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Weighted integral representations of harmonic functions in the unit disc by means of Mittag-Leffler type kernels

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Abstract. For weighted L^p-classes of functions harmonic in the unit disc, we obtain a family of weighted integral representations with weight function of the type $|w|^{2\varphi} \cdot (1-|w|^{2\rho})^{\beta}$.

Key Words: Harmonic Functions in the Unit Disc, Weighted Function Space, Weighted Integral Representation
Mathematics Subject Classification 2010: 30F15, 30C40, 30H20, 30E20

1 Introduction

It is well-known that the Cauchy integral formula has numerous applications in complex analysis.This formula makes it possible to reproduce values of holomorphic functions inside of a domain by integration of function along the boundary of the domain. First results are contained in [1, 2], where the values of holomorphic functions inside of a domain were obtained by integration of functions over the whole domain. In [3, 4] for the weighted spaces $H^p(\alpha)$ $(1 \le p < \infty, \alpha > -1)$ of functions f holomorphic in the unit disc $\mathbb{D}$ and satisfying the condition

$$\int_{\mathbb{D}} |f(\zeta)|^p (1-|\zeta|^2)^{\alpha} dudv < +\infty, \quad \zeta = u + iv,$$

the following result was established:

Theorem 1. *Each function $f \in H^p(\alpha)$ has the integral representation*

$$f(z) = \frac{\alpha+1}{\pi} \int_{\mathbb{D}} \frac{f(\zeta)(1-|\zeta|^2)^{\alpha}}{(1-z\overline{\zeta})^{2+\alpha}} dudv, \quad z \in \mathbb{D}, \tag{1}$$

$$\overline{f(0)} = \frac{\alpha+1}{\pi} \int_{\mathbb{D}} \frac{\overline{f(\zeta)}(1-|\zeta|^2)^{\alpha}}{(1-z\overline{\zeta})^{2+\alpha}} dudv, \quad z \in \mathbb{D}. \tag{2}$$

These representations had numerous applications in the theory of factorization of meromorphic functions in the unit disc (see [3, 4] as well as [5]).

Assume that $0 < p < +\infty, \rho > 0, \alpha > -1, \gamma > -1$. Denote by $L^p_{\alpha,\rho,\gamma}(\mathbb{D})$ the set of all complex-valued measurable functions $f(\zeta), \zeta \in \mathbb{D}$, for which

$$M^p_{\alpha,\rho,\gamma}(f) \equiv \iint\limits_{\mathbb{D}} |f(\zeta)|^p \left(1 - |\zeta|^{2\rho}\right)^{\alpha} |\zeta|^{2\gamma} dm(\zeta) < +\infty.$$

Also we will use the following notations:

$$H^p_{\alpha,\rho,\gamma}(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : M^p_{\alpha,\rho,\gamma}(f) < +\infty \right\},$$

$$h^p_{\alpha,\rho,\gamma}(\mathbb{D}) = \left\{ f \in h(\mathbb{D}) : M^p_{\alpha,\rho,\gamma}(f) < +\infty \right\},$$

where $H(\mathbb{D})$ and $h(\mathbb{D})$ are the sets of all holomorphic and harmonic functions in $\mathbb{D}$, respectively.

The spaces above were introduced in [6]. Moreover, for these spaces an analogue of representations (1) and (2) were written out by means of special reproducing kernels adapted to new weight functions (see [6] and [7]):

Let $\rho > 0, Re\beta > -1, Re\varphi > -1$ and $\mu = \dfrac{1+\varphi}{\rho}$. Then put

$$S_{\beta,\rho,\varphi}(z,\zeta) = \frac{\rho}{\pi\Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{\Gamma(\mu+\beta+1+\frac{k}{\rho})}{\Gamma(\mu+\frac{k}{\rho})} z^k \overline{\zeta}^k, \tag{3}$$

where $z \in \mathbb{D}$ and $\zeta \in \overline{\mathbb{D}}$.

The main properties of the kernel $S_{\beta,\rho,\varphi}(z,\zeta)$ can be summarized in the following theorem (see [6], [7]).

Theorem 2. *1. For all $z \in \mathbb{D}$ and $\zeta \in \overline{\mathbb{D}}$, the series $S_{\beta,\rho,\varphi}(z,\zeta)$ is absolutely convergent.*

2. For all $z \in \mathbb{D}$ and $\zeta \in \overline{\mathbb{D}}$,

$$|S_{\beta,\rho,\varphi}(z,\zeta)| \le \frac{const(\beta,\rho,\varphi)}{(1-|z|)^{2+Re\beta}}.$$

3. $S_{\beta,\rho,\varphi}(z,\zeta)$ can be majorated by a positive convergent series uniformly in $z \in K \subset \mathbb{D}$ and $\zeta \in \overline{\mathbb{D}}$, where K is a compact set.

4. For a fixed $\zeta \in \overline{\mathbb{D}}$, S is holomorphic in $z \in \mathbb{D}$. For a fixed $z \in \mathbb{D}$, S is antiholomorphic in $\zeta \in \mathbb{D}$ and continuous in $\zeta \in \overline{\mathbb{D}}$.

5. *For all* $z \in \mathbb{D}$ *and* $\zeta \in \overline{\mathbb{D}}$,

$$S_{\beta,\rho,\varphi}(z,\zeta) = \frac{\rho}{\pi\Gamma(\beta+1)} \int_0^{\infty} e^{-t} t^{\mu+\beta} E_\rho\left(t^{\frac{1}{\rho}} z\overline{\zeta}; \mu\right) dt,$$

where $E_\rho(\ \cdot\ ; \mu)$ *is the well-known Mittag-Leffler type entire function. Moreover, the function under the sign of the integral is majorated by a positive integrable function uniformly in* $z \in K \subset \mathbb{D}$ *and* $\zeta \in \overline{\mathbb{D}}$, *where* K *is a compact set.*

The corresponding generalization of (1) and (2) is formulated as follows (see [6], [7]):

Theorem 3. *Assume that* $1 \le p < +\infty, \rho > 0, \alpha > -1, \gamma > -1$, *complex numbers* β *and* φ *are satisfying the conditions*

$$Re\beta \ge \alpha, \quad Re\varphi \ge \gamma,$$

when $p = 1$ *and the conditions*

$$Re\beta > \frac{\alpha+1}{p} - 1, \quad Re\varphi > \frac{\gamma+1}{p} - 1,$$

when $p > 1$, *and put* $\mu = (\varphi+1)/\rho$. *Then each function* $f \in H^p_{\alpha,\rho,\gamma}(\mathbb{D})$ *has the following representation:*

$$f(z) = \iint_{\mathbb{D}} f(\zeta) S_{\beta,\rho,\varphi}(z,\zeta)(1-|\zeta|^{2\rho})^{\beta} |\zeta|^{2\varphi} dm(\zeta), \qquad z \in \mathbb{D},$$

and

$$\overline{f(0)} = \iint_{\mathbb{D}} \overline{f(\zeta)} S_{\beta,\rho,\varphi}(z,\zeta)\left(1-|\zeta|^{2\rho}\right)^{\beta} |\zeta|^{2\varphi} dm(\zeta), \qquad z \in \mathbb{D}.$$

In the present paper, we prove an analogous result of the theorem above for harmonic functions from the corresponding weighted L^p-spaces in the unit disc $\mathbb{D}$.

2 Necessary Estimates

First of all, we intend to strengthen some assertions of the Theorem 2.

Theorem 4. *Assume* $\rho > 0, Re\beta > -1, Re\varphi > -1, \mu = \dfrac{1+\varphi}{\rho}$, $z \in \mathbb{D}, \zeta \in \overline{\mathbb{D}}$ *and the kernel* $S_{\beta,\rho,\varphi}(z,\zeta)$ *is defined by* (3). *If*

$$-1 < a_1 \le Re\beta \le a_2 < +\infty, \qquad |Im\beta| \le A < +\infty,$$
$$-1 < b_1 \le Re\varphi \le b_2 < +\infty, \qquad |Im\varphi| \le B < +\infty$$

and $\zeta \in \overline{\mathbb{D}}$, $|z| \le \lambda < 1$, *then the expression of* $S_{\beta,\rho,\varphi}$ *can be uniformly majorated by the convergent series*

$$const(a_1, a_2, A, b_1, b_2, B, \rho) \sum_{k=0}^{\infty} (k+1)^{a_2+1} \lambda^k. \tag{4}$$

Moreover,

$$|S_{\beta,\rho,\varphi}(z,\zeta)| \le \frac{const(a_1, a_2, A, b_1, b_2, B, \rho)}{(1-|z||\zeta|)^{2+Re\beta}} \le \frac{const(a_1, a_2, A, b_1, b_2, B, \rho)}{(1-\lambda)^{2+a_2}}. \tag{5}$$

Proof. According to Stirling formula (see, for example, [8], pp. 158-159) there exist numbers a, b, $0 < a < b < +\infty$, such that

$$a \le \frac{|\Gamma(\mu+R)|}{R^{Re\mu+R-1/2} \cdot e^{-R}} \le b \tag{6}$$

uniformly in $\mu \in K \Subset \{\mu \in \mathbb{C} : Re\mu > 0\}$ and $0 < \delta \le R < +\infty$. Hence, in view of (6),

$$\begin{aligned}
|S_{\beta,\rho,\varphi}(z,\zeta)| &\le \frac{\rho}{\pi} \frac{|\Gamma(\mu+\beta+1)|}{|\Gamma(\beta+1)||\Gamma(\mu)|} \\
&+ \frac{\rho}{\pi|\Gamma(\beta+1)|} \sum_{k=1}^{\infty} \frac{|\Gamma(\mu+\beta+1+\frac{k}{\rho})|}{|\Gamma(\mu+\frac{k}{\rho})|} |z|^k \overline{|\zeta|}^k \\
&\le const(a_1, a_2, A, b_1, b_2, B, \rho) \\
&+ const(a_1, a_2, A, b_1, b_2, B, \rho) \sum_{k=1}^{\infty} \frac{\left(\frac{k}{\rho}\right)^{Re\mu+Re\beta+1+\frac{k}{\rho}-\frac{1}{2}} \cdot e^{-\frac{k}{\rho}}}{\left(\frac{k}{\rho}\right)^{Re\mu+\frac{k}{\rho}-\frac{1}{2}} \cdot e^{-\frac{k}{\rho}}} |z|^k |\zeta|^k \\
&= const(a_1, a_2, A, b_1, b_2, B, \rho) \\
&+ const(a_1, a_2, A, b_1, b_2, B, \rho) \sum_{k=1}^{\infty} k^{Re\beta+1} |z|^k |\zeta|^k \\
&\le const(a_1, a_2, A, b_1, b_2, B, \rho) \\
&+ const(a_1, a_2, A, b_1, b_2, B, \rho) \sum_{k=1}^{\infty} k^{a_2+1} |z|^k |\zeta|^k.
\end{aligned}$$

From here (4) follows. As to (5), it directly follows from the estimation

$$k^{Re\beta+1} \asymp \frac{\Gamma(k+Re\beta+2)}{\Gamma(Re\beta+2)\Gamma(k+1)}$$

and binomial expansion

$$\frac{1}{(1-x)^s} = \sum_{k=0}^{\infty} \frac{\Gamma(k+s)}{\Gamma(s)\Gamma(k+1)} \cdot x^k.$$

□

Corollary. *For a fixed $z \in \mathbb{D}$ and $\zeta \in \overline{\mathbb{D}}$, the kernel $S_{\beta,\rho,\varphi}(z,\zeta)$ is holomorphic in β and φ with $Re\beta > -1$ and $Re\varphi > -1$.*

Theorem 5. *Assume that $1 \le p < \infty$, $\rho > 0$, $\alpha > -1, \gamma > -1$ and $f \in L^p_{\alpha,\rho,\gamma}(\mathbb{D})$. Then there exists a positive function $\Phi(\zeta) \in L^1(\mathbb{D})$ such that*

$$\left| f(\zeta)(1-|\zeta|^{2\rho})^{\beta} |\zeta|^{2\varphi} S_{\beta,\rho,\varphi}(z,\zeta)\right| \le \Phi(\zeta), \qquad \zeta \in \mathbb{D},$$

uniformly in $z \in K \Subset \mathbb{D}$ and in β and φ satisfying the conditions

$$\alpha = a_1 \le Re\beta \le a_2, \quad |Im\beta| \le A,$$

$$\gamma = b_1 \le Re\varphi \le b_2, \quad |Im\varphi| \le B,$$

when $p = 1$ and the conditions

$$\frac{\alpha+1}{p} - 1 < a_1 \le Re\beta \le a_2, \quad |Im\beta| \le A,$$

$$\frac{\gamma+1}{p} - 1 < b_1 \le Re\varphi \le b_2, \quad |Im\varphi| \le B,$$

when $p > 1$.

Proof. Note that under the assumptions of the theorem, $|S_{\beta,\rho,\varphi}(z,\zeta)| \le const(a_1, a_2, A, b_1, b_2, B, \rho, K) < +\infty$, $\zeta \in \overline{\mathbb{D}}$ (see (5)). If $p = 1$, then

$$\begin{aligned}\left| f(\zeta)\cdot(1-|\zeta|^{2\rho})^{\beta}\cdot|\zeta|^{2\varphi}\cdot S_{\beta,\rho,\varphi}(z,\zeta)\right| &\le const\cdot|f(\zeta)|\cdot(1-|\zeta|^{2\rho})^{Re\beta}\cdot|\zeta|^{2Re\varphi} \\ &\le const\cdot|f(\zeta)|\cdot(1-|\zeta|^{2\rho})^{\alpha}\cdot|\zeta|^{2\gamma} = \Phi(\zeta) \in L^1(\mathbb{D}),\end{aligned}$$

while for $p > 1$ we can write

$$\begin{aligned}&\left| f(\zeta)\cdot(1-|\zeta|^{2\rho})^{\beta}\cdot|\zeta|^{2\varphi}\cdot S_{\beta,\rho,\varphi}(z,\zeta)\right| \\ &\quad\le const\cdot|f(\zeta)|\cdot(1-|\zeta|^{2\rho})^{Re\beta}\cdot|\zeta|^{2Re\varphi} \\ &\quad\le const\cdot|f(\zeta)|\cdot(1-|\zeta|^{2\rho})^{a_1}\cdot|\zeta|^{2b_1} \\ &\quad= const\cdot|f(\zeta)|\cdot(1-|\zeta|^{2\rho})^{\frac{\alpha}{p}}\cdot|\zeta|^{\frac{2\gamma}{p}}\cdot(1-|\zeta|^{2\rho})^{a_1-\frac{\alpha}{p}}\cdot|\zeta|^{2b_1-\frac{2\gamma}{p}} = \Phi(\zeta).\end{aligned}$$

It remains to show that $\Phi(\zeta) \in L^1(\mathbb{D})$. We have

$$\iint_{\mathbb{D}} \Phi(\zeta)dm(\zeta) \leq const \cdot \left(\iint_{\mathbb{D}} |f(\zeta)|^p (1-|\zeta|^{2\rho})^{\alpha} |\zeta|^{2\gamma} dm(\zeta) \right)^{1/p}$$
$$\left(\iint_{\mathbb{D}} (1-|\zeta|^{2\rho})^{q\left(a_1-\frac{\alpha}{p}\right)} |\zeta|^{2q\left(b_1-\frac{\gamma}{p}\right)} dm(\zeta) \right)^{1/q},$$

and the convergence of the second integral follows from the conditions of the theorem. □

3 Weighted integral representations for the main classes of holomorphic and harmonic functions

Theorem 6. *Assume* $1 \leq p < \infty$, $\rho > 0$, $\alpha > -1, \gamma > -1$, *complex numbers* β *and* φ *are satisfying the conditions*

$$Re\beta \geq \alpha, \quad Re\varphi \geq \gamma, \tag{7}$$

when $p = 1$ *and the conditions*

$$Re\beta > \frac{\alpha+1}{p} - 1, \quad Re\varphi > \frac{\gamma+1}{p} - 1, \tag{8}$$

when $p > 1$, *and put* $\mu = \dfrac{\varphi+1}{\rho}$. *Let* $f = u + iv \in H^p_{\alpha,\rho,\gamma}$. *Then the following integral representations hold: for all* $z \in \mathbb{D}$

$$f(z) = iv(0) + \iint_{\mathbb{D}} u(\zeta) \left(2S_{\beta,\rho,\varphi}(z,\zeta) - \frac{\rho}{\pi} \frac{\Gamma(\mu+\beta+1)}{\Gamma(\beta+1)\Gamma(\mu)} \right) (1-|\zeta|^{2\rho})^{\beta} |\zeta|^{2\varphi} dm(\zeta) \tag{9}$$

and

$$u(z) = \iint_{\mathbb{D}} u(\zeta) \left(S_{\beta,\rho,\varphi}(z,\zeta) + S_{\beta,\rho,\varphi}(\zeta,z) - \frac{\rho}{\pi} \frac{\Gamma(\mu+\beta+1)}{\Gamma(\beta+1)\Gamma(\mu)} \right) (1-|\zeta|^{2\rho})^{\beta} |\zeta|^{2\varphi} dm(\zeta) \tag{10}$$

Proof. First of all, note that

$$S_{\beta,\rho,\varphi}(0,\zeta)=S_{\beta,\rho,\varphi}(z,0)=\frac{\rho}{\pi}\frac{\Gamma(\mu+\beta+1)}{\Gamma(\beta+1)\Gamma(\mu)},\qquad z\in\mathbb{D},\quad \zeta\in\overline{\mathbb{D}}. \tag{11}$$

Further, due to Corollary 1, the expressions under the signs of the integrals are holomorphic in β and φ, $Re\beta>-1$, $Re\varphi>-1$ for fixed z and ζ. Hence, according to Theorem 5, for a fixed $z\in\mathbb{D}$, the right-hand side of (9) and (10) are also holomorphic in β and φ when (8) (or (7), depending on p) is satisfied. Thus, in view of the uniqueness theorem for holomorphic functions in two complex variables, we can additionally suppose (without loss of generality) that β and φ are real. According to [7, Theorem 4.2],

$$f(z)=\iint\limits_{\mathbb{D}} f(\zeta)S_{\beta,\rho,\varphi}(z,\zeta)(1-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi}dm(\zeta),\qquad z\in\mathbb{D}, \tag{12}$$

and

$$\overline{f(0)}=\iint\limits_{\mathbb{D}} \overline{f(\zeta)}S_{\beta,\rho,\varphi}(z,\zeta)\left(1-|\zeta|^{2\rho}\right)^{\beta}|\zeta|^{2\varphi}dm(\zeta),\qquad z\in\mathbb{D}. \tag{13}$$

Summation of (12) and (13) yields

$$f(z)+\overline{f(0)}=\iint\limits_{\mathbb{D}} 2u(\zeta)S_{\beta,\rho,\varphi}(z,\zeta)(1-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi}dm(\zeta). \tag{14}$$

According to (12) and (11),

$$f(0)=\iint\limits_{\mathbb{D}} f(\zeta)\frac{\Gamma(\mu+\beta+1)}{\Gamma(\beta+1)\Gamma(\mu)}\frac{\rho}{\pi}(1-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi}dm(\zeta),\qquad z\in\mathbb{D}.$$

Hence,

$$u(0)=\iint\limits_{\mathbb{D}} u(\zeta)\frac{\Gamma(\mu+\beta+1)}{\Gamma(\beta+1)\Gamma(\mu)}\frac{\rho}{\pi}(1-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi}dm(\zeta),\qquad z\in\mathbb{D}. \tag{15}$$

Combination of (14) and (15) immediately gives (9).

Further, taking the real parts in (14) we get

$$u(z)=\iint\limits_{\mathbb{D}} u(\zeta)\left(S_{\beta,\rho,\varphi}(z,\zeta)+S_{\beta,\rho,\varphi}(\zeta,z)\right)(1-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi}dm(\zeta)-u(0). \tag{16}$$

The formulas (16) and (15) together establish (10). □

Theorem 7. *Assume* $1 \le p < \infty$, $\rho > 0$, $\alpha > -1, \gamma > -1$, *complex numbers* β *and* φ *are satisfying the conditions*

$$Re\beta \ge \alpha, \quad Re\varphi \ge \gamma,$$

when $p = 1$ *and the conditions*

$$Re\beta > \frac{\alpha+1}{p} - 1, \quad Re\varphi > \frac{\gamma+1}{p} - 1,$$

when $p > 1$, *and put* $\mu = (\varphi+1)/\rho$. *For each* $u \in h^p_{\alpha,\rho,\gamma}(\mathbb{D})$, *the representation* (10) *holds.*

Proof. Repeating the argument from the proof of Theorem 6, we can assume additionally, that $\beta \in \mathbb{R}, \varphi \in \mathbb{R}$ and $\beta > 0$.

Since u is a harmonic function, there exists a holomorphic function f in $\mathbb{D}$ such that $u = Ref$ in $\mathbb{D}$. Fix an arbitrary $z_0 \in \mathbb{D}$ and denote $f_r(\zeta) = f(r\zeta)$. Obviously, $f_r \in H^p_{\alpha,\rho,\gamma}(\mathbb{D})$. As $u(r\zeta) = Ref(r\zeta)$, from (10) we obtain

$$u(rz_0) = \iint\limits_{\mathbb{D}} u(r\zeta)\left(S_{\beta,\rho,\varphi}(z_0,\zeta) + S_{\beta,\rho,\varphi}(\zeta,z_0) - \frac{\rho}{\pi}\frac{\Gamma(\mu+\beta+1)}{\Gamma(\beta+1)\Gamma(\mu)}\right)$$
$$(1-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi}dm(\zeta)$$
$$= \iint\limits_{|\zeta|<r} u(\zeta)\left(S_{\beta,\rho,\varphi}\left(z_0,\frac{\zeta}{r}\right) + S_{\beta,\rho,\varphi}\left(\frac{\zeta}{r},z_0\right) - \frac{\rho}{\pi}\frac{\Gamma(\mu+\beta+1)}{\Gamma(\beta+1)\Gamma(\mu)}\right)$$
$$\left(1-\frac{|\zeta|^{2\rho}}{r^{2\rho}}\right)^{\beta}\frac{|\zeta|^{2\varphi}}{r^{2\varphi}}\frac{dm(\zeta)}{r^2}$$
$$= \iint\limits_{\mathbb{D}} u(\zeta)\chi_r(\zeta)\left(S_{\beta,\rho,\varphi}\left(z_0,\frac{\zeta}{r}\right) + S_{\beta,\rho,\varphi}\left(\frac{\zeta}{r},z_0\right) - \frac{\rho}{\pi}\frac{\Gamma(\mu+\beta+1)}{\Gamma(\beta+1)\Gamma(\mu)}\right)$$
$$\left(1-\frac{|\zeta|^{2\rho}}{r^{2\rho}}\right)^{\beta}\frac{|\zeta|^{2\varphi}}{r^{2\varphi}}\frac{dm(\zeta)}{r^2} = I_r,$$

where $\chi_r(\zeta)$ is the characteristic function of the disc $\{\zeta : |\zeta| < r\}$ and I_r stands for the right-most integral in the expression above. Assume also that $0 < r_0 \le r < 1$ for some r_0. We intend to let $r \uparrow 1$ in the both sides of

$$u(r \cdot z_0) = I_r. \tag{17}$$

As a result, at the left-hand side of (17) we will obtain $u(z_0)$, while the right hand-side of (17) will coincide with the one in (10) for $z = z_0$.

Hence, it remains to show that such passage of the limit is legitimate. To do so, we will use the Lebesgue dominated convergence theorem. Using

(5) and the assumption $\beta > 0$, for the expression under the sign of integral I_r, we obtain

$$\Bigg| u(\zeta)\chi_r(\zeta)\left(S_{\beta,\rho,\varphi}\left(z_0,\frac{\zeta}{r}\right)+S_{\beta,\rho,\varphi}\left(\frac{\zeta}{r},z_0\right)-\frac{\rho}{\pi}\frac{\Gamma(\mu+\beta+1)}{\Gamma(\beta+1)\Gamma(\mu)}\right)$$
$$\left(1-\frac{|\zeta|^{2\rho}}{r^{2\rho}}\right)^{\beta}\frac{|\zeta|^{2\varphi}}{r^{2\varphi}}\frac{1}{r^2}\Bigg|$$
$$\leq \frac{|u(\zeta)|}{r^{2\rho\beta+2\varphi+2}}|\chi_r(\zeta)|(r^{2\rho}-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi}\left(\frac{const}{(1-|z_0|)^{\beta+2}}+\frac{\rho}{\pi}\frac{\Gamma(\mu+\beta+1)}{\Gamma(\beta+1)\Gamma(\mu)}\right)$$
$$\leq \frac{const}{r_0^{2\rho\beta+2\varphi+2}}|u(\zeta)|(1-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi}=const|u(\zeta)|(1-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi}\equiv\psi(\zeta).$$

We want to show that $\psi(\zeta) \in L^1(\mathbb{D})$, which is equivalent to show that $|u(\zeta)|(1-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi} \in L^1(\mathbb{D})$. For $p = 1$, we have

$$|u(\zeta)|(1-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi} \leq |u(\zeta)|(1-|\zeta|^{2\rho})^{\alpha}|\zeta|^{2\gamma} \in L^1(\mathbb{D}).$$

If $p > 1$, using Holder inequality, we get

$$\iint_{\mathbb{D}} |u(\zeta)|(1-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi}dm(\zeta)$$
$$=\iint_{\mathbb{D}} |u(\zeta)|(1-|\zeta|^{2\rho})^{\beta-\frac{\alpha}{p}}|\zeta|^{2\varphi-\frac{2\gamma}{p}}(1-|\zeta|^{2\rho})^{\frac{\alpha}{p}}|\zeta|^{\frac{2\gamma}{p}}dm(\zeta)$$
$$\leq\left(\iint_{\mathbb{D}} |u(\zeta)|^p(1-|\zeta|^{2\rho})^{\alpha}|\zeta|^{2\gamma}dm(\zeta)\right)^{1/p}$$
$$\left(\iint_{\mathbb{D}}(1-|\zeta|^{2\rho})^{q\left(\beta-\frac{\alpha}{p}\right)}|\zeta|^{2q\left(\varphi-\frac{\gamma}{p}\right)}dm(\zeta)\right)^{1/q}.$$

The convergence of the last integral follows from the conditions of the theorem. □

Remark. *In [9]-[13], one can find various interesting results relating to the weighted integral representations of harmonic functions.*

References

[1] W. Wirtinger, *ber eine minimumaufgabe im gebiet der analytischen functionen*, Monatshefte fur Math. und Phys. **39**(1932), pp. 377–384. https://doi.org/10.1007/bf01699078

[2] S. Bergman, *ber unendliche hermitische formen, die zu einem bereiche gehren, nebst anwendungenauf fragen der abbildung durch funktionen von zwei komplexen vernderlichen*, Math. Zeit. **29**(1929), pp. 641–677. https://doi.org/10.1007/bf01180554

[3] M. M. Djrbashian, *On the representability of certain classes of functions meromorphic in the unit disc*, Dokl. Akad. Nauk ArmSSR **3**(1945), no. 1, pp.3–9 (in Russian).

[4] M. M. Djrbashian, *On the problem of representability of analytic functions*, Soobshch. Inst. Matem. Mekh. Akad. Nauk ArmSSR **2**(1948), pp. 3–40 (in Russian).

[5] A. E. Djrbashian and F. A. Shamoyan, *Topics in the theory of A^p_α spaces*, Teubner-Texte zur Math. **105**(1988), Leipzig.

[6] M. M. Djrbashian, *Weighted integral representations of smooth or holomorphic functions in the unit disc and in the complex plane*, J. Contemp. Math. Analysis **28**(1993), no. 4, pp.1–27.

[7] F. V. Hayrapetyan, *Weighted integral representations of holomorphic functions in the unit disc by means of Mittag-Leffler type kernels*, Proc. NAS RA Math. **55**(2020), no. 4, pp. 15–30. https://doi.org/10.3103/s1068362320040032

[8] E. Titchmarsh, *The Theory of Functions*, Nauka, Moscow, 1980 (in Russian).

[9] M. Andersson, *Formulas for the L^2-minimal solutions of the $\partial\overline{\partial}$-equation in the unit ball of $\mathbb{C}^N$*, Math. Scand. **56**(1985), pp. 43–69. https://doi.org/10.7146/math.scand.a-12087

[10] A. H. Karapetyan, *Bounded projections in weighted function spaces in a generalized unit disc*, Annales Polonici Mathematici **62**(1995), no. 3, pp. 193–218. https://doi.org/10.4064/ap-62-3-193-218

[11] K. L. Avetisyan, *Bounded projections on mixed norm harmonic spaces*, Dokl. NAN Armenii **101** (2001), no. 3, pp. 211–215 (in Russian).

[12] K. L. Avetisyan, *Continuous inclusions and Bergman type operators in n-harmonic mixed norm spaces on the polydisc*, J. Math. Anal. Appl. **291** (2004), no. 2, pp. 727–740. https://doi.org/10.1016/j.jmaa.2003.11.039

[13] N. T. Gapoyan, *Bergman type operators on the spaces of functions in the ball from $\mathbb{C}^n$*, Candidate Dissertation, Yerevan, 2017 (in Russian).

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