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On normal solvability of a Dirichlet's type problem for improperly elliptic equation third order

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Abstract

We consider Dirichlet type problem in upper half-plane for improperly elliptic equation $u_{z\bar{z}^2} = 0$, with boundary functions from the class $L^1(\rho)$ $(\rho = (1 + |x|)^{-\alpha}, \alpha \geq 0)$. The solutions of the problem are obtained in explicit form.

Key Words: Improperly elliptic, Boundary value problem, Wight spaces, Dirichlet problem.
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1. Introduction

Note that two of three roots of the characteristic equation, corresponding to $u_{z\bar{z}^2} = 0$ belong to the upper half-plane and one to the lower. In this case the equation is called improperly elliptic. The boundary value problems for improperly elliptic equations in Hölder classes in bounded domains were studied in ([1] - [4]). In [5] in Hölder classes in half-plane the boundary-value problems for homogeneous equation with constant coefficients were studied. It should be noted that in these investigations the fact, that Cauchy type integral is a bounded operator in these spaces is essential. It is well known that Cauchy type integral is not bounded operator in L^1 and moreover in $L^1(\rho)$. In cases, when the boundary functions belong to L^1 class, the boundary condition should be interpreted in L^1–convergence sense ([6]-[9]).

Some results of this paper when α is noninteger where announced in [10]. In this paper the solutions of the Dirichlet type problem are obtained in explicit form. We will see, that the formula for the solution essentially depends on the fact that α integer or not.

2. Problem statement and preliminary lemmas

Let B_1 be the class of the continuously differentiable functions in the upper half-plane $\Pi^+ = \{z : Imz > 0\}$ of the complex plane, satisfying the conditions

$$\left|\frac{\partial^{p+q}u}{\partial z^p \partial \bar{z}^q}\right| < A|z|^N,\ Imz > y_0 > 0,\ p \geq 0,\ q \geq 0,\ p+q \leq 2,$$

where A is a constant depending on y_0, and N is the natural number depending on u,

$$\frac{\partial u}{\partial z} = \frac{1}{2}\Big(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\Big),\ \ \frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\Big).$$

In the Π^+ the following boundary value problem is considered: find the solution $u \in B_1$ of the equation

$$\frac{\partial^3 u}{\partial z \partial \bar{z}^2} = 0, \tag{1}$$

which satisfies conditions

$$\lim_{y\to+0} \|u(x,y) - f(x)\|_{L^1(\rho)} = 0 \tag{2}$$

$$\lim_{y\to+0} \left\|\Re\frac{\partial u(x,y)}{\partial y} - f_1(x)\right\|_{L^1(\rho)} = 0, \tag{3}$$

where $\rho(x) = (1+|x|)^{-\alpha},\ \alpha \geq 0,\ f_0(x),\ f_0'(x),\ f_1(x) \in L^1(\rho)$; $L^1(\rho)$ is the functional class with the following norm

$$\|f\|_{L^1(\rho)} = \int_{-\infty}^{\infty} |f(x)|\rho(x)dx < \infty.$$

To study the problem (1)-(3) the following lemmas are needed

Lemma 1 *Let $\delta \in (0,1)$ and*

$$I(x,y) = (x+i)^\delta \int_{-\infty}^{\infty} \frac{ydt}{(t+i)^\delta(t-x-iy)},\ \ y > 0 \tag{4}$$

then there exists a constant C, independent from x and y, such that

$$|I(x,y)| < C. \tag{5}$$

Proof. Let $I(x,y) = I_1(x,y) + I_2(x,y) + I_3(x,y)$, where

$$I_1(x,y) = (x+i)^\delta \int_{1}^{\infty} \frac{ydt}{(t+i)^\delta(t-x-iy)},$$

$$I_2(x,y) = (x+i)^\delta \int_{-\infty}^{-1} \frac{ydt}{(t+i)^\delta(t-x-iy)},$$

$$I_3(x,y) = (x+i)^\delta \int_{-1}^{1} \frac{ydt}{(t+i)^\delta(t-x-iy)}.$$

Replacing t by t^{-1} we get

$$I_1(x,y) = (x+i)^\delta \int_0^1 \frac{ydt}{(1+it)^\delta t^{1-\delta}(1-(x-iy)t)}$$
$$= \frac{(x+i)^\delta}{x-iy} \int_0^1 \frac{ydt}{(1+it)^\delta t^{1-\delta}(t-\frac{1}{x-iy})}.$$

Using Muskhelishvili's inquality ([11], p. 93) we get

$$\Big|\int_0^1 \frac{ydt}{(1+it)^\delta t^{1-\delta}(t-\frac{1}{x-iy})}\Big| < A_0|x-iy|^{1-\delta}.$$

Therefore, for $|x| > 2^{-1}$

$$|I_1(x,y)| \le \Big|\frac{x+i}{x-iy}\Big|^\delta < C.$$

If $|x| < 2^{-1}$, then $|t-x-iy| > 2^{-1}t$ and

$$|I_1(x,y)| < \int_1^\infty \frac{dt}{|t+i|^{1+\delta}}.$$

In the same way it may be proved, that $|I_2(x,y)| < C$. When $|x| < 2$ we get

$$|I_3(x,y)| < 3\int_{-1}^1 \frac{ydt}{|t-x-iy|}.$$

Therefore $|I_3(x,y)| < C$. Taking into account that the function

$$K(z) = \int_{-1}^1 \frac{dt}{(t+i)(t-z)}$$

is holomorphic in the infinity and $K(\infty) = 0$, then representing it by the Laurent series, we get

$$K(z) = \frac{A_1}{x-iy} + \frac{A_2}{(x-iy)^2} + ... = \frac{1}{x-iy}\Big(A_1 + \frac{A_2}{x-iy} + ...\Big)$$

and

$$|K(x-iy)| < \frac{A}{|x-iy|},\ |x| > 2,$$

where A is some constant. For $|x| > 2$ we have

$$|I_3(x,y))| < |x+i|^\delta \frac{B}{|x-iy|} < C.$$

The lemma is proved. $\square$

Lemma 2 *Let $f \in L^1(\rho)$, α be a noninteger, $n = [\alpha]$ and*

$$\Phi(f,z) = \frac{(z+i)^n}{2\pi i}\int_{-\infty}^\infty \frac{f(t)dt}{(t+i)^n(t-z)}, \tag{6}$$

then

$$\lim_{y\longrightarrow+0} y\|\Phi(f,x+iy)\|_{L^1(\rho)} = 0. \tag{7}$$

Proof. Let $f \in C^\delta(-\infty, \infty), \delta > 0$ be a finite function with support in $[-A, A]$. Out of segment $[-A, A]$ the function $\Phi(f, z)$ is holomorphic and $|\Phi(f, z)| = O(|z|^{n-1})$ in infinity.

$$y \int_{|x|>2A} |\Phi(f, x+iy)|\rho(x)dx < A_0 y \int_{|x|>2A} |x+iy|^{n-1}\rho(x)dx <$$

$$< A_0 y \int_{|x|>2A} \frac{dx}{|x+i|^{1+\{\alpha\}}}$$

The last integral converges ($\{\alpha\} > 0$) and

$$\lim_{y \longrightarrow +0} y \int_{|x|>2A} |\Phi(f, x+iy)|\rho(x)dx = 0.$$

Taking into account that $x \in [-A, A]$ the function $\Phi(f, z)$ is uniformly bounded [11] we get

$$\lim_{y \longrightarrow +0} y \int_{|x|<2A} |\Phi(f, x+iy)|\rho(x)dx = 0.$$

Thus, when $f \in C^\delta(-\infty, \infty)$, the finite function (7) has been proved. To prove the lemma we need the following inequality

$$y\|\Phi(f, x+iy)\|_{L^1(\rho)} < C\|f\|_{L^1(\rho)}. \tag{8}$$

In fact

$$\|\Phi(f, x+iy)\|_{L^1(\rho)} = \int_{-\infty}^{\infty} \left| \frac{(x+iy)^n}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)dt}{(t+i)^n(t-z)} \right| \frac{dx}{|x+i|^\alpha} <$$

$$< \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(t)}{|t+i|^{\alpha-\{\alpha\}}} \int_{-\infty}^{\infty} \frac{dx}{|x+i|^{\{\alpha\}}|t-x-iy|} dt.$$

According to lemma 1

$$|t+i|^{\{\alpha\}} \int_{-\infty}^{\infty} \frac{ydx}{|x+i|^{\{\alpha\}}|t-x-iy|} < C$$

so the validity of estimation (8) follows. Let now $f \in L^1(\rho)$ and $\epsilon > 0$ be arbitrary number, and a finite function $f_1 \in C^\delta(-\infty, \infty)$ satisfy inequality $\|f - f_1\| < \epsilon$. Then

$$y\|\Phi(f, x+iy)\|_{L^1(\rho)} \le y\|\Phi(f - f_1, x+iy)\|_{L^1(\rho)} + y\|\Phi(f_1, x+iy)\|_{L^1(\rho)} \le$$

$$\le C\epsilon + y\|\Phi(f, x+iy)\|_{L^1(\rho)},$$

and so as

$$\lim_{y \longrightarrow +0} y\|\Phi(f_1, x+iy)\|_{L^1(\rho)} = 0$$

and $\epsilon > 0$ is an arbitrary number, the lemma is proved. □

Lemma 3 *Let $f \in L^1(\rho)$. Then*

$$\lim_{y \longrightarrow +0} y\|\Phi'(f, x+iy)\|_{L^1(\rho)} = 0, \tag{9}$$

where $\Phi(f, x+iy)$ is the function defined by (6).

Proof. Let's begin with the proof of inequality

$$\sup_{y>0} y\|\Phi'(f,x+iy)\|_{L^1(\rho)} \le C\|f\|_{L^1(\rho)}. \tag{10}$$

Note that $\Phi'(f,z) = I_1(f,z) + I_2(f,z)$, where

$$I_1(f,z) = \frac{n(z+i)^{n-1}}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)dt}{(t+i)^n(t-z)}, \tag{11}$$

$$I_2(f,z) = \frac{(z+i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)dt}{(t+i)^n(t-z)^2}. \tag{12}$$

From (11) we have

$$y\int_{-\infty}^{\infty}|I_1(f,x+iy)|\rho(x)dx =$$

$$= \frac{ny}{2\pi}\int_{-\infty}^{\infty}\Big|(x+iy+i)^{n-1}\int_{-\infty}^{\infty}\frac{f(t)}{(t+i)^n}\frac{dt}{t-x-iy}\frac{dx}{|x+i|^\alpha}\Big| \le$$

$$\le Cny\int_{-\infty}^{\infty}\frac{f(t)}{|t+i|^\alpha}|t+i|^{\{\alpha\}}\int_{-\infty}^{\infty}\frac{dx}{|t-x-iy||x+i|^{1+\{\alpha\}}}dt.$$

Taking into consideration, that

$$|t+i|^{\{\alpha\}}\int_{-\infty}^{\infty}\frac{ydx}{|t-x-iy||x+i|^{1+\{\alpha\}}} \le$$

$$\le C_1\Big(\int_{-\infty}^{\infty}\frac{ydx}{|t-x-iy|^{1-\{\alpha\}}(1+|x|)^{1+\{\alpha\}}} + \int_{-\infty}^{\infty}\frac{ydx}{|t-x-iy|^2}+$$

$$+\int_{-\infty}^{\infty}\frac{ydx}{|x+i|^2}\Big) \le 2C_1\Big(\int_{-\infty}^{\infty}\frac{ydx}{|t-x-iy|^2} + \int_{-\infty}^{\infty}\frac{ydx}{|x+i|^2}\Big) < C_2,$$

we get finally

$$y\|I_1(f,x+iy)\|_{L^1(\rho)} < C\|f\|_{L^1(\rho)}. \tag{13}$$

Estimating (12) we get

$$y\int_{-\infty}^{\infty}|I_2(f,x+iy)|\rho(x)dx =$$

$$= \frac{y}{2\pi}\int_{-\infty}^{\infty}\Big|(x+iy+i)^n\int_{-\infty}^{\infty}\frac{f(t)}{(t+i)^n}\frac{dt}{(t-x-iy)^2}\frac{dx}{|x+i|^\alpha}\Big| \le$$

$$\le \int_{-\infty}^{\infty}\frac{f(t)}{|t+i|^\alpha}|t+i|^{\{\alpha\}}\int_{-\infty}^{\infty}\frac{ydx}{|t-x-iy|^2|x+i|^{\{\alpha\}}}dt.$$

As

$$|t+i|^{\{\alpha\}}\int_{-\infty}^{\infty}\frac{ydx}{|t-x-iy|^2|x+i|^{\{\alpha\}}} \le \int_{-\infty}^{\infty}\frac{ydx}{|t-x-iy|^2}+$$

$$+C_1\int_{-\infty}^{\infty}\frac{ydx}{|t-x-iy|^{2-\{\alpha\}}|x+i|^{\{\alpha\}}}dt \le \pi+$$

$$+C_2\Big(\int_{-\infty}^{\infty}\frac{ydx}{|t-x-iy|^2} + \int_{-\infty}^{\infty}\frac{ydx}{|x+i|^2}\Big) < C,$$

then

$$y\|I_2(f, x+iy)\|_{L^1(\rho)} < C\|f\|_{L^1(\rho)}. \tag{14}$$

From (13) and (14) follows (10). Let now $g(x) \in C^{1,\delta}$ be a finite function, the support of which is in the interval $[-A,\ A]$. In this case the function $\Phi(g, z)$ from (8) is analytic if $|z| > A$ and has the pole of order $n-2$ at infinity. Therefore $|I_1(g, z)| < C|z|^{n-2},\ |I_2(g, z)| < C|z|^{n-2},\ |z| > 2A$. From these estimates follows

$$y\int_{|x|>2A} |I_k(g, x+iy)|\rho(x)dx < Cy\int_{|x|>2A} \frac{dx}{(1+|x|)^{2+\{\alpha\}}},\ \ k = 1, 2.$$

From this inequality it may be deduced that

$$\lim_{y\to+0} y\int_{-\infty}^{\infty} |\Phi'(g, x+iy)|\rho(x)dx = 0.$$

Recalling one more time inequality (10) we achieve the proof of lemma. □

Lemma 4 *Let $f \in L^1(\rho)$ and*

$$\Phi_1(f, z) = \frac{(z+i)^n}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)dt}{(t+i)^n(t-z)^2}$$

Then

$$\lim_{y\longrightarrow+0} y\|\Phi_1(f, x+iy)\|_{L^1(\rho)} = 0,$$

Proof. Taking into account (12) and (14), we get

$$y\|\Phi_1(f, x+iy)\|_{L^1(\rho)} < C\|f\|_{L^1(\rho)}.$$

Now similar to the proof of lemma 3 we prove lemma 4. □

3 Investigation of a homogeneous problem

The general solution of the equation (1) may be represented in the form ([10])

$$u(z) = \varphi(z) + y\psi(z) + \overline{\omega(z)},\ \omega(i) = \omega\prime(i) = 0 \tag{15}$$

where $\varphi,\ \psi, \omega$ are analytic in Π^+ functions, uniquely defined by u. Substituting u into conditions (2), (3) we get the following boundary conditions

$$\lim_{y\to+0} \|\varphi(x+iy) + y\psi(x+iy) + \overline{\omega(x+iy)} - f_0\|_{L^1(\rho)} = 0, \tag{16}$$

$$\lim_{y\to+0} \|\Re(i\varphi(x+iy) + \psi(x+iy) + iy\psi'(x+iy) - i\overline{\omega(x+iy)} - f_1)\|_{L^1(\rho)} = 0, \tag{17}$$

for definition of functions $\varphi,\ \psi, \omega$. The problems (1)-(3) and (16)-(17) are equivalent. For investigation of the problem (1)-(3) we formulate the following theorem (see [10]):

Theorem A. Let the function u be the solution of the problem (1)-(3). Then it can be represented in the form (15), where

$$\varphi(z) = \frac{(z+i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_0(t)dt}{(t+i)^n(t-z)} + P_0(z) \tag{18}$$

$$\omega(z) = \frac{(z-i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{f_0(t)}dt}{(t-i)^n(t-z)} - \overline{P_0(\overline{z})} \tag{19}$$

$$\psi(z) = \frac{(z+i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_1(t)dt}{(t+i)^n(t-z)} + \\ + \frac{(z-i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_1(t)dt}{(t-i)^n(t-z)} - i\varphi'(z) - i\omega'(z) + P_1(z) \tag{20}$$

Here the polynomials P_0 and P_1 are uniquely determined by u and coefficients of P_1 are purely imaginary numbers.

3.1 At first, we consider the case of noninteger α.

Theorem 1 *If α is a noninteger number, then the homogeneous problem* (1)-(3) *has* $3n, n = [\alpha]$ *linearly independent solutions over the field of real numbers. These solutions can be represented in the following form*

$$u_k(z) = z^k - \overline{z}^k, \quad k = 2,3,...,n,$$

$$\widetilde{u}_k(z) = i((z^k - \overline{z}^k) - k(z-\overline{z})z^{k-1}), \quad k = 2,3,...,n+1,$$

$$v_k(z) = (z-\overline{z})z^k, \quad k = 0,1,,...,n-1,$$

$$\widetilde{v}_{n+1}(z) = z^{n+1} - \overline{z}^{n+1} - (n+1)(z-\overline{z})z^n \tag{21}$$

If $\alpha \leq 1$ the homogeneous problem has only a trivial solution.

Proof. Recalling theorem A from (15) and (18)-(20) we get ($f_0 \equiv f_1 \equiv 0$)

$$u(z) = P_0(z) - P_0(\overline{z}) + y(P_1(z) - i(P_0'(z) - \overline{P}_0'(z))), \tag{22}$$

where $P_0,\ P_1$ are some polynomials, and coefficients P_1 are purely imaginary. Denoting the coefficients of the polynomial P_0 by $a_k = a_k' + ia_k'',\ k = 0,1,...,m_0$, and the coefficients of the polynomial P_1 by $2ib_k,\ k = 0,1,...,m_1$, we represent u in the following form

$$u(z) = \sum_{k=1}^{m_0} a_k' u_k(z) + i\sum_{k=1}^{m_0} a_k''\widetilde{u}_k(z) + \sum_{k=1}^{m_1} b_k v_k(z). \tag{23}$$

The order of function $u_k(z)$ for fixed y if $|x| \to \infty$ is equal $k-1$, i.e. $|u_k(x+iy)| = yO(|x|^{k-1})$. Similarly $|v_k(x+iy)| = O(|x|^{k-1})$. As

$$\widetilde{u}_k(x+iy) = \widetilde{u}_k(iy) + \widetilde{u}_k'(iy)x ... + \frac{\widetilde{u}_k^{(k)}(iy)}{k!}x^k$$

and $\widetilde{u}_k^{(k)}(iy) = \widetilde{u}_k^{(k-1)}(iy) = 0,\ \widetilde{u}_k^{(k-2)}(iy) = 2k(k-1)y^2$, we get $|\widetilde{u}_k(x+iy)| = y^2 O(|x|^{k-2})$. The function u from (22) belongs to the class $L^1(\rho)$ if and only if the order doesn't exceed $n-1$, and this corresponds to $k = 2, 3, \ldots, n+1$. It should be noted that the order of this function $u(z)$ to doesn't exceed m, where $m = \max\{m_0 - 1, m_1\}$. Hence, in order the function $u(z)$ satisfy the homogeneous problem (2), (3), it is necessary that $m \leq n-1$, i.e. $m_0 \leq n$ and $m_1 \leq n-1$. For arbitrary $k > 0$ we have $z^{k+1} - \bar{z}^{k+1} = 2iy(k+1)x^k + \ldots$, $(z-\bar{z})z^k = 2iyx^k + \ldots$. Therefore for arbitrary real number a'_k , and $b_k = -(k+1)a'_k$, we have $|a'_k u_{k+1}(z) - b_k v_k(z)| = 2yO(|x|^k)$. It follows from this that $\widetilde{v}_{n+1}(z) = z^{n+1} - \bar{z}^{n+1} - (n+1)(z-\bar{z})z^n$ in infinity has the order of $n-1$. It also should be noted that $u_1(z) = v_0(z)$, $\widetilde{u}_1(z) \equiv 0$.

Let's prove that every function from (21) satisfies the conditions (2), (3). It is sufficient to consider the functions $u_n(z)$, $\widetilde{u}_{n+1}(z)$, $\widetilde{v}_{n-1}(z)$ and $\widetilde{v}_{n+1}(z)$. Since $|u_n(x+iy)| = 2yO(|x|^{n-1})$, $|v_{n-1}(x+iy) = 2yO(|x|^{n-1})$, $|\widetilde{u}_{n+1}(x+iy)| = y^2 O(|x|^{n-1})$ and $|\widetilde{v}_{n+1}(x+iy)| = y^2 O(|x|^{n-1})$ with fixed $y > 0$, thus

$$\lim_{y\to+0} \|u_n(x,y)\|_{L^1(\rho)} = \lim_{y\to+0} \|v_{n-1}(x,y)\|_{L^1(\rho)} =$$

$$= \lim_{y\to+0} \|\widetilde{u}_{n+1}(x,y)\|_{L^1(\rho)} = \lim_{y\to+0} \|\widetilde{v}_{n+1}(x,y)\|_{L^1(\rho)} = 0.$$

Further

$$\frac{\partial u_n(z)}{\partial y} = ni(z^{n-1} + \bar{z}^{n-1}),$$

$$\frac{\partial \widetilde{u}_{n+1}(z)}{\partial y} = -(n+1)(z^n + \bar{z}^n) + 2(n+1)z^n + 2i(n+1)nyz^{n-1},$$

$$\frac{\partial v_{n-1}(z)}{\partial y} = 2iz^{n-1} - 2y(n-1)z^{n-2},$$

$$\frac{\partial \widetilde{v}_{n+1}(z)}{\partial y} = -(n+1)i(z^n + \bar{z}^n) + 2i(n+1)z^n - 2(n+1)nyz^{n-1},$$

and therefore

$$|Re\frac{\partial v_{n-1}(z)}{\partial y}| = yO(|x|^{n-2}),\ |Re\frac{\partial \widetilde{v}_{n+1}(z)}{\partial y}| = yO(|x|^{n-1}),$$

so the functions $v_{n-1}(z)$, $\widetilde{v}_{n+1}(z)$ also satisfies the condition (3).

To complete the proof of the theorem we must prove that the system of functions (21) is linearly independent over the field of the real numbers. Let's prove the linearly independence of the system $u_k(z)$, $k = 2, 3, \ldots, n$, $\widetilde{u}_k(z)$, $k = 2, 3, \ldots, n+1$. In fact, assume for some real numbers A_k, B_k,

$$\sum_{k=2}^{n} A_k u_k(z) + \sum_{k=2}^{n+1} B_k \widetilde{u}_k(z) \equiv 0.$$

It follows from this identity that

$$\sum_{k=2}^{n} (A_k + iB_k)\bar{z}^k + iB_{n+1}\bar{z}^{n+1} = Q_1(z) + \bar{z}Q_2(z),$$

where $Q_1(z)$, $Q_2(z)$ are some polynomials. We get

$$\frac{\partial^2}{\partial \overline{z}^2}\left(\sum_{k=2}^{n}(A_k+iB_k)\overline{z}^k+iB_{n+1}\overline{z}^{n+1}\right)=0.$$

It is possible only if $A_k=0$, $k=2,3,\dots,n$, $B_k=0$, $k=2,3,\dots,n+1$. Similar argumentations confirm that the system (21) is linearly independent. □

3.2. The case when α is integer.

Theorem 2 *The following statements are true.*

a) Let $\alpha=0,\ 1$. *In this case the homogeneous problem* (1) - (3) *has only trivial solution.*

b) Let $\alpha=n=2,3,\dots$. *Then the homogeneous problem* (1) - (3) *has* $3n-4$, *linearly independent solutions over the field of real numbers. These solutions can be represented as*

$$u_k(z)=z^k-\overline{z}^k,\quad k=2,3,...,n-1,$$
$$\widetilde{u}_k(z)=i((z^k-\overline{z}^k)-k(z-\overline{z})z^{k-1}),\quad k=2,3,...,n,$$
$$v_k(z)=(z-\overline{z})z^k,\quad k=0,1,,...,n-2. \tag{24}$$

Proof. For the function $u(x+iy)\in L^1(\rho)$, if $\alpha=0,\ 1$, it is necessary that $|u(x+iy)|=o(1)$ for $|x|\to\infty$. It is possible only if in (23) $m_0=1$, $m_1=0$. Therefore in this case $u(x+iy)=2a_1'iy$. But this function doesn't satisfy the condition (2), so a) is proved. By the same way we have $|u_n(x+iy)|=yO|x|^{n-1}$, $u_n(x+iy)\notin L^1(\rho)$. Therefore $u_n(x+iy)$ doesn't satisfy the homogeneous conditions (2), (3). The proof of therem is completed. □

4 Investigation of a non homogeneous problem.

4.1. The case of noninteger α.

Theorem 3 *The problem* (1) - (3) *has a solution for any functions* $f_0(x)$, $f_0'(x)\in L^1(\rho)$, $f_1(x)\in L^1(\rho)$. *General solution can be represented as*

$$u(z)=u_0(z)+u_1(z), \tag{25}$$

where $u_0(z)$ *is the general solution of the homogeneous problem, and* $u_1(z)=\varphi_1(z)+y\psi_1(z)+\overline{\omega_1(z)}$, *where*

$$\varphi_1(z)=\frac{(z+i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_0(t)dt}{(t+i)^n(t-z)} \tag{26}$$

$$\omega_1(z)=\frac{(z-i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{f_0(t)}dt}{(t-i)^n(t-z)} \tag{27}$$

$$\psi_1(z)=\frac{(z+i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_1(t)dt}{(t+i)^n(t-z)}+ \\ +\frac{(z-i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_1(t)dt}{(t-i)^n(t-z)}-i\varphi'(z)-i\omega'(z). \tag{28}$$

Proof. We must prove that the function from (25) is a solution of the problem (1)-(3). Taking into account theorem 2, it is sufficient to show that the function $u_1(z)$ satisfies the statement. For this purpose let's consider the boundary-value problem in class of analytic functions in the upper half-plane.

$$\lim_{y\to+0} \|\varphi(x+iy)+\overline{\omega(x+iy)}-f_0(x)\|_{L^1(\rho)}=0, \quad (29)$$

$$\lim_{y\to+0} \|Re(i\varphi'(x+iy)+\psi(x+iy)+\omega'(x+iy))-f_1(x)\|_{L^1(\rho)}=0. \quad (30)$$

Let $\omega^-(z)=-\overline{\omega(\bar z)}$. From (29) we have

$$\lim_{y\to+0} \|\varphi(x+iy)-\omega^-(x-iy)-f_0(x)\|_{L^1(\rho)}=0.$$

This problem by [8] has a solution for any function $f_0(x)\in L^1(\rho)$, and the solution is represented in the form

$$\varphi(z)=\frac{(z+i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_0(t)dt}{(t+i)^n(t-z)}+P_0(z), \quad (31)$$

$$\omega(z)=\frac{(z-i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{f_0(t)}dt}{(t-i)^n(t-z)}-\overline{P_0}(z), \quad (32)$$

where $P_0(z)$ is a purely imaginary polynomial of order n. Denoting $\Psi^+(z)=i\varphi'(z)+\psi(z)+i\omega'(z), z\in\Pi^+$, $\Psi^-(z)=-\overline{\Psi(\bar z)}, z\in\Pi^-$, we get from (30)

$$\lim_{y\to+0} \|\Psi(x+iy)-\Psi^-(x-iy)-2f_1(x)\|_{L^1(\rho)}=0.$$

Using again the results of paper [8] we get

$$\Psi(z)=\frac{(z+i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_1(t)dt}{(t+i)^n(t-z)}+ \\ +\frac{(z-i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_1(t)dt}{(t-i)^n(t-z)}+P_1(z), \quad (33)$$

where the coefficients of the polynomial $P_1(z)$ are purely imaginary numbers. The functions $\varphi(z)$, $\omega(z)$, from (31) (32), and the function

$$\psi(z)=\frac{(z+i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_1(t)dt}{(t+i)^n(t-z)}+ \\ +\frac{(z-i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_1(t)dt}{(t-i)^n(t-z)}-i\varphi'(z)-i\omega'(z)+P_1(z)$$

are the solution of the problem (29), (30). By lemmas 3 and 4

$$\lim_{y\to+0} y\|\psi(x+iy)\|_{L^1(\rho)}=0, \quad (34)$$

$$\lim_{y\to+0} y\|\psi'(x+iy)\|_{L^1(\rho)}=0, \quad (35)$$

the functions (31), (32), and (33) are the solution of the problem (16), (17), implying that $u_1(z)$ satisfies conditions (2), (3). The theorem is proved. □

4.2. The case, if α is integer.

Lemma 5 *Let $\alpha = n$ be a natural number, $f_1(x)$ be a finite function, then $u(z) = \varphi_1(z) + y\widetilde{\psi_1}(z) + \overline{\omega_1(z)} + u_0(z)$ is the solution of the problem (1)-(3). Here $u_0(z)$ is the solution of the homogeneous problem, $\varphi_1(z)$, $\omega_1(z)$ are determined by formulas (26), (27), and $\widetilde{\psi_1}(z) = \psi_1(z) + A_0(z-\bar{z})z^{n-1}$, where $\psi_1(z)$ is determined by formula (28), and*

$$A_0 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} f_1(t)\left(\frac{1}{(t+i)^n} + \frac{1}{(t-i)^n}\right)dt. \tag{36}$$

Proof. Assume that $A_0 = 0$. We prove that in this case the function $u_1(z) = \varphi_1(z) + y\psi_1(z) + \overline{\omega_1(z)}$, where $\varphi_1(z)$, $\omega_1(z)$, $\psi_1(z)$ are determined by formulas (26), (27), (28), satisfies (2) and (3). Since

$$\lim_{y\to+0} \|\varphi(x+iy) + \overline{\omega(x+iy)} - f_0(x)\|_{L^1(\rho)} = 0,$$

it is sufficient to prove that

$$\lim_{y\to+0} y\|\psi_1(x+iy)\|_{L^1(\rho)} = 0. \tag{37}$$

Taking into account equalities

$$\lim_{y\to+0} y\|\varphi_1'(x+iy)\|_{L^1(\rho)} = \lim_{y\to+0} y\|\omega_1'(x+iy)\|_{L^1(\rho)} = 0,$$

we prove

$$\lim_{y\to+0} y\|\psi_2(x+iy)\|_{L^1(\rho)} = 0, \tag{38}$$

where

$$\psi_2(z) = \frac{(z+i)^n}{2\pi i}\int_{-\infty}^{\infty} \frac{f_1(t)dt}{(t+i)^n(t-z)} + \frac{(z-i)^n}{2\pi i}\int_{-\infty}^{\infty} \frac{f_1(t)dt}{(t-i)^n(t-z)}. \tag{39}$$

If $f_1(x)$ is a finite function, then the function $\psi_2(z)$ is analytic out of segment $[-A; A]$, $A > 0$ and the Laurent series of this function in the vicinity of an infinity has the form

$$\psi_2(x+iy) = A_0 z^{n-1} + \ldots,$$

where A_0 is determined by formula (36). The assumption $A_0 = 0$ implies

$$\psi_2(x+iy) = A_1(x+iy)^{n-2} + A_2(x+iy)^{n-3} + \ldots,$$

and $|\psi_2(x+iy)| \le C_0|x|^{n-2}$, $y \in [0;1]$, $|x| > 2A$, for some constant $C_0 > 0$. Therefore

$$\lim_{y\to+0} y\int_{|x|>2A} |\psi_2(x+iy)|\rho(x)dx = 0.$$

Taking into account equality

$$\lim_{y\to+0} y\int_{|x|<2A} |\psi_2(x+iy)|\rho(x)dx = 0,$$

we get (38) (if $A_0 = 0$). Now let $A_0 \neq 0$. Let's prove that the solution of problem (1)-(3) is the function

$$u_1(z) = \varphi_1(z) + y\psi_1(z) + \overline{\omega_1(z)} + \frac{A_0 i}{2} v_{n-1}(z), \tag{40}$$

where $v_{n-1}(z) = (z - \bar{z})z^{n-1}$. As

$$y\psi_2(z) + \frac{A_0 i}{2} v_{n-1}(z) = y\left(\psi_2(z) - A_0 z^{n-1}\right),$$

and in the vicinity of a infinity $|\psi_2(z) - A_0 z^{n-1}| = O\left(|z|^{n-2}\right)$, now in a similar way

$$\lim_{y \to +0} \|y\psi_2(x+iy) + \frac{A_0 i}{2} v_{n-1}(x+iy)\|_{L^1(\rho)} = 0.$$

The lemma is proved. □

Theorem 4 *Let α be a natural number that is more than 1. In this case the problem (1)-(3) has a solution for any functions $f_0(x)$, $f_0'(x) \in L^1(\rho)$, $f_1(x) \in L^1(\rho)$. The general solution may be represented in the form*

$$u(z) = \varphi_1(z) + y\widetilde{\psi_1}(z) + \overline{\omega_1(z)} + u_0(z)$$

Here $u_0(z)$ is the general solution of the homogeneous problem, $\varphi_1(z)$, $\omega(z)$ are determined by formulas (26), (27), and $\widetilde{\psi_1}(z) = \psi_1(z) + A_0(z - \bar{z})z^{n-1}$, where $\psi_1(z)$ is determined by formula (28), and A_0 by formula (36).

Proof. Let $A_0 = 0$. Let's prove that function

$$u_1(z) = \varphi_1(z) + y\psi_1(z) + \overline{\omega_1(z)},$$

where $\varphi_1(z)$, $\omega_1(z)$, $\psi_1(z)$ are determined by formulas (26), (27), (28), satisfies conditions (2) and (3). We have

$$\lim_{y \to +0} \|\varphi(x+iy) + \overline{\omega(x+iy)} - f_0(x)\|_{L^1(\rho)} = 0,$$

therefore it is sufficient to prove that

$$\lim_{y \to +0} y\|\psi_1(x+iy)\|_{L^1(\rho)} = 0. \tag{41}$$

Taking into account

$$\lim_{y \to +0} y\|\varphi_1'(x+iy)\|_{L^1(\rho)} = \lim_{y \to +0} y\|\omega_1'(x+iy)\|_{L^1(\rho)} = 0,$$

(41) can be replaced with

$$\lim_{y \to +0} y\|\psi_2(x+iy)\|_{L^1(\rho)} = 0, \tag{42}$$

where

$$\psi_2(z) = \Psi_2(f_1 : z) =$$

$$= \frac{(z+i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_1(t)dt}{(t+i)^n(t-z)} + \frac{(z-i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_1(t)dt}{(t-i)^n(t-z)}.$$

Now

$$\Psi_2(f_1 : z) = I_1(f_1 : z) + I_2(f_1 : z),$$

where

$$I_1(f_1 : z) = \frac{(z+i)^n-(z-i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_1(t)dt}{(t-i)^n(t-z)},$$

$$I_2(f_1 : z) = \frac{(z+i)^n}{2\pi i}\int_{-\infty}^{\infty} f_1(t)\left(\frac{1}{(t+i)^n}+\frac{1}{(t-i)^n}\right)\frac{dt}{t-z}.$$

Let's estimate the norm $\|I_1(f_1 : z)\|_{L^1(\rho)}$.

$$y\int_{-\infty}^{\infty}|I_1(f_1 : x+iy)|\rho(x)dx <$$

$$< Cy\int_{-\infty}^{\infty}\frac{1}{|x+i|}\left|\int_{-\infty}^{\infty}\frac{f_1(ft)}{(t-i)^n}\frac{dt}{t-x-iy}\right|dx \le$$

$$C\int_{-\infty}^{\infty}\frac{f_1(t)}{|t+i|^n}\int_{-\infty}^{\infty}\frac{ydx}{|t-x-iy||x+i|}dt < C\|f_1\|_{L^1(\rho)}. \quad (43)$$

Integrating by parts we get

$$I_2(f_1 : z) = \frac{(fz+i)^n}{2\pi i}\int_{-\infty}^{\infty}\widetilde{f_1}(t)\frac{dt}{(t-z)^2},$$

where

$$\widetilde{f_1}(t) = \int_{-\infty}^{t} f_1(\tau)\left(\frac{1}{(\tau+i)^n}+\frac{1}{(\tau-i)^n}\right)d\tau.$$

The function $\widetilde{f_1}(t) \in L_1(\rho)$ if $n = 2, 3, \ldots$ and by lemma 5

$$\lim_{y\to+0} y\|I_2(f_1 : x+iy)\|_{L^1(\rho)} = 0. \quad (44)$$

Now let $f_{1\varepsilon}(x) \in C^{\delta}(-\infty; +\infty)$ be a finite function, such that

$$\|f_{1\varepsilon}(x) - f_1(x)\|_{L^1(\rho)} < \varepsilon, \quad (45)$$

$$\int_{-\infty}^{\infty} f_{1\varepsilon}(t)\left(\frac{1}{(t+i)^n}+\frac{1}{(t-i)^n}\right)dt = 0.$$

Denoting

$$\Psi_{2\varepsilon}(f_{1\varepsilon} : z) = \frac{(z+i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_{1\varepsilon}(t)dt}{(t+i)^n(t-z)} +$$

$$+\frac{(z-i)^n}{2\pi i}\int_{-\infty}^{\infty}\frac{f_{1\varepsilon}(t)dt}{(t-i)^n(t-z)}$$

we get

$$\|\Psi_2(f_1 : x+iy)\|_{L^1(\rho)} \le y\|\Psi_2(f_1 : x+iy) - \Psi_{2\varepsilon}(f_{1\varepsilon} : x+iy)\|_{L^1(\rho)} +$$

$$+y\|\Psi_{2\varepsilon}(f_{1\varepsilon} : x+iy)\|_{L^1(\rho)} \le y\|I_1(f_1 - f_{1\varepsilon} : x+iy)\|_{L^1(\rho)} +$$

$$+y\|I_2(f_1 - f_{1\varepsilon} : x+iy)\|_{L^1(\rho)} + y\|\Psi_{2\varepsilon}(f_{1\varepsilon} : x+iy)\|_{L^1(\rho)} \leq$$

$$\leq C\|f_1 - f_{1\varepsilon}\|_{L^1(\rho)} + y\|I_2(f_1 - f_{1\varepsilon} : x+iy)\|_{L^1(\rho)} + y\|\Psi_{2\varepsilon}(f_{1\varepsilon} : x+iy)\|_{L^1(\rho)}.$$

Since

$$\lim_{y\to+0} y\|I_2(f_1 - f_{1\varepsilon} : x+iy)\|_{L^1(\rho)} = 0,$$

and from lemma 5

$$\lim_{y\to+0} y\|\Psi_{2\varepsilon}(f_{1\varepsilon} : x+iy)\|_{L^1(\rho)} = 0,$$

and taking into account (43) and (44), we get (41). Now let $A_0 \neq 0$. Let's prove that the function

$$u_1(z) = \varphi_1(z) + y\psi_1(z) + \overline{\omega_1(z)} + \frac{A_0 i}{2} v_{n-1}(z),$$

where $v_{n-1}(z) = (z-\bar{z})z^{n-1}$ is the solution of the problem (1)-(3). For this purpose let's choose the finite function $f_{1\varepsilon}(x)$ so that the following equality holds

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}(f_1(t) - f_{1\varepsilon}(t))\left(\frac{1}{(t+i)^n} + \frac{1}{(t-i)^n}\right)dt = 0. \qquad (46)$$

If $f_1(x)$ substituted by $f_1(x) - f_{1\varepsilon}(x)$ in statement (3), the problem (1)-(3) will have the solution $u_1(z) = \varphi_1(z) + y\psi_1(z) + \overline{\omega_1(z)}$, where $\varphi_1(z)$, $\omega_1(z)$ are determined by formulas (26), and (27), and $\psi_1(z) = \psi_1(f_1 - f_{1\varepsilon} : z)$. For the function $f_{1\varepsilon}(x)$ the problem (1)-(3), according to lemma 5, has a solution, hence the problem (1)-(3) has a solution for the function $f_1(x)$, and this solution may be represented in the from mentioned in the theorem. The theorem is proved. □

Theorem 5 *Let* $\alpha = 0, 1$. *If the function* $f_1(x)$ *is such that* $\widetilde{f}_1(x) \in L^1(\rho)$, *then the problem* (1)-(3) *is uniquely solvable. The solution can be represented by* (25), *where* $P_0(z) \equiv P_1(z) \equiv 0$.

Proof. Using notation of Theorem th4 and taking into account that $\widetilde{f}_1(t) \in L^1(\rho)$ we get

$$\lim_{y\to+0} y\|I_2(f_1 : x+iy)\|_{L^1(\rho)} = 0$$

and therefore

$$\lim_{y\to+0} y\|\psi_1(f_1 : x+iy)\|_{L^1(\rho)} = 0.$$

The theorem is proved. □

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