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Exponential decay for a strain gradient porous thermoelasticity with second sound

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Abstract. In this paper, we consider a strain gradient porous elastic bar subjected to a thermal disturbance modelled by Cattaneo's law for heat conduction. We use the semigroup approach to prove the existence of a unique weak solution. Although the thermal dissipation induced by the second sound thermoelasticity is weaker than that caused by the classical heat conduction, we prove that the solution decays exponentially.

Key Words: Strain gradient, thermoelasticity, second sound, exponential decay
Mathematics Subject Classification 2010: 35B40, 35G46, 35Q74, 74F05, 93D20

1 Introduction

In the present paper, we are concerned with the longtime behavior of the following system of equations

$$\begin{cases} \rho u_{tt} = a u_{xx} + b\phi_x - c u_{xxxx} - d\phi_{xxx} - \delta\theta_x & \text{in } (0,\pi)\times[0,\infty[, \\ J\phi_{tt} = d u_{xxx} + \beta\phi_{xx} - \xi\phi - b u_x + m\theta - \mu\phi_t & \text{in } (0,\pi)\times[0,\infty[, \\ c^*\theta_t = -q_x - \delta u_{xt} - m\phi_t & \text{in } (0,\pi)\times[0,\infty[, \\ \tau q_t + q + \kappa\theta_x = 0 & \text{in } (0,\pi)\times[0,\infty[, \end{cases} \tag{1}$$

where u, ϕ, θ and q are the transversal displacement, the volume fraction, the difference of temperature from an equilibrium reference value and the heat flux of a one dimensional elastic material of length π, respectively. The coefficients ρ, J, a, c, c^*, β, ξ, τ, μ and κ are positive constitutive constants. The coefficients b, d, m and δ are the coupling constants that are different from zero but their signs does not matter in the analysis. In addition, to ensure that the energy functional associated to the system (1) is a positive definite form, we assume that the coefficients a, b, c, d, β and ξ satisfy the inequalities

$$a\xi > b^2 \qquad \text{and} \qquad c\beta > d^2. \tag{2}$$

In this paper, a strain gradient theory has been employed for the mechanical modelling of the micro-structure of an elastic bar with voids in the presence of thermal effects. In the literature, the theory of elastic materials with voids was established by Nunziato and Cowin [6, 7, 30] and has been applied to elastic materials with small voids or vacuo porous. This theory is characterized by the fact that the bulk density at any point in the material is the product of two scalar fields, the matrix material density and the volume fraction.

It is worth noting that when dealing with classical elasticity, the microscopic structure is ignored, only the macroscopic properties are analyzed, and the equations field are in the form of strain-stress relations. However, when non-classical elasticity is considered, as for nonsimple materials, the microstructure takes part in the constitutive equations, thence higher-order derivatives of the displacement appear in the basic postulates. This theory is called the strain-gradient or nonsimple elasticity, and it was introduced by Mindlin [25] and Toupin [39]. Its linear form was proposed by Green and Rivlin [17] and Mindlin and Eshel [26].

In the last few years, a huge number of contributions have investigated the existence and the time behavior of solutions of porous and thermoelastic systems. Let us introduce some of them.

Early in this century, Quintanilla [32] considered the classical porous elastic system

$$\begin{cases} \rho_0 u_{tt} = \mu u_{xx} + \beta \phi_x, \\ \rho_0 \kappa \phi_{tt} = \alpha \phi_{xx} - \beta u_x - \xi \phi - \tau \phi_t, \end{cases} \tag{3}$$

and proved that the porous damping $\tau\phi_t$ is not strong enough to produce an exponential stability, only a slow decay result has been obtained. Later, in 2017, Apalara [2] and, independently, Santos *et al.* [35] established the exponential stability of (3) provided that the wave speeds of the equations in (3) are equal

$$\frac{\mu}{\rho_0} = \frac{\alpha}{\rho_0 \kappa}. \tag{4}$$

The same result was obtained by Santos *et al.* [35] if the damping $-\tau\phi_t$ is replaced by the elastic damping γu_t in the first equation.

To exponentially stabilize the system (3), several dissipation mechanisms have been examined, and different stability results have been obtained. Magaña and Quintanilla [22] obtained a slow decay if only the dissipation γu_{txx} is present in the first equation of (3), or it combined with a thermal dissipation by coupling the system (3) with the heat equation ($\tau = 0$). They also obtained the same result when coupling the system (3) with micro temperature. Apalara [3] and Feng [11] considered the cases of a memory term and past history, respectively. They obtained a general rate of decay depending on the relaxation function.

Messaoudi and Fareh [23, 24], Soufyane and co-authors [36, 37, 38] and Feng and Yin [9] examined the combination of a thermal dissipation with a memory term. Feng [10] and Khochemane [18] considered the coupling of thermal and history dissipations. General, exponential and polynomial rates of decay were obtained.

An inhomogeneous system in the presence of temperature and micro temperature was considered by Feng *et al.* [8]. Under an appropriate assumptions on the system coefficients, they proved the exponential stability of the solution.

Regarding the nonsimple elasticity theory, Fernández Sare *et al.* [13] considered the system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} + \alpha u_{xxxx} - \beta\theta_x = 0, \\ c\theta_t - \kappa\theta_{xx} - \beta u_{tx} = 0, \end{cases} \tag{5}$$

where the non-simple elasticity is coupled with the classical heat equation. They obtained an exponential decay for different boundary conditions.

It is well known that the heat part in the second equation of (5) is obtained via Fourier's law of thermal conduction

$$q + \kappa\theta_x = 0,$$

which leads to the paradox of infinite wave speed. Aouadi [1] replaced Fourier's law with the Gurtin–Pipkin one

$$q = -\int_{-\infty}^{0} g(t-s)\theta(x,s)ds,$$

where g is a relaxation function. He extended the result of [13] to the case of a hereditary heat conduction.

Liu *et al.* [19] combined the nonsimple elasticity with porosity. They considered the system

$$\begin{cases} \rho u_{tt} = a u_{xx} + b\varphi_x - c u_{xxxx} - d\varphi_{xxx}, \\ J\varphi_{tt} = d u_{xxx} + \beta\varphi_{xx} - \xi\varphi - b u_x, \end{cases}$$

with different dissipation mechanisms and obtained an exponential stability for the dissipations δu_{txx}, $-\alpha u_{txxxx}$ in the first equation or $m\varphi_{txx}$ in the second equation. However, the dissipation $-\tau\varphi_t$ in the second equation leads to a slow decay.

Recently, Fernández *et al.* [12] studied the system (1) in the case of Fourier's law of heat conduction ($\tau = 0$). They used a semigroup approach and proved the existence of a unique mild solution that decays exponentially.

In the system (1), the heat conduction is given through Cattaneo's law

$$\tau q_t + q + \kappa\theta_x = 0,$$

which renders the system fully hyperbolic. Consequently, the heat propagation is to be viewed as a wave-like propagation rather than a diffusion phenomenon. A wave-like thermal disturbance is referred to as the second sound (where the first sound being the usual sound). A non-classical theory predicting the occurrence of such disturbances are known as a thermoelasticity with finite wave speeds or a second sound thermoelasticity. The theory of second sound thermoelasticity was developed by Lord and Shulman [21] to settle the paradox of infinite speed caused by Fourier's law of heat conduction.

The non-simple thermoelasticity with a second sound was considered by Muñoz Rivera and Vega [29]. They studied the system

$$\begin{cases} \rho u_{tt} - \gamma u_{ttxx} - \mu u_{xx} + \alpha u_{xxxx} - \beta\theta_x = 0, & \text{in } \ ]0,\ell[\times\mathbb{R}_+, \\ c\theta_t + q_x - \beta u_{tx} = 0, & \text{in } \ ]0,\ell[\times\mathbb{R}_+, \\ \tau q_t + q + \kappa\theta_x = 0, & \text{in } \ ]0,\ell[\times\mathbb{R}_+, \end{cases} \tag{6}$$

with boundary conditions of the form

$$u(0,t) = u_{xx}(0,t) = u(\ell,t) = u_{xx}(\ell,t) = 0, \quad \theta_x(0,t) = \theta_x(\ell,t) = 0,$$

and proved that the semigroup associated with the solution of (6) is not exponentially stable for $\gamma \geq 0$.

Close to the system considered in this paper, Fernàndez Sare and Muñoz Rivera [14] studied the time behavior of the plate system in $\mathbb{R}^2$,

$$\begin{cases} \rho u_{tt} - \mu\Delta u_{tt} + \gamma\Delta^2 u + \alpha\Delta\theta = 0, & \text{in } \Omega, \\ c\theta_t + \kappa div q - \alpha\Delta u_t = 0, & \text{in } \Omega, \\ \tau q_t + \kappa_0 q + \kappa\nabla\theta = 0, & \text{in } \Omega. \end{cases} \tag{7}$$

They proved that the semigroup associated to the solutions of (7) is exponentially stable if and only if $\mu \neq 0$. Moreover, for $\mu = 0$, they obtained the optimal rate of decay t^{-2}.

We should notice here, that replacing Fourier's law by Cattaneo's one is not obviously profitable, because the dissipativeness due to the heat conduction induced by Cattaneo's law are generally weaker than those induced by Fourier's law. For example, Fernández Sare and Racke [34] have shown that the Timoshenko system coupled with the heat conduction of a second sound loses the exponential stability property. Also, Quintanilla and Racke [33] proved that the thermoelastic plate changed the behavior from exponential to non-exponential stability when changing from Fourier's law to Cattaneo's law in the modelling of heat conduction (see also [15] and [28]).

Finally, we note that the system studied in this paper on the one hand is analogous to the system (6) with $\gamma = 0$, which is not exponentially stable, and on the other hand, its isothermal part coincides with the system studied by Fernández *et al.* in [12]. Here we prove that the porous dissipation $\mu\phi_t$ together with the weak dissipation caused by the heat conduction of a second sound, leads to an exponential stability regardless any assumption on the wave speeds. Our result improves the result of [29] and extends the result of [12].

To make the problem given by the system (1) well determined, we impose the following boundary and initial conditions:

$$\begin{aligned} u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0,\ t \geq 0, \\ \phi_x(0,t) = \phi_x(\pi,t) = q(0,t) = q(\pi,t) = 0,\ t \geq 0, \end{aligned} \tag{8}$$

and

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = v_0(x), \phi(x,0) = \phi_0(x), & \text{in } x \in [0,\pi], \\ \phi_t(x,0) = \varphi_0(x), \theta(x,0) = \theta_0(x), q(x,0) = q_0(x) & \text{in } x \in [0,\pi]. \end{cases} \tag{9}$$

From the boundary conditions on q and equation $(1)_4$, we have

$$\theta_x\,(t,0) = \theta_x\,(t,\pi) = 0.$$

Note that since the Neumann boundary conditions are assumed for ϕ and θ, we are not able to apply Poincaré's inequality for them. To allow the application of the aforementioned inequality, we proceed as follows.

Integrating $(1)_2$ and $(1)_3$ with respect to x and taking into account the boundary conditions, we obtain

$$\frac{d^2}{dt^2}\int_0^\pi \phi\,(x,t)\,dx = -\frac{\xi}{J}\int_0^\pi \phi\,(x,t)\,dx + \frac{m}{J}\int_0^\pi \theta\,(x,t)\,dx - \frac{\mu}{J}\frac{d}{dt}\int_0^\pi \phi\,(x,t)\,dx \tag{10}$$

and

$$c^*\frac{d}{dt}\int_0^\pi \theta dx = -m\frac{d}{dt}\int_0^\pi \phi dx. \tag{11}$$

Solving (11), we get

$$\int_0^\pi \theta\,(t,x)\,dx = -\frac{m}{c^*}\int_0^\pi \phi\,(t,x)\,dx + C.$$

From the initial conditions, we compute

$$C = \int_0^\pi \theta_0\,(x)\,dx + \frac{m}{c^*}\int_0^\pi \phi_0\,(x)\,dx.$$

Plugging $\int_0^\pi \theta\,(t,x)\,dx$ into (10), we infer

$$\frac{d^2}{dt^2}\int_0^\pi \phi\,(x,t)\,dx + \frac{\mu}{J}\frac{d}{dt}\int_0^\pi \phi\,(x,t)\,dx + \left(\frac{\xi}{J} + \frac{m^2}{Jc^*}\right)\int_0^\pi \phi\,(x,t)\,dx = -C. \tag{12}$$

The solution of (12) is then

$$\int_0^\pi \phi(x,t)\,dx = y_H(t) + y_P(t)$$

where y_P and y_H are, respectively, a particular solution of (12) and the general solution of the homogeneous equation associated to (12). Therefore,

$$\int_0^\pi \theta(t,x)\,dx = -\frac{m}{c^*}\left(y_H(t) + y_P(t)\right) + C.$$

Thus, if we set

$$\widetilde{\phi}(x,t) = \phi(x,t) - y_H(t) - y_P(t),$$

$$\widetilde{\theta}(x,t) = \theta(x,t) + \frac{m}{c^*}\left(y_H(t) + y_P(t)\right) - C,$$

we reach

$$\int_0^\pi \widetilde{\phi}(x,t)\,dx = 0 \qquad \text{and} \qquad \int_0^\pi \widetilde{\theta}(x,t)\,dx = 0,$$

which allows the application of Poincaré's inequality. Moreover, $(u,\widetilde{\phi},\widetilde{\theta},q)$ satisfies the system (1) with the boundary conditions (9). In the sequel, we work with $(u,\widetilde{\phi},\widetilde{\theta},q)$, but for the convenience, we write (u,ϕ,θ,q).

The energy functional associated with the solution to the problem (1), (8), (9) is given by

$$E(t) := \frac{1}{2}\int_0^L \Big[\rho|u_t|^2 + a|u_x|^2 + c|u_{xx}|^2 + J\,|\phi_t|^2 + \xi\,|\phi|^2 + \beta\,|\phi_x|^2 + c^*|\theta|^2 + \frac{\tau}{\kappa}|q|^2 + 2bRe(u_x\phi) + 2dRe(u_{xx}\phi_x)\Big]\,dx.$$

Note that from (2), the energy $E(t)$ is a positive definite form. Moreover, taking the L^2-product of $(1)_1$ by u_t, $(1)_2$ by ϕ_t, $(1)_3$ by θ, and $(1)_4$ by $\frac{1}{\kappa}q$, adding the obtained equations and using integration by parts, we get

$$\frac{dE}{dt}(t) = -\frac{1}{\kappa}\int_0^L |q|^2 dx - \mu\int |\phi_t|^2\,dx,$$

which shows the dissiptiveness of the energy $E(t)$.

The rest of the paper is organized as follows. In Section 2, we prove the well-posedness of the problem. Section 3 is devoted to the prove of the exponential stability of the solution of (1).

2 Well-posedness

In this section, we will prove that the problem determined by the system (1), the boundary conditions (8) and the initial conditions (9) has a unique solution. Our main tools are the following two theorems from the theory of semigroup of contractions operators in a Hilbert space $\mathcal{H}$.

Theorem 1 *[41] Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ be the infinitesimal generator of a C_0–semigroup $\{S(t); t \geq 0\}$ on $\mathcal{H}$. Then, for each $\xi \in D(\mathcal{A})$ and each $t \geq 0$, we have $S(t)\xi \in D(\mathcal{A})$, and the mapping*

$$t \longrightarrow S(t)\xi$$

is C^1 on $[0, +\infty)$ and satisfies

$$\frac{d}{dt}(S(t)\xi) = \mathcal{A}S(t)\xi = S(t)\mathcal{A}\xi. \tag{13}$$

Theorem 1 means that $u(t) = S(t)\xi$ is the strong solution to the abstract Cauchy problem

$$\begin{cases} u_t(t) = \mathcal{A}u(t), \ t > 0, \\ u(0) = \xi. \end{cases}$$

Theorem 2 (Lumer–Phillips) *[31, 41] Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ be a densely defined operator. Then $\mathcal{A}$ generates a C_0–semigroup of contractions on $\mathcal{H}$ if and only if*

(i) $\mathcal{A}$ is dissipative;

(ii) there exists $\lambda > 0$ such that $\lambda I - \mathcal{A}$ is surjective.

To rewrite the problem (1), (8)–(9) in the settings of Theorem 2, we first define the space

$$\mathcal{H} := \Big\{(u, v, \phi, \psi, \theta, q) \in \Big(H^2(0,\pi) \cap H_0^1(0,\pi)\Big) \times L^2(0,\pi) \times H_*^1(0,\pi) \times L_*^2(0,\pi) \times L_*^2(0,\pi) \times L_0^2(0,\pi)\Big\},$$

where

$$L_*^2(0,\pi) := \left\{\varphi \in L^2(0,\pi)\,; \int_0^\pi \varphi(x)dx = 0\right\},$$

$$H_*^m(0,\pi) = H^m(0,\pi) \cap L_*^2(0,\pi)$$

and

$$L_0^2(0,\pi) := \left\{\varphi \in L^2(0,\pi)\,; \varphi(0) = \varphi(\pi) = 0\right\}.$$

It is clear that $L_*^2(0,\pi)$, $L_0^2(0,\pi)$ and $H_*^1(0,\pi)$ are Hilbert spaces, and hence, $\mathcal{H}$ is a Hilbert space too.

The Hilbert space $\mathcal{H}$ is equipped with the inner product

$$\langle U,U^*\rangle_{\mathcal{H}}=\int_0^L\left[\rho v\overline{v}^*+au_x\overline{u}_x^*+cu_{xx}\overline{u}_{xx}^*+J\psi\overline{\psi}^*+\beta\phi_x\overline{\phi}_x^*+\xi\phi\overline{\phi}^*+c^*\theta\overline{\theta}^*\right]dx$$
$$+\int_0^L\left[\frac{\tau}{\kappa}q\overline{q}^*+b\left(u_x\overline{\phi}^*+\phi\overline{u}_x^*\right)+d\left(u_{xx}\overline{\phi}_x^*+\phi_x\overline{u}_{xx}^*\right)\right]dx,$$

where $U=(u,v,\phi,\psi,\theta,q)^T$ and $U^*=(u^*,v^*,\phi^*,\psi^*,\theta^*,q^*)^T$. The corresponding norm in $\mathcal{H}$ is

$$\|U\|_{\mathcal{H}}^2=\int_0^L\left[\rho|v|^2+a|u_x|^2+c|u_{xx}|^2+J\,|\psi|^2+\xi\,|\phi|^2+\beta\,|\phi_x|^2+c^*|\theta|^2\right.$$
$$\left.+\frac{\tau}{\kappa}|q|^2+2bRe\,\langle u_x,\phi\rangle+2dRe\,\langle u_{xx},\phi_x\rangle\right]dx.$$

Secondly, we introduce the new variables $v=u_t$ and $\psi=\phi_t$. Then the system (1) becomes

$$\begin{cases} u_t=v,\\ v_t=\dfrac{1}{\rho}\left(au_{xx}+b\phi_x-cu_{xxxx}-d\phi_{xxx}-\delta\theta_x\right),\\ \phi_t=\psi,\\ \psi_t=\dfrac{1}{J}\left(du_{xxx}+\beta\phi_{xx}-\xi\phi-bu_x+m\theta-\mu\psi\right),\\ \theta_t=-\dfrac{1}{c^*}\left(q_x+\delta v_x+m\psi\right)\\ q_t=-\dfrac{1}{\tau}\left(q+\kappa\theta_x\right), \end{cases}$$

which, with the initial conditions (9), can be written in the form of an abstract Cauchy problem as follows

$$\begin{cases} U_t=\mathcal{A}U,\\ U(0)=U_0, \end{cases}\tag{14}$$

where $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\longrightarrow\mathcal{H}$, is the operator given by

$$\mathcal{A}=\begin{pmatrix} 0 & I & 0 & 0 & 0 & 0\\ \frac{a}{\rho}D^2-\frac{c}{\rho}D^4 & 0 & \frac{b}{\rho}D-\frac{d}{\rho}D^3 & 0 & -\frac{\delta}{\rho}D & 0\\ 0 & 0 & 0 & I & 0 & 0\\ \frac{d}{J}D^3-\frac{b}{J}D & 0 & \frac{\beta}{J}D^2-\frac{\xi}{J}I & -\frac{\mu}{J}I & \frac{m}{J}I & 0\\ 0 & -\frac{\delta}{c^*}D & 0 & -\frac{m}{c^*}I & 0 & -\frac{1}{c^*}D\\ 0 & 0 & 0 & 0 & -\frac{\kappa}{c^*}D & -\frac{1}{\tau}I \end{pmatrix}\tag{15}$$

where $D^j = \dfrac{\partial^j}{\partial x^j}$ and I is the identity operator. The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \Big\{ U \in \mathcal{H} : v \in H^2 \cap H_0^1, \phi \in H_*^2, \psi, \theta \in H_*^1, q \in H_0^1, \\ (cu_x + d\phi) \in H_*^3, (du_x + \beta\phi)_x \in H_*^1 \cap L_0^2 \Big\}.$$

The well-posedness of the problem (1), (8), (9) is given by the following theorem.

Theorem 3 *For any $U_0 = (u_0, u_1, \phi_0, \phi_1, \theta_0, q_0) \in \mathcal{H}$, the problem* (1), (8), (9) *has a unique mild solution (u, ϕ, θ, q) such that*

$$u \in C\left([0, +\infty[\,; H^2(0, L) \cap H_0^1(0, L)\right), \ \phi \in C\left([0, +\infty[\,; H_*^1(0, L)\right), \\ \theta \in C\left([0, +\infty[\,; L^2(0, L)\right), q \in C\left([0, +\infty[\,; L^2(0, L)\right).$$

According to Theorem 1 and the Lumer–Phillips theorem, the proof of Theorem 3 is a consequence of the following lemma.

Lemma 1 *The operator $\mathcal{A}$ defined by* (15) *is the infinitesimal generator of a C_0–semigroup of contractions on $\mathcal{H}$.*

Proof. It suffices to prove that the operator $\mathcal{A}$ is dissipative and maximal.

A straightforward calculation shows that

$$Re \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -\frac{1}{\kappa} \int_0^\pi |q|^2 dx - \mu \int_0^\pi |\psi|^2 \, dx \le 0,$$

which proves the dissipativeness of $\mathcal{A}$.

Further, we show that $0 \in \rho(\mathcal{A})$. Let $F = (f^1, f^2, f^3, f^4, f^5, f^6)^T \in \mathcal{H}$ be given, and find $U = (u, v, \phi, \psi, \theta, q)^T \in D(\mathcal{A})$ such that $\mathcal{A}U = F$, that is,

$$\begin{cases} v = f^1, \\ au_{xx} + b\phi_x - cu_{xxxx} - d\phi_{xxx} - \delta\theta_x = \rho f^2, \\ \psi = f^3, \\ du_{xxx} + \beta\phi_{xx} - \xi\phi - bu_x + m\theta - \mu\psi = Jf^4, \\ q_x + \delta v_x + m\psi = -c^* f^5, \\ q + \kappa\theta_x = -\tau f^6. \end{cases} \tag{16}$$

From $(16)_1$, $(16)_3$ and $(16)_5$, we have

$$v = f^1 \in H^2(0, \pi) \cap H_0^1(0, \pi), \tag{17}$$

$$\psi = f^3 \in H_*^1(0, \pi) \tag{18}$$

and

$$q_x = -c^* f^5 - \delta f_x^1 - m f^3 \in L^2(0, \pi). \tag{19}$$

The theory of ordinary differential equations implies that

$$q(x) = -c^\star \int_0^x f^5(s)ds - \delta f^1(x) - m \int_0^x f^3(s)ds \in H^1(0,\pi).$$

Clearly, $q(0) = q(\pi) = 0$. Then

$$q \in H_0^1(0,\pi). \tag{20}$$

On the other hand, plugging q just obtained in $(16)_6$, we get

$$\theta_x = -\frac{1}{\kappa}(\tau f^6 + q) \in L^2(0,\pi).$$

Using the usual theory of ordinary differential equations again, we conclude that

$$\theta \in H^1(0,\pi).$$

By substituting the values of ψ and θ in $(16)_2$ and $(16))_4$, we get

$$\begin{cases} au_{xx} + b\phi_x - cu_{xxxx} - d\phi_{xxx} = \rho f^2 + \delta\theta_x, \\ du_{xxx} + \beta\phi_{xx} - \xi\phi - bu_x = Jf^4 - m\theta + \mu f^3. \end{cases} \tag{21}$$

Note that

$$g_1 = \rho f^2 + \delta\theta_x \in L^2(0,\pi) \quad \text{and} \quad g_2 = Jf^4 - m\theta + \mu f^3 \in L^2_*(0,\pi).$$

To prove the solvability of (21), we take the L^2–inner product of the equations of (21) by u^* and ϕ^*, respectively; then, using the integration by parts and adding the two equations, we get

$$B\left((u,\phi)^T, (u^*,\phi^*)^T\right) = L\left((u^*,\phi^*)^T\right), \tag{22}$$

where

$$\begin{aligned} B\left((u,\phi)^T, (u^*,\phi^*)^T\right) &= c\int_0^\pi u_{xx}\overline{u^*_{xx}}dx + a\int_0^\pi u_x\overline{u^*_x}dx + \beta\int_0^\pi \phi_x\overline{\phi^*_x}dx \\ &\quad + b\int_0^\pi \left(\phi\overline{u^*_x} + u_x\overline{\phi^*}\right)dx + d\int_0^\pi \left(\phi_x\overline{u^*_{xx}} + u_{xx}\overline{\phi^*_x}\right)dx \\ &\quad + \xi\int_0^\pi \phi\overline{\phi^*}dx \end{aligned}$$

and

$$L\left((u^*,\phi^*)^T\right) = \int_0^\pi \left(\rho f^2 + \delta\theta_x\right)\overline{u^*}dx + \int_0^\pi \left(Jf^4 - m\theta + \mu f^3\right)\overline{\phi^*}dx.$$

Clearly, $B(\cdot,\cdot)$ is a bilinear and continuous form, and $L(\cdot)$ is a linear and continuous form over

$$\mathcal{W} = \left(H^2(0,\pi) \cap H_0^1(0,\pi)\right) \times H^1_*(0,\pi).$$

Moreover,

$$B\left((u,\phi)^T,(u,\phi)^T\right) = c\int_0^\pi |u_{xx}|^2\,dx + a\int_0^\pi |u_x|^2\,dx + \beta\int_0^\pi |\phi_x|^2\,dx + \xi\int_0^\pi |\phi|^2\,dx + 2b\,\mathrm{Re}\int_0^\pi (\phi\overline{u_x})\,dx + 2d\,\mathrm{Re}\int_0^\pi (\phi_x\overline{u_{xx}})\,dx.$$

It is easy to show that

$$a\int_0^\pi |u_x|^2\,dx + \xi\int_0^\pi |\phi|^2\,dx + 2b\,\mathrm{Re}\int_0^\pi (\phi\overline{u_x})\,dx \geq \frac{1}{2}\left(a-\frac{b^2}{\xi}\right)\int_0^\pi |u_x|^2\,dx + \frac{1}{2}\left(\xi-\frac{b^2}{a}\right)\int_0^\pi |\phi|^2\,dx.$$

Thus, using (2), we arrive at

$$B\left((u,\phi)^T,(u,\phi)^T\right) \geq \frac{1}{2}\left(a-\frac{b^2}{\xi}\right)\int_0^\pi |u_x|^2\,dx + \frac{1}{2}\left(\xi-\frac{b^2}{a}\right)\int_0^\pi |\phi|^2\,dx + \frac{1}{2}\left(c-\frac{d^2}{\beta}\right)\int_0^\pi |u_{xx}|^2\,dx + \frac{1}{2}\left(\beta-\frac{d^2}{c}\right)\int_0^\pi |\phi_x|^2\,dx,$$

which shows that $B(\cdot,\cdot)$ is coercive. Then, using the Lax–Milgram theorem, we infer that the problem (21) has a unique solution

$$(u,\phi) \in \left(H^2(0,\pi)\cap H_0^1(0,\pi)\right)\times H_*^1(0,\pi). \tag{23}$$

Moreover, by taking $(u^*,\phi^*)=(u^*,0)$ in (22), we arrive at

$$\int_0^\pi (cu_x+d\phi)_x\,\overline{u_{xx}^*}dx = \int_0^\pi \left(au_{xx}+b\phi_x+\rho f^2+\delta\theta_x\right)\overline{u^*}dx,\ \forall u^*\in H_0^1(0,\pi),$$

which means that

$$cu_x+d\phi\in H^3(0,\pi)$$

with

$$(cu_x+d\phi)_{xxx} = \left(au_{xx}+b\phi_x+\rho f^2+\delta\theta_x\right).$$

Clearly, $cu_x+d\phi\in L_*^2(0,\pi)$. Therefore,

$$cu_x+d\phi\in H_*^3(0,\pi). \tag{24}$$

Similarly, if we take $(u^*,\phi^*)=(0,\phi^*)$, $\phi^*\in H_*^1$, we get

$$\int_0^\pi (du_x+\beta\phi)_x\,\overline{\phi_x^*}dx = -\int_0^\pi \left(bu_x+\xi\phi-Jf^4+m\theta-\mu f^3\right)\overline{\phi^*}dx. \tag{25}$$

Here, we are not able to apply the elliptic regularity theory.

Let $\Psi \in H_0^1(0,\pi)$ and set $\phi^* = \Psi - \int_0^\pi \Psi(x,t)\,dx$. Then, $\phi^* \in H_*^1(0,\pi)$. Consequently, for any $\Psi \in H_0^1$,

$$\int_0^\pi (du_x + \beta\phi)_x \overline{\Psi_x} dx = -\int_0^\pi \left(bu_x + \xi\phi - Jf^4 + m\theta - \mu f^3\right) \overline{\Psi} dx.$$

Therefore,

$$du_x + \beta\phi \in H^2(0,\pi)$$

with

$$(du_x + \beta\phi)_{xx} = \left(bu_x + \xi\phi - Jf^4 + m\theta - \mu f^3\right) = r(x).$$

As above, $du_x + \beta\phi \in L_*^2(0,\pi)$, which gives

$$du_x + \beta\phi \in H_*^2(0,\pi). \tag{26}$$

From (24) and (26), we deduce that

$$\phi \in H_*^2(0,\pi). \tag{27}$$

On other hand, since $(du_x + \beta\phi)_{xx} = r(x)$, we have,

$$\int_0^\pi (du_x + \beta\phi)_{xx} \overline{\varphi} dx = \int_0^\pi r(x) \overline{\varphi} dx, \qquad \varphi \in H^1(0,\pi).$$

Integration by parts and the fact that $H_*^1(0,\pi) \subset H^1(0,\pi)$ yield

$$(du_x + \beta\phi)_x(\pi)\overline{\varphi}(\pi) - (du_x + \beta\phi)_x(0)\overline{\varphi}(0) - \int_0^\pi (du_x + \beta\phi)_x \overline{\varphi}_x dx$$
$$= \int_0^\pi r(x)\overline{\varphi} dx,$$

for any $\varphi \in H_*^1(0,\pi)$. Thus, (25) leads to

$$(du_x + \beta\phi)_x(\pi)\overline{\varphi}(\pi) - (du_x + \beta\phi)_x(0)\overline{\varphi}(0) = 0, \qquad \varphi \in H_*^1(0,\pi).$$

Consequently,

$$(du_x + \beta\phi)_x(\pi)\overline{\varphi}(\pi) = (du_x + \beta\phi)_x(0) = 0. \tag{28}$$

Using $(16)_4$, the boundary conditions and the above results, we infer that

$$\int_0^\pi \theta(x,t)\,dx = \frac{1}{m}\int_0^\pi \left(Jf^4 - du_{xxx} - \beta\phi_{xx} + \xi\phi + bu_x + \mu\psi\right) dx = 0.$$

Thus,

$$\theta \in H_*^1(0,\pi). \tag{29}$$

Combining (17), (18), (20) and (23)–(29), we conclude that $(u, v, \phi, \varphi, q, \theta) \in D(\mathcal{A})$, and thus, $0 \in \rho(\mathcal{A})$. Moreover, using a geometric series argument, we prove that $\lambda I - \mathcal{A} = \mathcal{A}(\lambda\mathcal{A}^{-1} - I)$ is invertible for $|\lambda| < \|\mathcal{A}^{-1}\|$, and hence, $\lambda \in \rho(\mathcal{A})$. This completes the proof that $\mathcal{A}$ is the infinitesimal generator of a C_0–semigroup of contractions. The Lumer–Phillips theorem ensures the existence of a unique solution to the problem (14) satisfying the statements of Theorem 3. □

3 Exponential stability

In this section, we state and prove our main result. To achieve this, we use a tool based on Prüss's and Gearhart's theorems. First, we recall the following theorem due to Gearhart [16, 20].

Theorem 4 *A C_0–semigroup of contractions $S(t) = e^{-\mathcal{A}t}$, generated by an operator $\mathcal{A}$ in a Hilbert space $\mathcal{H}$, is exponentially stable if and only if*

(i) $i\mathbb{R} = \{i\lambda, \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A})$;

(ii) $\overline{\lim\limits_{|\lambda|\to\infty}} \|(i\lambda I - \mathcal{A})^{-1}\| < \infty$.

Now, we state our main result.

Theorem 5 *Let (u, ϕ, θ, q) be the solution of* (1), (8), (9). *Then there exist two positive constants η and ω, independent of t and the initial data, such that*

$$E(t) \leq \eta e^{-\omega t}, \qquad t > 0.$$

The proof of Theorem 5 will be established through the following two lemmas.

Lemma 2 *The set $i\mathbb{R} = \{i\lambda; \lambda \in \mathbb{R}\}$ is contained in $\rho(\mathcal{A})$.*

Proof. The operator $\mathcal{A}^{-1}$ is compact. Indeed, let (F_n) be a bounded sequence in $\mathcal{H}$ and let (U_n) be the sequence in $D(\mathcal{A})$ such that $F_n = \mathcal{A}U_n$. Since $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H})$, the sequence $(U_n) = (\mathcal{A}^{-1}F_n)$ is bounded in $\mathcal{H}$. Therefore, there exists a constant $C > 0$ such that

$$\|U_n\|_{D(\mathcal{A})} = \|U_n\|_{\mathcal{H}} + \|\mathcal{A}U_n\|_{\mathcal{H}} \leq C.$$

Thus, (U_n) is bounded in $D(\mathcal{A})$. Using the fact that the injection of $H^m(0;\pi)$ into $H^j(0;\pi)$ is compact for $m > j$, we infer that we can extract a convergent subsequence $U_r = (u_r, v_r, \phi_r, \psi_r, \theta_r, q_r)$ with a limit $U = (u, v, \phi, \psi, \theta, q) \in \mathcal{H}$.

Suppose that there exists $\lambda \in \mathbb{R}$ ($\lambda \neq 0$) such that $i\lambda \in \sigma(\mathcal{A})$. As $\mathcal{A}^{-1}$ is compact, $i\lambda$ must be an eigenvalue of $\mathcal{A}$. Then, there exists a vector $U = (u, v, \phi, \psi, \theta, q) \neq 0$ such that $i\lambda U - \mathcal{A}U = 0$, that is,

$$\begin{aligned}
i\lambda u - v &= 0, \\
i\lambda\rho v - au_{xx} - b\phi_x + cu_{xxxx} + d\phi_{xxx} + \delta\theta_x &= 0, \\
i\lambda\phi - \psi &= 0, \\
i\lambda J\psi - du_{xxx} - \beta\phi_{xx} + \xi\phi + bu_x - m\theta + \mu\psi &= 0, \\
i\lambda c^*\theta + q_x + \delta v_x + m\psi &= 0, \\
i\lambda\tau q + q + \kappa\theta_x &= 0.
\end{aligned} \tag{30}$$

Further, we have

$$\langle i\lambda U - \mathcal{A}U, U\rangle = \frac{1}{\kappa}\int_0^{\pi} |q|^2 dx + \mu \int_0^{\pi} |\psi|^2\, dx = 0.$$

Then $q = \psi = 0$. Moreover, $(30)_3$, $(30)_6$ and $(30)_5$ give $\phi = 0$, $\theta = 0$ and $v_x = 0$. Then $(30)_1$ yields $u_x = 0$, and, finally, $u = v = 0$. Thus, we come to a contradiction, and the proof of Lemma 2 is complete. □

Lemma 3 *The operator $\mathcal{A}$ defined by* (15) *satisfies*

$$\overline{\lim_{|\lambda|\to\infty}} \left\|(i\lambda I - \mathcal{A})^{-1}\right\| < \infty.$$

Proof. It suffices to prove that there exists a positive constant C such that

$$\| \left(i\lambda I - \mathcal{A}\right)^{-1} F\| \leq C\|F\| \qquad \text{for all } \lambda \in \mathbb{R}.$$

Let $\lambda \in \mathbb{R}$ be given and let $F = (f^1, f^2, f^3, f^4, f^5, f^6) \in \mathcal{H}$. Then there exists a unique $U = (u, v, \phi, \psi, \theta, q)^T \in D(\mathcal{A})$ such that $(i\lambda I - \mathcal{A})U = F$, that is,

$$\begin{cases} i\lambda u - v = f^1, \\ i\lambda \rho v - a u_{xx} - b\phi_x + c u_{xxxx} + d\phi_{xxx} + \delta\theta_x = \rho f^2, \\ i\lambda\phi - \psi = f^3, \\ i\lambda J\psi - d u_{xxx} - \beta\phi_{xx} + \xi\phi + b u_x - m\theta + \mu\psi = J f^4, \\ i\lambda c^*\theta + q_x + \delta v_x + m\psi = c^* f^5, \\ i\lambda\tau q + q + \kappa\theta_x = \tau f^6, \end{cases} \tag{31}$$

First, recall that

$$Re\langle (i\lambda I - \mathcal{A})U, U\rangle_{\mathcal{H}} = \frac{1}{\kappa}\int_0^L |q|^2 dx + \mu \int |\psi|^2 = Re\langle F, U\rangle_{\mathcal{H}}.$$

Therefore,

$$\frac{1}{\kappa}\int_0^L |q|^2 dx + \mu \int |\psi|^2 \leq \|F\| \, \|U\| \, . \tag{32}$$

Taking the L^2–product of $(31)_2$ by u, $(31)_4$ by ϕ, and then using $(31)_1$ and $(31)_3$, we obtain

$$a\left\|u_x\right\|^2 + c\left\|u_{xx}\right\|^2 + \beta\left\|\phi_x\right\|^2 + \xi\left\|\phi\right\|^2 + 2b\,\mathrm{Re}\langle u_x, \phi\rangle + 2d\,\mathrm{Re}\langle u_{xx}, \phi_x\rangle$$

$$= \rho\left\langle f^2, u\right\rangle + \rho\left\langle v, f^1\right\rangle + J\left\langle f^4, \phi\right\rangle + J\left\langle \psi, f^3\right\rangle + \rho\left\|v\right\|^2 + J\left\|\psi\right\|^2$$

$$+\delta\left\langle \theta, u_x\right\rangle + m\left\langle \theta, \phi\right\rangle - \mu\left\langle \psi, \phi\right\rangle .$$

The Cauchy–Schwarz and Young's inequalities lead to

$$\frac{a}{2}\|u_x\|^2 + c\|u_{xx}\|^2 + \beta\|\phi_x\|^2 + \frac{\xi}{2}\|\phi\|^2 + 2b\,\mathrm{Re}\langle u_x, \phi\rangle + 2d\,\mathrm{Re}\langle u_{xx}, \phi_x\rangle \\ \leq C\|F\|\|U\| + \rho\,\|v\|^2 + (\frac{\delta^2}{2a} + \frac{m^2}{\xi})\|\theta\|^2. \tag{33}$$

At this point, we have to estimate $\|v\|_{L^2}$. To this end, we define the functions φ, w, z and y as solutions to the following problems:

$$-\varphi_{xx} = v \qquad \varphi_x(0) = \varphi_x(\pi) = 0, \tag{34}$$

$$-w_{xx} = \theta \qquad w_x(0) = w_x(\pi) = 0, \tag{35}$$

$$-z_{xx} = f^5 \qquad z(0) = z(\pi) = 0, \tag{36}$$

$$-y_{xx} = f^2 \qquad y(0) = y(\pi) = 0. \tag{37}$$

Multiplying (36) by z in $L^2(0,\pi)$ and using the Cauchy–Schwarz and Poincaré's inequalities, we get

$$\|z_x\| \leq C_p\left\|f^5\right\|, \tag{38}$$

Similarly,

$$\|y_x\| \leq C_p\left\|f^2\right\|, \tag{39}$$

where C_p is Poincaré's constant.

Next, we multiply $(31)_5$ by φ_x in $L^2(0,\pi)$ to obtain

$$\langle i\lambda c^*\theta, \varphi_x\rangle + \langle q_x, \varphi_x\rangle + \delta\langle v_x, \varphi_x\rangle + m\langle\psi, \varphi_x\rangle = c^*\langle f^5, \varphi_x\rangle. \tag{40}$$

Let us estimate each term in (40).

Integration by parts, (34) and (35) lead to

$$I_1 = \langle i\lambda c^*\theta, \varphi_x\rangle = -c^*\langle i\lambda w_{xx}, \varphi_x\rangle = c^*\langle i\lambda w_x, \varphi_{xx}\rangle = c^*\langle w_x, i\lambda v\rangle. \tag{41}$$

From $(31)_2$, we get

$$i\lambda v = \frac{1}{\rho}\left(au_{xx} + b\phi_x - cu_{xxxx} - d\phi_{xxx} - \delta\theta_x + \rho f^2\right).$$

Plugging $i\lambda v$ in (41) and using (37), we obtain

$$I_1 = \frac{ac^*}{\rho}\langle w_x, u_{xx}\rangle + \frac{bc^*}{\rho}\langle w_x, \phi_x\rangle - \frac{cc^*}{\rho}\langle w_x, u_{xxxx}\rangle \\ -\frac{dc^*}{\rho}\langle w_x, \phi_{xxx}\rangle - \frac{\delta c^*}{\rho}\langle w_x, \theta_x\rangle - c^*\langle w_x, y_{xx}\rangle.$$

Integration by parts and (35) give

$$I_1 = \frac{ac^*}{\rho}\langle\theta, u_x\rangle + \frac{bc^*}{\rho}\langle\theta, \phi\rangle + \frac{cc^*}{\rho}\langle\theta_x, u_{xx}\rangle + \frac{dc^*}{\rho}\langle\theta_x, \phi_x\rangle - \frac{\delta c^*}{\rho}\|\theta\|^2 - c^*\langle\theta, y_x\rangle. \tag{42}$$

Similarly, we obtain

$$I_2 = \langle q_x, \varphi_x \rangle = -\langle q, \varphi_{xx} \rangle = \langle q, v \rangle, \tag{43}$$

$$I_3 = \delta \langle v_x, \varphi_x \rangle = -\delta \langle v, \varphi_{xx} \rangle = \delta \|v\|^2 \tag{44}$$

and

$$I_4 = c^* \langle f^5, \varphi_x \rangle = -c^* \langle z_{xx}, \varphi_x \rangle = c^* \langle z_x, \varphi_{xx} \rangle = -c^* \langle z_x, v \rangle . \tag{45}$$

Substituting (42)–(45) into (40), we get

$$\begin{aligned} \delta \|v\|^2 = \quad & -\frac{ac^*}{\rho} \langle \theta, u_x \rangle - \frac{bc^*}{\rho} \langle \theta, \phi \rangle - \frac{cc^*}{\rho} \langle \theta_x, u_{xx} \rangle - \frac{dc^*}{\rho} \langle \theta_x, \phi_x \rangle \\ & + \frac{\delta c^*}{\rho} \|\theta\|^2 + c^* \langle \theta, y_x \rangle - \langle q, v \rangle - m \langle \psi, \varphi_x \rangle - c^* \langle z_x, v \rangle . \end{aligned} \tag{46}$$

Young's inequality yields

$$\frac{ac^*}{\rho} |\langle \theta, u_x \rangle| \leq \frac{a\delta}{24\rho} \|u_x\|^2 + \frac{6ac^{*2}}{\delta\rho} \|\theta\|^2$$

and

$$\frac{bc^*}{\rho} |\langle \theta, \phi \rangle| \leq \frac{\xi\delta}{24\rho} \|\phi\|^2 + \frac{6b^2 c^{*2}}{\xi\delta\rho} \|\theta\|^2.$$

Replacing θ_x obtained from $(31)_6$, we get

$$\begin{aligned} \frac{cc^*}{\rho} |\langle \theta_x, u_{xx} \rangle| = \quad & \frac{cc^*}{\rho\kappa} |\langle \tau f^6 - i\lambda\tau q - q, u_{xx} \rangle| \\ & \leq \frac{cc^*\tau}{\rho\kappa} |\langle f^6, u_{xx} \rangle| + \left(\frac{cc^*\lambda\tau}{\rho\kappa} + \frac{cc^*}{\rho\kappa} \right) |\langle q, u_{xx} \rangle| \end{aligned}$$

and

$$\begin{aligned} \frac{dc^*}{\rho} |\langle \theta_x, \phi_x \rangle| = \quad & \frac{dc^*}{\rho\kappa} |\langle \tau f^6 - i\lambda\tau q - q, \phi_x \rangle| \\ & \leq \frac{dc^*\tau}{\rho\kappa} |\langle f^6, \phi_x \rangle| + \left(\frac{dc^*\lambda\tau}{\rho\kappa} + \frac{dc^*}{\rho\kappa} \right) |\langle q, \phi_x \rangle| . \end{aligned}$$

Thus, the Cauchy–Schwarz and Young's inequalities lead to

$$\frac{cc^*}{\rho} |\langle \theta_x, u_{xx} \rangle| \leq \frac{cc^*\tau}{\rho\kappa} \|f^6\| \|u_{xx}\| + \frac{4cc^{*2} (1+\lambda\tau)^2}{\delta\rho\kappa^2} \|q\|^2 + \frac{c\delta}{16\rho} \|u_{xx}\|^2 ,$$

and

$$\frac{dc^*}{\rho} |\langle \theta_x, \phi_x \rangle| \leq \frac{dc^*\tau}{\rho\kappa} \|f^6\| \|\phi_x\| + \frac{4d^2 c^{*2} (1+\lambda\tau)^2}{\beta\delta\rho\kappa^2} \|q\|^2 + \frac{\beta\delta}{16\rho} \|\phi_x\|^2 .$$

Using (32), we infer that

$$\frac{cc^*}{\rho}\left|\langle\theta_x, u_{xx}\rangle\right| \leq \frac{c\delta}{16\rho}\left\|u_{xx}\right\|^2 + C\|F\|\|U\|$$

and

$$\frac{dc^*}{\rho}\left|\langle\theta_x, \phi_x\rangle\right| \leq \frac{\beta\delta}{16\rho}\left\|\phi_x\right\|^2 + C\|F\|\|U\|.$$

Similarly, the Cauchy–Schwarz and Young's inequalities, (38) and (39) entail

$$c^*\Big(|\langle\theta, y_x\rangle| + |\langle z_x, v\rangle|\Big) \leq c^* C_p\Big(\|\theta\|\|f^2\| + \|f^5\|\|v\|\Big) \leq C\|F\|\|U\|,$$

and (32) gives

$$|\langle q, v\rangle| \leq \frac{\delta}{4}\left\|v\right\|^2 + C\left\|F\right\|\left\|U\right\|.$$

Next, Young's and Poincaré's inequalities yield

$$\begin{aligned} m|\langle\psi, \varphi_x\rangle| &\leq \varepsilon\|\varphi_x\|^2 + \frac{m^2}{4\varepsilon}\|\psi\|^2 \leq \varepsilon C_p\|\varphi_{xx}\|^2 + \frac{m^2}{4\varepsilon}\|\psi\|^2 \\ &\leq \frac{\delta}{4}\|v\|^2 + C\|F\|\|U\|. \end{aligned}$$

Finally, we substitute these estimates into (46), to get

$$\begin{aligned} 2\rho\left\|v\right\|^2 \leq \;& \frac{a}{6}\left\|u_x\right\|^2 + \frac{\xi}{6}\|\phi\|^2 + \frac{c}{4}\left\|u_{xx}\right\|^2 + \frac{\beta}{4}\left\|\phi_x\right\|^2 \\ &+ \left(\frac{24ac^{*2}}{\delta^2} + \frac{24b^2c^{*2}}{\xi\delta^2} + 4c^*\right)\left\|\theta\right\|^2 + C\left\|F\right\|\left\|U\right\|. \end{aligned} \tag{47}$$

Adding (33) and (47), we obtain

$$\begin{aligned} \rho\left\|v\right\|^2 + \frac{a}{3}\left\|u_x\right\|^2 + \frac{3c}{4}\left\|u_{xx}\right\|^2 + \frac{3\beta}{4}\left\|\phi_x\right\|^2 + \frac{\xi}{3}\left\|\phi\right\|^2 + c^*\left\|\theta\right\|^2 \\ + J\|\psi\|^2 + \frac{\tau}{\kappa}\|q\|^2 + 2b\,\mathrm{Re}\left\langle u_x, \phi\right\rangle + 2d\,\mathrm{Re}\left\langle u_{xx}, \phi_x\right\rangle \\ \leq \Big(\frac{24ac^{*2}}{\delta^2} + \frac{24b^2c^{*2}}{\xi\delta^2} + \frac{\delta^2}{2a} + \frac{m^2}{\xi} + 5c^*\Big)\|\theta\|^2 + C\left\|F\right\|\left\|U\right\|. \end{aligned} \tag{48}$$

The next step is to estimate $\left\|\theta\right\|_{L^2}$. To this end, we define the functions w and z as in (35) and (36), and α, η as the solutions to the following problems:

$$-\alpha_{xx} = q, \qquad \alpha\left(0\right) = \alpha_x\left(\pi\right) = 0, \tag{49}$$

$$-\eta_{xx} = f^6, \qquad \eta\left(0\right) = \eta\left(\pi\right) = 0. \tag{50}$$

Multiplying the equations (49) by α and (50) by η in $L^2(0,\pi)$, using the integration by parts, the Cauchy–Schwarz and Poincaré's inequalities, we obtain

$$\|\alpha_x\| \le C_p \|q\|, \tag{51}$$
$$\|\eta_x\| \le C_p \left\|f^6\right\|. \tag{52}$$

Taking the L^2–inner product of $(31)_6$ by w_x, using (49), (50), (35) and the integration by parts, we get

$$\tau \langle \alpha_x, i\lambda\theta\rangle - \langle \alpha_x, \theta\rangle + \kappa \|\theta\|^2 = -\tau \langle \eta_x, \theta\rangle.$$

Replacing $i\lambda\theta$ obtained from $(31)_5$, we get

$$\begin{aligned}\tau \langle \alpha_x, i\lambda\theta\rangle &= \tau \left\langle \alpha_x, f^5\right\rangle - \frac{\tau}{c^*}\langle \alpha_x, q_x\rangle - \frac{\delta\tau}{c^*}\langle \alpha_x, v_x\rangle - \frac{m\tau}{c^*}\langle \alpha_x, \psi\rangle \\ &= -\tau \langle q, z_x\rangle - \frac{\tau}{c^*}\|q\|^2 - \frac{\delta\tau}{c^*}\langle q, v\rangle - \frac{m\tau}{c^*}\langle \alpha_x, \psi\rangle.\end{aligned}$$

Thus,

$$\kappa \|\theta\|^2 = -\tau \langle \eta_x, \theta\rangle + \tau \langle q, z_x\rangle + \frac{\tau}{c^*}\|q\|^2 + \frac{\delta\tau}{c^*}\langle q, v\rangle + \frac{m\tau}{c^*}\langle \alpha_x, \psi\rangle + \langle \alpha_x, \theta\rangle.$$

Let us estimate each term in the obtained relation. First, we have

$$\tau\Big(|\langle \eta_x, \theta\rangle| + |\langle q, z_x\rangle|\Big) \le C \|F\| \|U\|.$$

Young's inequality, (51) and (32) give

$$\left|\frac{\tau\delta}{c^*}\langle q, v\rangle\right| \le C_\varepsilon \|F\|\|U\| + \varepsilon\|v\|^2,$$

$$|\langle \alpha_x, \theta\rangle| \le C\|F\|\|U\| + \frac{\kappa}{2}\|\theta\|^2,$$

and

$$\frac{m\tau}{c^*}|\langle \alpha_x, \psi\rangle| \le C\|F\|\|U\|.$$

Therefore,

$$\frac{\kappa}{2}\|\theta\|^2 \le C_\varepsilon \|F\|\|U\| + \varepsilon\|v\|^2. \tag{53}$$

Multiplying (53) by $\left(\frac{48ac^{*2}}{\kappa\delta^2} + \frac{48b^2c^{*2}}{\kappa\xi\delta^2} + \frac{\delta^2}{a\kappa} + \frac{2m^2}{\kappa\xi} + \frac{10c^*}{\kappa}\right)$ and choosing ε such that $\left(\frac{48ac^{*2}}{\kappa\delta^2} + \frac{48b^2c^{*2}}{\kappa\xi\delta^2} + \frac{\delta}{a\kappa} + \frac{2m^2}{\kappa\xi} + \frac{10c^*}{\kappa}\right)\varepsilon = \frac{\rho}{2}$, we obtain

$$\left(\frac{24ac^{*2}}{\delta^2} + \frac{24b^2c^{*2}}{\xi\delta^2} + \frac{\delta}{2a} + \frac{m^2}{\xi} + 5c^*\right)\|\theta\|^2 \le C\|F\|\|U\| + \frac{\rho}{2}\|v\|^2. \tag{54}$$

Substituting (54) into (48), we get

$$\frac{\rho}{2}\|v\|^2+\frac{a}{3}\|u_x\|^2+\frac{3c}{4}\|u_{xx}\|^2+\frac{3\beta}{4}\|\phi_x\|^2+\frac{\xi}{3}\|\phi\|^2+c^*\|\theta\|^2+J\|\psi\|^2+\frac{\tau}{\kappa}\|q\|^2$$
$$+2b\operatorname{Re}\langle u_x,\phi\rangle+2d\operatorname{Re}\langle u_{xx},\phi_x\rangle\le C\|F\|\|U\|,$$

that is,

$$\frac{1}{3}\|U\|^2\le C\|F\|\|U\|,$$

which gives

$$\|U\|\le C\|F\|,$$

and the proof of Lemma 3 is completed. □

The proof of Theorem 5 is a consequence of the Lemmas 2 and 3.

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