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On w-extensions of the abelian semigroups

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Abstract

In this paper introduced a criterion of weight extension on semigroups.

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Introduction

Let H be a subsemigroup of commutative additive group without torsion Γ. A map ν on the semigroup H with values in the extended half line $\overline{\mathbb{R}}_+ = [0, \infty]$ is called a *weight* if $\nu(a + b) = \nu(a) + \nu(b)$, for all $a, b \in H$. A weight ν is called *finite*, if $\nu : H \to [0, \infty)$. By $W(H)$ we denote the set of all weights on H.

All the semigroups in this paper are assumed to contain an identity element 0 of the group Γ, $\nu(0) = 0$, for all $\nu \in W(H)$ and the semigroup H is assumed to not contain any non-trivial subgroup.

K. Ross [1] (see also [2]) obtained a criterion for weight ν on the subsemigroup $H\,(H \subset S)$ to be extended as a weight on the semigroup S in terms of order relation on H, generated by S. The question of weight extension is strongly connected with the question of multiplicative linear functionals extension from algebra A_H to algebra A_S(see [3], [4]).

Definition. A semigroup S is called a *w-extension* of subsemigroup H if every weight $\nu \in W(H)$ can be extended as a weight $\tilde{\nu}$ on S.

In this paper we define an angle between elements of the semigroup H and give a criterion of w-extension by this definition.

1 Angles on semigroups

Consider two additive semigroups $H \subset S$. We define a partial order in the semigroup H generated by S as follows: $a < b$, $a, b \in H$, if $a + c = b$, for some $c \in S \setminus \{0\}$.

The following theorem gives a necessary and sufficient condition for weight ν to be an extendable.

Theorem 1.1 *([1], [2]).The weight $\nu \in W(H)$ has an extension on S if and only if ν preserves an order relation on H generated by S.*

Consider a semigroup H, $H \subset \Gamma$. The elements $a, b \in H$ are called *independent* on $\mathbb{Z}$ if $n \cdot a + m \cdot b \neq 0$, $n, m \in \mathbb{Z} \setminus \{0\}$. Given an independent elements a, $b \in H$ on $\mathbb{Z}$, let

$$\Gamma_{a,b} = \{n \cdot a + m \cdot b | n,\ m \in \mathbb{Z}\} \subset \Gamma.$$

One can easily check that $\Gamma_{a,b}$ is a group. Define a mapping $\Phi_{a,b} : \Gamma_{a,b} \to \mathbb{Z} \times \mathbb{Z}$ by letting $\Phi_{a,b}(n \cdot a + m \cdot b) = (n, m)$, for all $n, m \in \mathbb{Z}$. Clearly, $\Phi_{a,b}$ is a homomorphism. Let $H_{a,b} = \Phi_{a,b}(\Gamma_{a,b} \cap H)$, $H_{a,b}$ is a subsemigroup in $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$.

Definition 1.2 *Let $a, b \in H$ be an independent elements on $\mathbb{Z}$. The angle of a minimal cone in $\mathbb{R}^2$ containing $H_{a,b}$ is called the angle between a and b in H.*

Since $a, b \in H$, the points $(1, 0)$ and $(0, 1)$ belong to $H_{a,b}$. Therefore, $(n, m) \in H_{a,b}$, for all $n, m \in \mathbb{Z}_+$. Denote by $K^H_{a,b}$ the minimal cone containing $H_{a,b}$ and by $\delta_H(a, b)$ the angle between a and b.

Similarly, one can define a semigroup $S_{a,b} = \Phi_{a,b}(\Gamma_{a,b} \cap S)$, a cone $K^S_{a,b}$ and an angle $\delta_S(a, b)$.

Since $H \subset S$, we have $K^H_{a,b} \subset K^S_{a,b}$ and $\delta_H(a, b) \leq \delta_S(a, b)$.

2 Main Result

We are now able to state the main result.

Theorem 2.1 *Let $0 \in S$. The following conditions are equivalent:*

1. *every weight ν on H can be extended to a weight $\tilde{\nu}$ on S;*

2. *$\delta_H(a, b) = \delta_S(a, b)$, for all $a, b \in H$.*

Proof. $1 \Rightarrow 2$.

Conversely, assume that there are independent elements $a, b \in H$ on $\mathbb{Z}$ such that $\delta_H(a, b) < \delta_S(a, b)$. It means that the cone $K^H_{a,b}$ is a proper subset of the cone $K^S_{a,b}$. Then there is a line l with the following properties: $(0, 0) \in l$, $l \subset K^S_{a,b}$, the line l and the cone $K^H_{a,b}$ have only one common point $(0, 0)$.

Consider the function ν on $K^H_{a,b}$ defined by the equality: $\nu(c) = pr_l(c)$, for all $c \in K^H_{a,b}$, i.e. $\nu(c)$ is a distance between $c \in K^H_{a,b}$ and a line l. Clearly, ν is an additive positive function on $K^H_{a,b}$, i.e. ν is a weight. Since $H_{a,b}$ is a subsemigroup of $K^H_{a,b}$, a restriction function $\nu|_{H_{a,b}}$ is also a weight.

Prove that $\nu|_{H_{a,b}}$ doesn't have an extension to a weight on $S_{a,b}$.

Indeed, let the element $c \in S_{a,b}$ and the cone $K^H_{a,b}$ lie on the opposite sides from the line l. Suppose $c = (n, m)$ and $k = |n| + |m|$. The points $d = (-k, k)$ and $c + d$ belong to $H_{a,b}$. We have that $pr_l(c + d) = pr_l(d) - pr_l(c)$. Hence $\nu(c + d) < \nu(d)$. Therefore, a weight ν doesn't preserve an order relation. Hence, $\delta_H(a, b) = \delta_S(a, b)$.

$2 \Rightarrow 1$.

Conversely, suppose there is a weight $\nu \in W(H)$ that doesn't have an extension on S. According to the theorem 1.1, there are elements $a, b \in H$ such that $a < b$ ($b = a + c,\ c \in S$) and $\nu(a) > \nu(b)$.

We claim that a and b are independent elements on $\mathbb{Z}$. Indeed, if we suppose the contrary, i.e. $na + mb = 0$, for some $n, m \in \mathbb{Z}$, then $(n + m)a + mc = 0$. Without loss of generality we can assume that $m > 0$. Then $n + m < 0$ and $mc = -(n + m)a$ is an element in H. Therefore $\nu(mb) > \nu(ma)$. Hence $\nu(b) > \nu(a)$ contrary to the hypothesis on ν. We also prove that $mc \notin H$, for all $m \in \mathbb{Z} \setminus \{0\}$.

For the mapping $\Phi_{a,b} : \Gamma_{a,b} \to \mathbb{Z} \times \mathbb{Z}$ we have $\Phi_{a,b}(a) = (1, 0)$, $\Phi_{a,b}(b) = (0, 1)$, $\Phi_{a,b}(c) = (-1, 1)$. Therefore, $\mathbb{Z}_+ \times \mathbb{Z}_+$ is a subsemigroup of the semigroup $H_{a,b}$. Show that the intersection of the half-plane $\{(x, y) \in \mathbb{R}^2 :\ x + y < 0\}$ and $H_{a,b}$ is the empty set. Suppose, in the contrary, $(l, k) \in H_{a,b}$ and $l + k < 0$. Then $-k - l > 0$ and $(-k - l)(1, 0) = (-k - l, 0)$ belongs to the semigroup $H_{a,b}$. Hence, $(l, k) + (-k - l, 0) = (-k, k) = k(-1, 1) \in H_{a,b}$. We observe that $kc \in H_{a,b}$ in contradiction with the hypothesis on c. Hence $\{(x, y) \in \mathbb{R}^2 : x + y < 0\} \cap H_{a,b} = \emptyset$.

If $\delta_H(a, b) = \delta_S(a, b)$, i.e. $K^H_{a,b} = K^S_{a,b}$, then the half-line $\{(-t, t) :\ t \in \mathbb{R}_+\}$ belongs to a minimal cone $K^H_{a,b}$. Therefore, for any $\epsilon > 0$ there exists an element $(n, m) \in H_{a,b}$, such that $n < 0$, $m > 0$ and $\left|-1 + \frac{n}{m}\right| < \epsilon$. Hence $na + mb \in H$ and $\nu(na + mb) + \nu(-na) = \nu(mb)$.

Consequently,

$$m\nu(b) - (-n)\nu(a) \geq 0,$$

$$\nu(b) \geq \left(1 - \frac{n}{m}\right)\nu(a) + \nu(a) \geq (1 - \epsilon)\nu(a).$$

Taking a limit by $\epsilon \to 0$, we obtain that $\nu(b) \geq \nu(a)$ in contradiction with the assumption $\nu(a) > \nu(b)$. $\square$

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