerevan State University*
Alex Manoogian 1, 0025, Yerevan, Armenia.
khachatryanis@yahoo.com

Dedicated to the memory of Prof. V. B. Lidskii

Abstract

We consider a linear bounded operator in infinite dimensional separable Hilbert space satisfying some conditions. We prove formulas that can be used to calculate the index of this operator.

Key Words: Operator, index, trace, absolute norm.
Mathematics Subject Classification 2000: Primary 58J20 Secondary 45P05 .

Let V be a linear bounded operator, acting in an infinite dimensional Hilbert space H, V^*— the adjoint operator, $\ker V$ and $\ker V^*$ — their null-spaces, $\dim(\ker V)$ and $\dim(\ker V^*)$— dimensions of corresponding subspaces. The index of the operator V is the number

$$\operatorname{ind} V = \dim(\ker V) - \dim(\ker V^*)\,, \tag{1}$$

assuming that these dimensions are finite.

Under some conditions on V we prove some formulas, which can be used to calculate $\operatorname{ind} V$. The conditions on V and the received formulas for $\operatorname{ind} V$ differ from the known ones (see [1]).

Lemma 1 *Let I be the identity operator and $\lambda \neq 0$ be some number. Then*

$$ker\,(V^*V) = ker\,V, \quad ker\,(VV^*) = ker\,V^*, \tag{2}$$

$$ker\,(V^*V-\lambda I)\cap ker\,V = \{0\}, \quad ker\,(VV^*-\lambda I)\cap ker\,V^* = \{0\}, \tag{3}$$

$$V(ker\,(V^*V-\lambda I)) = ker\,(VV^*-\lambda I), \quad V^*(ker\,(VV^*-\lambda I)) = ker\,(V^*V-\lambda I)\,, \tag{4}$$

$$dim(ker\,(V^*V-\lambda I)) \,=\, dim(ker\,(VV^*-\lambda I))\,. \tag{5}$$

Proof. If $x \in \ker(V^*V)$, then $(Vx, Vx) = (V^*Vx, x) = 0$. Therefore $Vx = 0$. It follows that $x \in \ker V$ and $\ker(V^*V) \subset \ker V$. From this and from obvious inclusion $\ker V \subset \subset \ker(V^*V)$ we get the first equality of (2). The second equality of (2) can be proved in the same way. Let $x \in \ker V$ and $x \neq 0$. Then $V^*Vx - \lambda x = -\lambda x \neq 0$, i. e. $x \notin \ker(V^*V - \lambda I)$. This implies the first equality of (3). The second equality of (3) can be proved in the same way. If $y \in \ker(V^*V - \lambda I)$, then for $z = Vy$ we have $VV^*z - \lambda z = V(V^*Vy - \lambda y) = 0$, i. e. $z \in \ker(VV^* - \lambda I)$. Thus, $V(\ker(V^*V - \lambda I)) \subset \ker(VV^* - \lambda I)$. Let $x \in \ker(VV^* - \lambda I)$ and $y = \frac{1}{\lambda} V^*x$. Then $x = Vy$ and $V^*Vy - \lambda y = \frac{1}{\lambda} V^*(VV^*x - \lambda x) = 0$, i. e. $y \in \ker(V^*V - \lambda I)$ and $x \in V(\ker(V^*V - \lambda I))$. Therefore, $\ker(VV^* - \lambda I) \subset V(\ker(V^*V - \lambda I))$. From these inclusions we obtain the first equality of (4). The second equality of (4) can be proved by similar reasoning. If $x \in ker(V^*V - \lambda I)$, then $(Vx, Vy) = \lambda(x, y)$ for all $y \in H$, and if $x \in ker(VV^* - \lambda I)$, then $(V^*x, V^*y) = \lambda(x, y)$. Hence, using (3) and (4) we obtain that the operator V transforms any orthogonal non zero system of elements from $\ker(V^*V - \lambda I)$ into an orthogonal non zero system of elements from $\ker(VV^* - \lambda I)$, and V^* performs this transformation in reversed order. Consequently, (5) holds.

Theorem 1 *Let for some number $c \neq 0$ operators $A = V^*V - cI$ and $B = VV^* - cI$ be compact. Then $c > 0$, and if the number $-c$ is an eigenvalue of the multiplicity κ' for A and of the multiplicity κ'' for B (we do not exclude cases $\kappa' = 0$ or $\kappa'' = 0$), then*

$$ind\, V = \kappa' - \kappa''. \tag{6}$$

Moreover, if the space H is separable and one of the operators A and B is Hilbert–Schmidt operator, then the other one is also Hilbert–Schmidt operator, and for their absolute norms $N(A)$ and $N(B)$ the equality

$$ind\, V = \tfrac{1}{c^2}\{N^2(A) - N^2(B)\} \tag{7}$$

holds. If one of the operators A or B is nuclear, then the other one is also nuclear, and for their traces $sp\, A$ and $sp\, B$ the equality

$$ind\, V = \frac{1}{c}\{sp\, B - sp\, A\} = \frac{1}{c} sp\,(B - A) = \frac{1}{c}\, sp\,(VV^* - V^*V) \tag{8}$$

holds.

Proof. The spectrum $\sigma(A)$ of any compact operator A is at most a countable and bounded set containing zero, and any non-zero element of this set is an eigenvalue of finite multiplicity. Moreover, if the set $\sigma(A)$ is infinite, then zero is the only limiting point of $\sigma(A)$. Evidently the spectrum of the operator V^*V is the set $\sigma(V^*V) = \{\lambda + c : \lambda \in \sigma(A)\}$. Thus $c \in \sigma(V^*V)$. Since V^*V is a non-negative self-adjoint operator, then $c > 0$, and A is self-adjoint. Similar statements are true for operators B and VV^*. Particularly $\sigma(VV^*) = \{\lambda + c : \lambda \in \sigma(B)\}$. From this and the statement (5) it follows that $\sigma(A) \backslash \{-c\} =$

$= \sigma(B)\backslash\{-c\}$, and if the number $\lambda \neq -c$ is an eigenvalue for one of the operators A and B, then λ is an eigenvalue of the same multiplicity for the other one (in the case $\lambda \neq 0$ this multiplicity is finite). Let $-c$ be an eigenvalue of multiplicity κ' for A and of multiplicity κ'' for B. These multiplicities, evidently are finite and

$$\kappa' = \dim(\ker(V^*V)), \quad \kappa'' = \dim(\ker(VV^*)).$$

From here, by (1) and (2) we get (6). Put $\sigma = \sigma(A)\backslash\{-c, 0\} = \sigma(B)\backslash\{-c, 0\}$. Any number $\lambda \in \sigma$ is an eigenvalue of the same multiplicity $\kappa(\lambda)$ for both operators A and B. Let A be a Hilbert–Schmidt operator, i. e. have finite absolute norm $N(A)$ (see [2], pp. 96—103, 208—212). Since the operator A is self-adjoint, then (see [2], p. 209)

$$N^2(A) = c^2\kappa' + \sum_{\lambda \in \sigma} \lambda^2 \kappa(\lambda).$$

Hence the absolute norm $N(B)$ of the operator B is also finite and

$$N^2(B) = c^2\kappa'' + \sum_{\lambda \in \sigma} \lambda^2 \kappa(\lambda).$$

Thus $N^2(A) - N^2(B) = c^2(\kappa' - \kappa'')$. From this and relation (6) we get (7).

Let the operator A be nuclear (see [2], p. 208—212), i. e.

$$\sum_{\lambda \in \sigma} |\lambda|\, \kappa(\lambda) < \infty.$$

Then the operator B is also nuclear. According to the theorem of V. B. Lidskii (see [2], p. 212; [3], p. 101; [4]), for $\operatorname{sp} A$ and $\operatorname{sp} B$ the following equalities

$$\operatorname{sp} A = -c\kappa' + \sum_{\lambda \in \sigma} \lambda\, \kappa(\lambda), \quad \operatorname{sp} B = -c\kappa'' + \sum_{\lambda \in \sigma} \lambda\, \kappa(\lambda)$$

are true. Hence $\operatorname{sp} B - \operatorname{sp} A = c(\kappa' - \kappa'')$ and by (6) we get (8).

The theorem is proved.

Consider in the space $L^2(a, b)$ with finite or infinite interval (a, b) the following integral operator $\mathcal{K}$:

$$(\mathcal{K}x)(\xi) = \int_a^b K(\xi, \eta)\, x(\eta)\, d\eta, \quad x \in L^2(a, b), \ \ \xi \in (a, b),$$

where the function $K(\xi, \eta)$ satisfies the following condition:

$$\int_a^b \int_a^b |K(\xi, \eta)|^2\, d\eta\, d\xi < \infty.$$

It is known (see [2], pp. 101—102), that $\mathcal{K}$ is Hilbert–Schmidt operator and its absolute norm $N(\mathcal{K})$ is equal to

$$N^2(\mathcal{K}) = \int_a^b \int_a^b |K(\xi, \eta)|^2\, d\eta\, d\xi.$$

If the operator $\mathcal{K}$ is self-adjoint, then $K(\xi,\,\eta)=\overline{K(\eta,\,\xi)}$. For the sake of definiteness we consider the case, where the self-adjoint compact operator $\mathcal{K}$ has an infinite set of eigenvalues. Enumerate non-zero eigenvalues λ_n $(n=1,\,2,\,\cdots)$ in order of decreasing moduli: $|\lambda_1|\geqslant|\lambda_2|\geqslant\cdots$, repeating each eigenvalue according to its multiplicity. Denote by φ_n $(n=1,\,2,\,\cdots)$ the orthonormal set of corresponding eigenfunctions: $\mathcal{K}\varphi_n=\lambda_n\varphi_n$. It is known (see [2], pp. 102, 209), that

$$N^2(\mathcal{K})=\sum_{n=1}^{\infty}\lambda_n^2\,,$$

$$K(\xi,\,\eta)\;=\;\sum_{n=1}^{\infty}\lambda_n\,\varphi_n(\xi)\,\overline{\varphi_n(\eta)}\,, \tag{9}$$

and the functional series in (9) converges in the space $L^2((a,\,b)\times(a,\,b))$.

Let the self-adjoint operator $\mathcal{K}$ be nuclear. Then

$$\sum_{n=1}^{\infty}|\lambda_n|<\infty\,,\quad sp\,\mathcal{K}=\sum_{n=1}^{\infty}\lambda_n\,.$$

We extend each function $x\in L^2(a,\,b)$ onto $\mathbb{R}=(-\infty,\,\infty)$, putting $x(\xi)=0$ for $\xi\notin(a,\,b)$. We extend also the function $K(\xi,\,\eta)$ onto $\mathbb{R}^2$, putting $K(\xi,\,\eta)=0$ for $(\xi,\,\eta)\notin(a,\,b)\times(a,\,b)$. By (9) we have

$$K(\xi+t,\,\xi)\;=\;\sum_{n=1}^{\infty}\lambda_n\,\varphi_n(\xi+t)\,\overline{\varphi_n(\xi)}\,, \tag{10}$$

and the functional series on variables ξ and t converges in the space $L^2(\mathbb{R}^2)$. Indeed, this fact follows from the equality:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big|\sum_{n=p}^{m}\lambda_n\,\varphi_n(\xi+t)\,\overline{\varphi_n(\xi)}\Big|^2dt\,d\xi\;=\;\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big|\sum_{n=p}^{m}\lambda_n\,\varphi_n(\eta)\,\overline{\varphi_n(\xi)}\Big|^2d\eta\,d\xi\;=$$

$$=\sum_{n=p}^{m}\sum_{j=p}^{m}\lambda_n\lambda_j\Big|\int_{-\infty}^{\infty}\overline{\varphi_n(\xi)}\,\varphi_j(\xi)\,d\xi\Big|^2\;=\;\sum_{n=p}^{m}\lambda_n^2$$

which is valid for any positive integers $p<m$.

Besides, for any t the functional series on the variable ξ in (10) converges in the space $L^1(\mathbb{R})$. Indeed, this follows from the estimate:

$$\int_{-\infty}^{\infty}\Big|\sum_{n=p}^{m}\lambda_n\,\varphi_n(\xi+t)\,\overline{\varphi_n(\xi)}\Big|\,d\xi\;\leqslant\;\sum_{n=p}^{m}|\lambda_n|\int_{-\infty}^{\infty}|\varphi_n(\xi+t)\,\overline{\varphi_n(\xi)}\,|\,d\xi\;\leqslant$$

$$\leqslant\;\sum_{n=p}^{m}|\lambda_n|\Big(\int_{-\infty}^{\infty}|\varphi_n(\xi+t)|^2\,d\xi\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{\infty}|\varphi_n(\xi)|^2\,d\xi\Big)^{\frac{1}{2}}\;=$$

$$=\;\sum_{n=p}^{m}|\lambda_n|\int_{-\infty}^{\infty}|\varphi_n(\xi)|^2\,d\xi\;=\;\sum_{n=p}^{m}|\lambda_n|\,.$$

Define the function $K(\xi,\,\xi)$ by the equality

$$K(\xi,\,\xi)\;=\;\sum_{n=1}^{\infty}\lambda_n\,|\varphi_n(\xi)|^2, \tag{11}$$

where the functional series converges in the space $L^1(\mathbb{R})$. Evidently,

$$\int\limits_a^b K(\xi,\,\xi)\,d\xi \;=\; \sum_{n=1}^{\infty} \lambda_n \;=\; \operatorname{sp}\mathcal{K}\,. \tag{12}$$

Taking into account (10) and (11), we get

$$\int\limits_{-\infty}^{\infty} |K(\xi+t,\,\xi)-K(\xi,\,\xi)|\,d\xi \;=\; \int\limits_{-\infty}^{\infty} \Big|\sum_{n=1}^{\infty} \lambda_n\,\overline{\varphi_n(\xi)}\{\varphi_n(\xi+t)-\varphi_n(\xi)\}\Big|\,d\xi \;\leqslant$$

$$\leqslant \sum_{n=1}^{\infty} |\lambda_n| \int\limits_{-\infty}^{\infty} |\overline{\varphi_n(\xi)}\{\varphi_n(\xi+t)-\varphi_n(\xi)\}|\,d\xi \;\leqslant$$

$$\leqslant \sum_{n=1}^{\infty} |\lambda_n| \Big(\int\limits_{-\infty}^{\infty} |\varphi_n(\xi)|^2\,d\xi\Big)^{\frac{1}{2}} \Big(\int\limits_{-\infty}^{\infty} |\varphi_n(\xi+t)-\varphi_n(\xi)|^2\,d\xi\Big)^{\frac{1}{2}} \;=$$

$$=\; \sum_{n=1}^{\infty} |\lambda_n| \Big(\int\limits_{-\infty}^{\infty} |\varphi_n(\xi+t)-\varphi_n(\xi)|^2\,d\xi\Big)^{\frac{1}{2}}.$$

But (see [5], pp. 499—502)

$$\lim_{t\to 0} \int\limits_{-\infty}^{\infty} |\varphi_n(\xi+t)-\varphi_n(\xi)|^2\,d\xi \;=\; 0\,,$$

$$\Big(\int\limits_{-\infty}^{\infty} |\varphi_n(\xi+t)-\varphi_n(\xi)|^2\,d\xi\Big)^{\frac{1}{2}} \;\leqslant$$

$$\leqslant\; \Big(\int\limits_{-\infty}^{\infty} |\varphi_n(\xi+t)|^2\,d\xi\Big)^{\frac{1}{2}} + \Big(\int\limits_{-\infty}^{\infty} |\varphi_n(\xi)|^2\,d\xi\Big)^{\frac{1}{2}} \;=\; 2\,.$$

Hence

$$\lim_{t\to 0} \int\limits_{-\infty}^{\infty} |K(\xi+t,\,\xi)-K(\xi,\,\xi)|\,d\xi \;=\; 0\,.$$

Thus for any finite or infinite interval $(\alpha,\,\beta)$ the equality

$$\int\limits_{\alpha}^{\beta} K(\xi,\,\xi)\,d\xi \;=\; \lim_{h\to 0} \frac{1}{h} \int\limits_0^h \int\limits_{\alpha}^{\beta} K(\xi+t,\,\xi)\,d\xi\,dt \tag{13}$$

holds. Particularly,

$$\operatorname{sp}\mathcal{K} \;=\; \lim_{h\to 0} \frac{1}{h} \int\limits_0^h \int\limits_a^b K(\xi+t,\,\xi)\,d\xi\,dt\,. \tag{14}$$

It is evident that if the function $K(\xi,\,\eta)$ is continuous in the domain $(a,\,b)\times(a,\,b)$, then the function $K(\xi,\,\xi)$, defined in the usual sense, satisfies the equality (13) for any finite interval $(\alpha,\,\beta)$. Thus, for the function $K(\xi,\,\xi)$ the equality (12) is also true, as

$$\operatorname{sp}\mathcal{K} \;=\; \lim_{h\to 0} \frac{1}{h} \int\limits_0^h \int\limits_{-\infty}^{\infty} K(\xi+t,\,\xi)\,d\xi\,dt \;=$$

$$=\; \lim_{\alpha\to-\infty}\,\lim_{\beta\to\infty} \Big(\lim_{h\to 0} \frac{1}{h} \int\limits_0^h \int\limits_{\alpha}^{\beta} K(\xi+t,\,\xi)\,d\xi\,dt\Big).$$

It is clear that above assertions remain true also for the case, where the set of eigenvalues of $\mathcal{K}$ is finite.

Note that in the case of finite interval $[a, b]$ and continuous function $K(\xi, \eta)$ the formula (12) has been obtained also in [3], pp. 115–118.

Corollary 1 *Let $H = L^2(a, b)$ with finite or infinite interval (a, b), and for some number $c > 0$ the operators $A = V^*V - cI$ and $B = VV^* - cI$ are integral operators, defined for $x \in L^2(a, b)$ by*

$$(A\,x)(\xi) = \int_a^b \mathcal{A}(\xi, \eta)\, x(\eta)\, d\eta\,, \quad (B\,x)(\xi) = \int_a^b \mathcal{B}(\xi, \eta)\, x(\eta)\, d\eta\,, \quad \xi \in (a, b)\,,$$

where the functions $\mathcal{A}(\xi, \eta)$ and $\mathcal{B}(\xi, \eta)$ satisfy the following conditions:

$$\int_a^b \int_a^b |\mathcal{A}(\xi, \eta)|^2\, d\eta\, d\xi < \infty\,, \quad \int_a^b \int_a^b |\mathcal{B}(\xi, \eta)|^2\, d\eta\, d\xi < \infty\,.$$

Then

$$ind\, V \;=\; \tfrac{1}{c^2} \int_a^b \int_a^b \{|\mathcal{A}(\xi, \eta)|^2 - |\mathcal{B}(\xi, \eta)|^2\}\, d\eta\, d\xi\,.$$

Besides, if operators A and B are nuclear, then

$$ind\, V \;=\; \lim_{h \to 0} \tfrac{1}{hc} \int_0^h \int_a^b \{\mathcal{B}(\xi + t, \xi) - \mathcal{A}(\xi + t, \xi)\}\, d\xi\, dt$$

(we suppose that $\mathcal{A}(\xi, \eta)$ and $\mathcal{B}(\xi, \eta)$ are equal to zero outside of $(a, b) \times (a, b)$), and if the functions $\mathcal{A}(\xi, \eta)$ and $\mathcal{B}(\xi, \eta)$ are continuous in $(a, b) \times (a, b)$, then

$$ind\, V \;=\; \tfrac{1}{c} \int_a^b \{\mathcal{B}(\xi, \xi) - \mathcal{A}(\xi, \xi)\}\, d\xi\,. \tag{15}$$

As an example of a bounded linear operator V in $L^2(0, \infty)$, for which $A = V^*V - I$ and $B = VV^* - I$ are integral operators, we can take the operator, defined for $x \in L^2(0, \infty)$ by

$$(V\,x)(\xi) \;=\; \tfrac{1}{\sqrt{2\pi}} \sum_{k=0}^{m} \int_0^\infty x(\eta)\, S_k(\eta)\, e^{i\omega_k \xi \eta}\, d\eta\,, \quad \xi \in (0, \infty)\,, \tag{16}$$

where $m \geqslant 1$ is a integer, i is the imaginary unit,

$$\omega_k \;=\; exp(\tfrac{i\pi k}{m})\,, \quad k \;=\; 0,\, 1,\, \cdots,\, m\,,$$

the functions $S_k(\eta)$ are continuous and bounded on $(0, \infty)$, with $S_m(\eta) \equiv 1$, $|S_0(\eta)| \equiv 1$, the function $S_0(\eta)$ has continuous and integrable on $(0, \infty)$ derivative $S_0'(\eta)$, and the limits $S_0(0)$ and $S_0(\infty)$ of $S_0(\eta)$ at $\eta \to 0$ and $\eta \to \infty$ are real numbers.

The adjoint operator V^* is defined by the formula

$$(V^*\,x)(\xi) \;=\; \tfrac{1}{\sqrt{2\pi}} \sum_{k=0}^{m} \overline{S_k(\xi)} \int_0^\infty x(\eta)\, e^{-i\overline{\omega}_k \xi \eta}\, d\eta\,.$$

It is easy to see that

$$(A\,x)(\xi) \;=\; \int\limits_0^\infty \mathcal{A}(\xi,\,\eta)\,x(\eta)\,d\eta\,, \quad (B\,x)(\xi) \;=\; \int\limits_0^\infty \mathcal{B}(\xi,\,\eta)\,x(\eta)\,d\eta\,,$$

where

$$\mathcal{A}(\xi,\,\eta) \;=\; \tfrac{1}{2\pi i} \sum_{k,j=0}^{m} \tfrac{\overline{S_k(\xi)}\,S_j(\eta)}{\overline{\omega}_k\xi\,-\,\omega_j\eta}\,,$$

$$\mathcal{B}(\xi,\,\eta) \;=\; \tfrac{1}{2\pi i(\xi\,+\,\eta)}\Big\{\int\limits_0^\infty \overline{S_0'(t)}\,e^{-it(\xi\,+\,\eta)}\,dt \;-\; \int\limits_0^\infty S_0'(t)\,e^{it(\xi\,+\,\eta)}\,dt\Big\}\,+$$

$$+\,\tfrac{1}{2\pi} \sum_{k=1}^{m-1} \int\limits_0^\infty \big\{S_k(t)\,e^{it(\omega_k\xi\,+\,\eta)} \;+\; \overline{S_k(t)}\,e^{-it(\xi\,+\,\overline{\omega}_k\eta)}\big\}\,dt\,+$$

$$+\,\tfrac{1}{2\pi} \sum_{k=1}^{m-1} \int\limits_0^\infty \big\{S_k(t)\,\overline{S_0(t)}\,e^{it(\omega_k\xi\,-\,\eta)} \;+\; \overline{S_k(t)}\,S_0(t)\,e^{it(\xi\,-\,\overline{\omega}_k\eta)}\big\}\,dt\,+$$

$$+\,\tfrac{1}{2\pi} \sum_{k,j=1}^{m-1} \int\limits_0^\infty \overline{S_k(t)}\,S_j(t)\,e^{it(\omega_j\xi\,-\,\overline{\omega}_k\eta)}\,dt\,.$$

Under some additional restrictions on the functions S_k, the equality (15) can be proved and reduced to the form

$$\operatorname{ind} V \;=\; \tfrac{1}{2\pi i} \int\limits_0^\infty \tfrac{S_0'(\xi)}{S_0\xi)}\,d\xi \,-\, \tfrac{1}{4}(S_0(\infty) - S_0(0))\,.$$

In the case of $m = 1$ at least one of the operators V and V^* has inverse even if the function S_0 is only measurable and bounded (see [6]).

Operator of the form (16) arise in the investigations of the scattering inverse problem for differential operator of order $2m$, and the equality (15) expresses a relation between scattering data (see [7]—[9]).

References

[1] B. V. Fedosov, Index theorems, In *Partial differential equations, 8 (Russian)*, Itogi Nauki i Tekhniki, pages 165–268, 275. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1991.

[2] N. I. Akhiezer and I. M. Glazman *Theory of linear operators in Hilbert space,* Moscow, Nauka, 1966 (in Russian).

[3] I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators,* Translations of Mathematical Monographs, Vol. 18, AMS, Providence, R.I. 1969.

[4] V. B. Lidskii, *Non self-adjoint operators, having trace,* DAN SSSR, **125**(1959), no. 3, 485–488 (in Russian).

[5] L. A. Lusternik and V. J. Sobolev, *Elements of functional analysis*, Moscow, Nauka, 1965 (in Russian).

[6] I. G. Khachatryan, *On the invertibility of some Fourier type integral operators*, Proceedings of Yerevan State University, 2004, No 2, pp. 3–6 (in Russian).

[7] I. G. Khachatryan, *On a trace formula*, Izvestia AN Arm SSR, Matematika, **20**(1985), no 1, pp. 41–52 (in Russian).

[8] V. A. Marchenko, *Sturm-Liouville operators and applications*, Naukova Dumka, Kiev, 1977 (in Russian).

[9] Z. S. Agranovich and V. A. Marchenko, *The inverse problem of the scattering theory*, Kharkov, 1960 (in Russian).