Armenian Journal of Mathematics
Volume 2, Number 2, 2009, 65–72

On properties of fuzzy ideals in po-semigroup

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Abstract

In this paper some properties of fuzzy ideals and fuzzy ideal extensions in a po-semigroup have been investigated.

Key Words: Fuzzy left(right) ideal, Fuzzy semiprime, Fuzzy ideal extensions.
Mathematics Subject Classification 2000: 20M12, 03F55, 08A72

1 Introduction

The important concept of fuzzy set has been introduced by Lofti Zadeh [8]. Since then many papers on fuzzy sets appeared showing its importance in different fields of mathematics. Rosenfeld [6] introduced the concept of fuzzy subgroups. Kuroki [2, 3] defined the fuzzy left and right ideals in a semigroup. The idea of fuzzy bi-ideals in a semigroup has been introduced by Liu [5]. N. Kehayopulu and M. Tsingelis [1] introduced the notion of fuzzy bi-ideals in po-semigroups(ordered semigroups). In this paper different properties of fuzzy ideals and its extensions in a po-semigroup have been investigated and in this connection different types of regularities in a po-semigroup have also been considered.

2 Some Preliminaries

Definition 2.1 *[4] A po-semigroup(ordered semigroup) is an ordered set* $(G, \leq)$ *which is a semigroup such that for* $a, b \in G, a \leq b \Rightarrow xa \leq xb$ *and* $ax \leq bx$*.*

Definition 2.2 *[4] Let G be a po-semigroup. A non-empty subset A of G is said to be right(resp. left) ideal of G if (i) $AG \subseteq A$(resp. $GA \subseteq A$), (ii) $x \in A$ and $y \leq x$ imply that $y \in A$.*

Definition 2.3 *[4] A non-empty subset A of a po-semigroup G is said to be an ideal if is a right ideal as well as left ideal of G.*

Definition 2.4 *[4] A po-semigroup G is called left(right) regular if for every $a \in G$ there exists $x \in G$ such that $a \leq xa^2$ (resp. $a \leq a^2x$).*

Definition 2.5 *[4] A po-semigroup G is called regular if for every $a \in G$ there exists $x \in G$ such that $a \leq axa$.*

Definition 2.6 *[4] A po-semigroup G is called completely regular if for every $a \in G$ there exists $x \in G$ such that $a \leq a^2xa^2$.*

Definition 2.7 *[4] A po-semigroup G is called intra regular if for every $a \in G$ there exist $x, y \in G$ such that $a \leq xa^2y$.*

Definition 2.8 *[4] A po-semigroup G is called left simple if for every $a, b \in G$ there exist $x, y \in G$ such that $b \leq xa$ and $a \leq yb$.*

Definition 2.9 *[4] A po-semigroup G is called simple if for every $a, b \in G$ there exist $x, y \in G$ such that $a \leq xby$.*

Definition 2.10 *[1] If $(G, ., \leq)$ be a po-semigroup, a fuzzy subset μ of G is called a fuzzy subsemigroup of G if $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ $\forall x, y \in G$.*

Definition 2.11 *[1] A fuzzy subset μ of a po-semigroup G is called a fuzzy left ideal of G if*

(i) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ $\forall x, y \in G$.
(ii) $\mu(xy) \geq \mu(y)$ $\forall x, y \in G$.

Definition 2.12 *[1] A fuzzy subset μ of a po-semigroup G is called a fuzzy right ideal of G if*

(i) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ $\forall x, y \in G$.
(ii) $\mu(xy) \geq \mu(x)$ $\forall x, y \in G$.

Definition 2.13 *[1] A fuzzy subset μ of a po-semigroup G is called a fuzzy ideal of G, if it is both fuzzy left ideal and fuzzy right ideal of G.*

Definition 2.14 *[1] A fuzzy subsemigroup μ of a po-semigroup G is called a fuzzy bi-ideal of G if*

(i) $x \leq y \Rightarrow \mu(x) \geq \mu(y)\ \forall x, y \in G$.
(ii) $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}\ \forall x, y, z \in G$.

Definition 2.15 *[1] Let G be a po-semigroup and μ, λ be two fuzzy subsets of G. Then the product $\mu \circ \lambda$ of μ and λ is defined as*

$$\begin{aligned}(\mu \circ \lambda)(x) &= \sup_{x=yz}[\min\{\mu(y), \lambda(z)\} : y, z \in G] \\ &= 0, \text{ if for any } y, z \in G, x \neq yz.\end{aligned}$$

Definition 2.16 *[1] A fuzzy subset μ of a po-semigroup G is called a fuzzy semiprime in G if $\mu(x) \geq \mu(x^2)\ \forall x \in G$.*

Definition 2.17 *[7] Let G be a po-semigroup, μ is a fuzzy subset of G and $x \in G$. The fuzzy subset $< x, \mu >: G \to [0, 1]$ is defined by $< x, \mu > (y) = \mu(xy)$ is called the extension of μ by x.*

3 Fuzzy Ideals in a Po-semigroup

Theorem 3.1 *Let I be a non-empty subset of a po-semigroup G and λ_I be the characteristic function of I. Then I is a subsemigroup of G if and only if λ_I is a fuzzy subsemigroup of G.*

Proof. The proof is similar to that of po-group. So we omit it. □

Theorem 3.2 *Let I be a non-empty subset of a po-semigroup G and λ_I be the characteristic function of I. Then I is a left ideal(right ideal, ideal) of G if and only if λ_I is a fuzzy left ideal(resp. fuzzy right ideal, fuzzy ideal) of G.*

Proof. Let I be a left ideal of G. Let $x, y \in G$. Then $xy \in I$ if $y \in I$. It follows that $\lambda_I(xy) = 1 \geq \lambda_I(y)$. In the case when $y \notin I$, then $\lambda_I(xy) \geq \lambda_I(y) = 0$.

Let $x, y \in G$ be such that $x \leq y$. If $y \in I$, then $\lambda_I(y) = 1$. Since $x \leq y \in I$, we have $x \in I$, then $\lambda_I(x) = 1$. Thus $\lambda_I(x) \geq \lambda_I(y)$. If $y \notin I$, then $\lambda_I(y) = 0$. Hence we have $\lambda_I(x) \geq 0 = \lambda_I(y)$. So λ_I is a fuzzy left ideal of G.

Conversely, let λ_I be a fuzzy left ideal of G. Let x, $y \in G$ and $y \in I$. Then $\lambda_I(y) = 1$. Again $\lambda_I(xy) \geq \lambda_I(y)$. So, $\lambda_I(xy) = 1$. Consequently, $xy \in I$. Now if $x \leq y$ and $y \in I$, then $\lambda_I(y) = 1$ and $\lambda_I(x) \geq \lambda_I(y)$. So, $\lambda_I(x) = 1$. This implies that $x \in I$. □

Theorem 3.3 *Let I be a non-empty subset of a po-semigroup G and λ_I be the characteristic function of I. Then I is a bi-ideal of G if and only if λ_I is a fuzzy bi-ideal of G.*

Proof. Let I be a bi-ideal of G. Let $x, y, z \in G$. Then $xyz \in I$ if $x, z \in I$. It follows that $\lambda_I(xyz) = 1 = \min\{\lambda_I(x), \lambda_I(z)\}$. Let either $x \notin I$ or $z \notin I$. Then *Case* (i) : If $xyz \notin I$, then $\lambda_I(xyz) \geq 0 = \min\{\lambda_I(x), \lambda_I(z)\}$. *Case* (ii) : If $xyz \in I$, then $\lambda_I(xyz) = 1 \geq 0 = \min\{\lambda_I(x), \lambda_I(z)\}$. Again, let $x, y \in G$ be such that $x \leq y$. If $y \in I$, then $\lambda_I(y) = 1$. Since $x \leq y \in I$, we have $x \in I$, then $\lambda_I(x) = 1$. Thus $\lambda_I(x) \geq \lambda_I(y)$. If $y \notin I$, then $\lambda_I(y) = 0$. Since $x \in G$, we have $\lambda_I(x) \geq 0$. Thus $\lambda_I(x) \geq \lambda_I(y)$. Hence λ_I is a fuzzy bi-ideal of G.

Conversely, let λ_I be a fuzzy bi-ideal of G. Let $x, y, z \in G$ be such that $x, z \in I$. Then $\lambda_I(x) = \lambda_I(z) = 1$. Also $\lambda_I(xyz) \geq \min\{\lambda_I(x), \lambda_I(z)\} = 1$. Hence $xyz \in I$. Again let $x \in I$ and $G \ni y \leq x$. Then $\lambda_I(x) = 1$. Since λ_I is a fuzzy bi-ideal of G, then $1 = \lambda_I(x) \leq \lambda_I(y)$. Since $y \in G$, we have $\lambda_I(y) \leq 1$. Hence $\lambda_I(y) = 1$ whence $y \in I$. Consequently, I is a bi-ideal of G. □

Theorem 3.4 *A fuzzy subset μ of a po-semigroup G is a fuzzy subsemigroup of G if and only if $\mu \circ \mu \subseteq \mu$.*

Proof. If $\mu \circ \mu \subseteq \mu$, then for $x, y \in G$, we have $\mu(xy) \geq (\mu \circ \mu)(xy) \geq \min\{\mu(x), \mu(y)\}$. So μ is a fuzzy subsemigroup of G.

Conversely, suppose μ is a fuzzy subsemigroup of G. Then for $x \in G$, $(\mu \circ \mu)(x) = \sup_{x=yz}[\min\{\mu(y), \mu(z)\}]$. Since μ is a fuzzy subsemigroup, $\mu(yz) \geq \min\{\mu(y), \mu(z)\}\ \forall y, z \in G$. In particular, $\mu(yz) \geq \min\{\mu(y), \mu(z)\}\ \forall y, z \in G$, with $x = yz$.

Hence

$$\mu(x) \geq \sup_{x=yz}[\min\{\mu(y), \mu(z)\}] = (\mu \circ \mu)(x).$$

□

Theorem 3.5 *In a po-semigroup G the following are equivalent. (i) μ is a fuzzy left ideal of G. (ii) $\tau \circ \mu \subseteq \mu$ and $x \leq y \Rightarrow \mu(x) \geq \mu(y)\ \forall x, y \in G$, where τ is the characteristic function of G.*

Proof. Let us assume that (i) holds. Let $a \in G$. In the case when there exist elements $x, y \in G$ such that $a = xy$, then, since μ is a fuzzy left ideal of G, we have

$$\begin{aligned}(\tau \circ \mu)(a) &= \sup_{a=xy}[\min\{\tau(x), \mu(y)\}] \\ &\leq \sup_{a=xy}[\min\{1, \mu(xy)\}] \\ &= \min\{1, \mu(a)\} = \mu(a)\end{aligned}$$

Otherwise $(\tau \circ \mu)(a) = 0 \leq \mu(a)$. Thus $\tau \circ \mu \subseteq \mu$. By the definition of fuzzy left ideal of G, we have $x \leq y \Rightarrow \mu(x) \geq \mu(y)\ \forall x, y \in G$.

Conversely, let us assume that (ii) holds. Let $x, y \in G$. Since $\tau \circ \mu \subseteq \mu$, we have

$$\begin{aligned}\mu(xy) &\geq (\tau \circ \mu)(xy) \geq \min\{\tau(x), \mu(y)\} \\ &= \min\{1, \mu(y)\} = \mu(y)\end{aligned}$$

Thus μ is a fuzzy left ideal of G. This completes the proof. $\square$

Similarly we can deduce the following theorem.

Theorem 3.6 *In a po-semigroup G the following are equivalent. (i) μ is a fuzzy right ideal of G. (ii) $\mu \circ \tau \subseteq \mu$ and $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ $\forall x, y \in G$, where τ is the characteristic function of G.*

Now by combining the above two theorems we easily deduce the following theorem.

Theorem 3.7 *In a po-semigroup G the following are equivalent. (i) μ is a fuzzy two-sided ideal of G. (ii) $\mu \circ \tau \subseteq \mu, \tau \circ \mu \subseteq \mu$ and $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ $\forall x, y \in G$, where τ is the characteristic function of G.*

Theorem 3.8 *In a po-semigroup G the following are equivalent. (i) μ is a fuzzy bi-ideal of G. (ii) $\mu \circ \mu \subseteq \mu$, $\quad \mu \circ \tau \circ \mu \subseteq \mu$ and $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ $\forall x, y \in G$, where τ is the characteristic function of G.*

Proof. Let us assume that (i) holds. Since μ is a fuzzy bi-ideal of G, then it is obvious that $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ $\forall x, y \in G$. Since μ is a fuzzy subsemigroup of G, $\mu \circ \mu \subseteq \mu$ (cf. Theorem 3.4). Let $a \in G$. If there are no $x, y \in G$ such that $a = xy$ then $(\mu \circ \tau \circ \mu)(a) = 0 \leq \mu(a)$.

On the other hand if there exist $x, y, p, q \in G$ such that $a = xy$ and $x = pq$, then

$$\begin{aligned}(\mu \circ \tau \circ \mu)(a) &= \sup_{a=xy}[\min\{(\mu \circ \tau)(x), \mu(y)\}] \\ &= \sup_{a=xy}[\min\{\sup_{x=pq}\{\min\{\mu(p), \tau(q)\}\}, \mu(y)\}] \\ &= \sup_{a=xy}[\min\{\sup_{x=pq}\{\min\{\mu(p), 1\}\}, \mu(y)\}].\end{aligned}$$

$$= \sup_{\substack{a = xy \\ x = p'q}} [\min\{\mu(p'),\ \mu(y))\}].$$

Now since μ is a fuzzy bi-ideal, $\min\{\mu(p'), \mu(y)\} \leq \mu(p'qy)$.

Hence

$$\sup_{\substack{a=xy \\ x=p'q}}[\min\{\mu(p'), \mu(y))\}] \leq \mu(p'qy) = \mu(xy) = \mu(a).$$

Hence $\mu \circ \tau \circ \mu \subseteq \mu$.

Conversely, let us assume that (ii) holds. Since $\mu \circ \mu \subseteq \mu$, μ is a fuzzy subsemigroup of G (cf. Theorem 3.4). Let $x, y, z \in G$. Since $\mu \circ \tau \circ \mu \subseteq \mu$, we see that

$$\begin{aligned}\mu(xyz) &\geq (\mu \circ \tau \circ \mu)(xyz) \\ &\geq \min\{(\mu \circ \tau)(xy), \mu(z)\} \\ &\geq \min[\min\{\mu(x), \tau(y)\}, \mu(z)] \\ &= \min[\min\{\mu(x), 1\}, \mu(z)] \\ &= \min\{\mu(x), \mu(z)\}\end{aligned}$$

Hence μ is a fuzzy bi-ideal of G. This completes the proof. □

Theorem 3.9 *If μ is a fuzzy left ideal of a left regular po-semigroup G, then μ is fuzzy semiprime in G.*

Proof. Let G be a left regular po-semigroup. Then for any $a \in G$ there exists some $x \in G$ such that $a \leq xa^2$. Now

$$\mu(a) \geq \mu(xa^2) \geq \mu(a^2)(\text{since } \mu \text{ is a fuzzy left ideal})$$

Hence μ is fuzzy semiprime in G. □

Theorem 3.10 *If μ is a fuzzy right ideal of a right regular po-semigroup G, then μ is fuzzy semiprime in G.*

Proof. Let G be a right regular semigroup. Then for any $a \in G$ there exists some $x \in G$ such that $a \leq a^2x$. Now

$$\mu(a) \geq \mu(a^2x) \geq \mu(a^2)(\text{since } \mu \text{ is a fuzzy right ideal})$$

Hence μ is fuzzy semiprime in G. □

Theorem 3.11 *If μ is a fuzzy ideal of an intra regular po-semigroup G, then μ is fuzzy semiprime in G.*

Proof. Let G be an intra regular po-semigroup. Then for any $a \in G$ there exist some $x, y \in G$ such that $a \leq xa^2y$. Now

$$\mu(a) \geq \mu(xa^2y) \geq \mu(a^2y) \geq \mu(a^2) \ (\text{since } \mu \text{ is a fuzzy ideal})$$

Hence μ is fuzzy semiprime in G. □

Theorem 3.12 *If μ is a fuzzy ideal of an intra regular po-semigroup G, then $\mu(ab) = \mu(ba)$ $\forall a, b \in G$.*

Proof. By Theorem 3.11,

$$\mu(ab) \geq \mu((ab)^2) = \mu(a(ba)b) \geq \mu(ba).$$

Again

$$\mu(ba) \geq \mu((ba)^2) = \mu(b(ab)a) \geq \mu(ab).$$

Hence $\mu(ab) = \mu(ba)$ $\forall a, b \in G$. □

Definition 3.13 *[1] A po-semigroup G is said to be fuzzy left simple if every fuzzy left ideal of G is constant.*

Theorem 3.14 *If a po-semigroup G is left simple, then G is fuzzy left simple.*

Proof. Let μ be a fuzzy left ideal of the po-semigroup G and $a, b \in \dot{G}$. Since G is left simple, then there exist $x, y \in G$ such that $a \leq yb, b \leq xa$. Now

$$\mu(a) \geq \mu(yb) \geq \mu(b)(\text{since } \mu \text{ is a fuzzy left ideal})$$

Again,

$$\mu(b) \geq \mu(xa) \geq \mu(a)(\text{since } \mu \text{ is a fuzzy left ideal})$$

Thus μ is a constant function. Hence G is fuzzy left simple. □

Theorem 3.15 *If a po-semigroup G is simple, then every fuzzy interior ideal of G is constant.*

Proof. Let μ be a fuzzy interior ideal of the po-semigroup G and $a, b \in \dot{G}$. Since G is simple, then there exist $x, y \in G$ such that $a \leq xby$. Now

$$\mu(a) \geq \mu(xby) \geq \mu(b)(\text{since } \mu \text{ is a fuzzy interior ideal})$$

It can be seen in a similar way that $\mu(b) \geq \mu(a)$.

Since a and b are arbitrary elements, we conclude that μ is a constant function. □

4 Fuzzy Ideal Extensions in a Po-semigroup

Theorem 4.1 *Let G be a right regular po-semigroup and μ be a fuzzy right ideal of G. Then the fuzzy ideal extension $< x, \mu >$ is fuzzy semiprime in G, for all $x \in G$.*

Proof. Let μ be a fuzzy right ideal of G and $a \in \dot{G}$. Since G is right regular, then there exists $y \in G$ such that $a \leq a^2y$. Now

$$< x, \mu > (a) = \mu(xa) \geq \mu(xa^2y)$$
$$\geq \mu(xa^2) =< x, \mu > (a^2)$$

Hence $< x, \mu >$ is fuzzy semiprime in G. □

In a similar way we can deduce the following theorem.

Theorem 4.2 *Let G be a commutative regular po-semigroup and μ be a fuzzy ideal of G. Then the fuzzy ideal extension $< x, \mu >$ is fuzzy semiprime in G, for all $x \in G$.*

Theorem 4.3 *Let G be an intra regular commutative po-semigroup and μ be a fuzzy ideal of G. Then the fuzzy ideal extension $< x, \mu >$ is fuzzy semiprime in G, for all $x \in G$.*

Proof. Let μ be a fuzzy ideal of the po-semigroup G and $a \in \dot{G}$. Since G is intra regular, there exist $z,\ y \in G$ such that $a \leq za^2y$. Now

$$< x, \mu > (a) = \mu(xa) \geq \mu(xza^2y) = \mu(zxa^2y) \geq \mu(xa^2) =< x,\ \mu > (a^2).$$

Hence the fuzzy ideal extension $< x, \mu >$ is fuzzy semiprime in G. □

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