

# An atomic decomposition of variable Besov and Triebel-Lizorkin spaces

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## Abstract

An atomic decomposition of variable Besov and Triebel-Lizorkin spaces is given.

*Key Words:* variable exponent, Besov space, Triebel-Lizorkin space, atomic decomposition, Maximal operator

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## Introduction

Let  $p$  be a measurable function on  $\mathbb{R}^n$  with range in  $[1, \infty)$ .  $L^{p(\cdot)}(\mathbb{R}^n)$  denotes the set of all measurable functions  $f$  on  $\mathbb{R}^n$  such that for some  $\lambda > 0$ ,

$$\int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

The set becomes a Banach function space when equipped with the norm

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

These spaces are referred to as variable Lebesgue spaces, since they generalized the standard Lebesgue spaces. Note that one can define variable Lebesgue spaces on any measurable subset of  $\mathbb{R}^n$ , see [16]. However, in this paper we only work on the whole space  $\mathbb{R}^n$ .

Denote  $\mathcal{P}(\mathbb{R}^n)$  the set of all measurable functions  $p$  on  $\mathbb{R}^n$  with range in  $[1, \infty)$  such that

$$1 < p_- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x), \quad \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x) = p_+ < \infty.$$

In the classical Lebesgue spaces one can work with  $L^p$  where  $0 < p < 1$ . In this paper, we also consider analogous spaces with variable exponents. Define  $\mathcal{P}^0(\mathbb{R}^n)$  to be the set of all measurable functions  $p$  on  $\mathbb{R}^n$  with range in  $(0, \infty)$  such that

$$p_- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x) > 0, \quad \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x) = p_+ < \infty.$$

Given  $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ , one can define the space  $L^{p(\cdot)}(\mathbb{R}^n)$  as above. This is equivalent to defining it to be the set of all functions  $f$  such that  $|f|^{p_0} \in L^{q(\cdot)}(\mathbb{R}^n)$ , where  $0 < p_0 < p_-$ , and  $q(x) = \frac{p(x)}{p_0} \in \mathcal{P}(\mathbb{R}^n)$ . Then one can define a quasi-norm on this space by

$$\|f\|_{L^{p(\cdot)}} = \||f|^{p_0}\|_{L^{q(\cdot)}}^{1/p_0}.$$

Let  $f \in L^1_{loc}(\mathbb{R}^n)$ . Then the standard Hardy-Littlewood maximal function of  $f$  was defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where  $B$  denotes balls in  $\mathbb{R}^n$ , and  $|B|$  is the volume of ball  $B$ .

It is well known that the boundedness of the Hardy-Littlewood maximal operator on Lebesgue spaces plays a key role in classical analysis. So does it on variable exponent Lebesgue spaces. Let  $\mathcal{B}(\mathbb{R}^n)$  be the set of all  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  such that the Hardy-Littlewood maximal operator  $M$  is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ . There are some sufficient conditions on  $p(\cdot)$  for  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ , see examples [5, 6, 8, 17, 18]. If the maximal operator  $M$  is bounded on space  $L^{p(\cdot)}(\mathbb{R}^n)$ , then many classical operators such as singular integrals and commutators are also bounded on the same space  $L^{p(\cdot)}(\mathbb{R}^n)$ , see [4] and references therein.

In recent decades, many attention has payed to the study of variable Lebesgue spaces, the corresponding variable Sobolev spaces  $W^{k,p(\cdot)}(\mathbb{R}^n)$  and variable Bessel potential spaces  $L^{s,p(\cdot)}(\mathbb{R}^n)$  with  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ . In fact, likewise to the classical situation, variable Bessel potential spaces  $L^{m,p(\cdot)}(\mathbb{R}^n)$  coincide to variable Sobolev spaces  $W^{m,p(\cdot)}(\mathbb{R}^n)$  when  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$  and  $m$  is any integer. This result was obtained firstly by Almeida and Samko in [3] and again by Gurka, Harjulehto and Nekvinda in [12]. Moreover, these spaces have been applied to partial differential equations and the calculus of variation, see [1, 3, 4, 5, 6, 7, 8, 10, 11, 12, 16, 17, 18, 19, 21, 22, 29] and references therein.

It is well known that Besov and Triebel-Lizorkin spaces include many classical spaces as special cases, for example, the Hölder spaces, the Sobolev spaces, the Bessel potential spaces, the Zygmund spaces, the local Hardy spaces and the space  $\operatorname{bmo}(\mathbb{R}^n)$ . All the above mentioned spaces have been studied intensively and applied in many fields of analysis, such as ordinary and partial differential equations; see for examples, [2, 20, 23, 24, 25, 26].

Inspired by the mentioned references, similar to classical Besov and Triebel-Lizorkin spaces, the author introduced the variable Besov spaces and Triebel-Lizorkin spaces in [27]. In fact, the author obtained a characterization of these new spaces by maximal operator. Then in [28], the author proved that if  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$  then the variable Bessel potential

spaces  $L^{s,p(\cdot)}(\mathbb{R}^n)$  and variable Triebel-Lizorkin space  $F_{p(\cdot),2}^s(\mathbb{R}^n)$  are coincident for  $s \in \mathbb{R}$ , a relationship similar to the classical setting.

It is known that atomic decomposition is an important property for classical function spaces. In this paper, we will consider the atomic decomposition of the variable Besov and Triebel-Lizorkin spaces. Our results will be stated in the next section. At the end of this section we recall the definition of these spaces.

Let  $\mathcal{S}(\mathbb{R}^n)$  be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on  $\mathbb{R}^n$ . Let  $\mathcal{S}'(\mathbb{R}^n)$  be the set of all the tempered distribution on  $\mathbb{R}^n$ . If  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ , then  $\widehat{\varphi}$  denotes the Fourier transform of  $\varphi$ , and  $\varphi^\vee$  denotes the inverse Fourier transform of  $\varphi$ . Let  $\Phi \in \mathcal{S}(\mathbb{R}^n)$  satisfy the following conditions:

$$\text{supp } \widehat{\Phi} \subset B(0, 1), \quad \text{and} \quad \text{supp } \widehat{\Phi} = 1 \text{ on } B(0, 1/2).$$

Set  $\Phi_j(x) = 2^{nj}\Phi(2^jx)$ ,  $x \in \mathbb{R}^n$ , for  $j \in \mathbb{Z}^n$ . We also put

$$\theta_j = \Phi_j(x) - \Phi_{j-1}(x).$$

Denote  $\theta_0 = \Phi$ . It follows that

$$\sum_{j=0}^{\infty} \widehat{\theta}_j(\xi) \equiv 1.$$

**Definition 1** Let  $s \in \mathbb{R}$ ,  $0 < q \leq \infty$ ,  $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ . Suppose  $\theta_j$ ,  $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$  as above, then the variable exponent Triebel-Lizorkin space  $F_{p(\cdot),q}^s(\mathbb{R}^n)$  is the collection of  $f \in S'(\mathbb{R}^n)$  such that  $\|f\|_{F_{p(\cdot),q}^s} < \infty$ , where the norm of  $f$  is

$$\|f\|_{F_{p(\cdot),q}^s} = \left\| \left\{ 2^{sj} \theta_j * f \right\}_0^\infty \right\|_{L^{p(\cdot)}(\ell_q)}.$$

Similarly, the variable exponent Besov spaces  $B_{p(\cdot),q}^s(\mathbb{R}^n)$  is the collection of  $f \in S'(\mathbb{R}^n)$  such that  $\|f\|_{B_{p(\cdot),q}^s} < \infty$ , where the norm of  $f$  in this space is

$$\|f\|_{B_{p(\cdot),q}^s} = \left\| \left\{ 2^{sj} \theta_j * f \right\}_0^\infty \right\|_{\ell_q(L^{p(\cdot)})}.$$

Here  $L^{p(\cdot)}(\ell_q)$  and  $\ell_q(L^{p(\cdot)})$  are the spaces of all sequences  $\{g_j\}$  of measurable functions on  $\mathbb{R}^n$  with finite quasi-norms

$$\|\{g_j\}\|_{L^{p(\cdot)}(\ell_q)} = \left\| \left\| \{g_j\}\|_{\ell_q} \right\|_{L^{p(\cdot)}} = \left\| \left( \sum_{j=1}^{\infty} |g_j(x)|^q \right)^{\frac{1}{q}} dx \right\|_{L^{p(\cdot)}},$$

and

$$\|\{g_j\}\|_{\ell_q(L^{p(\cdot)})} = \left\| \left\| \{g_j\}\|_{L^{p(\cdot)}} \right\|_{\ell_q} = \left( \sum_{j=1}^{\infty} \|g_j\|_{L^{p(\cdot)}}^q \right)^{\frac{1}{q}}.$$

Throughout, the letter  $C$  denotes positive constants, but it may change line to line. Constants may in general depend on all fixed parameters, and sometimes we show this dependence explicitly by writing, e.g.  $C_N$ .

# 1. Atomic decomposition

Set

$$x_{jk} = 2^{-j}k \text{ for } j \in \mathbb{N}_0 \text{ and } k \in \mathbb{Z}^n,$$

then we define dyadic cubes  $Q_{jk}$  by

$$Q_{jk} = x_{jk} + 2^{-j}[0, 1)^n.$$

The characteristic function of  $Q_{jk}$  is denoted by  $\chi_{jk}$ .

For any cube  $Q$ , its side is denoted by  $l(Q)$ , and for  $\lambda > 0$ ,  $\lambda Q$  denotes the cube concentric to  $Q$  with side  $\lambda l(Q)$ . Denote  $3Q_{jk}$  by  $\tilde{Q}_{jk}$ , and  $\tilde{\chi}_{jk}$  denotes the characteristic function for  $\tilde{Q}_{jk}$ .

**Definition 2** *Let  $S, T$  be a nonnegative integer. A function  $a_{jk}$  in  $C^S(\mathbb{R}^n)$  is called a  $(S, T)$  atom for a cube  $Q_{jk}$  if it satisfies the following conditions:*

$$\begin{aligned} \text{supp } a_{jk} &\subset \tilde{Q}_{jk}; \\ \sup_x 2^{-j|\gamma|} |D^\gamma a_{jk}(x)| &\leq 1 \text{ for all } |\gamma| \leq S; \\ \text{if } j > 0, \int_{\mathbb{R}^n} x^\gamma a_{jk}(x) dx &= 0 \text{ for all } |\gamma| \leq T. \end{aligned}$$

Now we can state our first result.

**Theorem 1** *Let  $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$  with  $0 < p_0 < p_-$  such that  $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$ .*

(i) *Let  $f$  belong to  $B_{p(\cdot), q}^s$ , then there exist  $(S, T)$  atoms  $\{a_{jk}\}$  for the dyadic  $\{Q_{jk}\}$ , and coefficients  $\{t_{jk}\}$  such that*

$$f = \sum_{j=0}^{\infty} 2^{-js} u_j \text{ in } \mathcal{S}' \text{ with } u_j = \sum_{k \in \mathbb{Z}^n} t_{jk} a_{jk},$$

and denoting  $t_j = \sum_{k \in \mathbb{Z}^n} t_{jk} \chi_{jk}$ , there is a constant  $C$  independent of  $f$  such that

$$\|\{t_j\}_0^\infty\|_{\ell^q(L^{p(\cdot)})} = \left( \sum_{j=0}^{\infty} \left\| \sum_{k \in \mathbb{Z}^n} |t_{jk} \chi_{jk}| \right\|_{L^{p(\cdot)}}^q \right)^{1/q} \leq C \|f\|_{B_{p(\cdot), q}^s}. \quad (1)$$

(ii) *Let  $f$  belong to  $F_{p(\cdot), q}^s$ , then there exist  $(S, T)$  atoms  $\{a_{jk}\}$  for the dyadic  $\{Q_{jk}\}$ , and coefficients  $\{t_{jk}\}$  such that*

$$f = \sum_{j=0}^{\infty} 2^{-js} u_j \text{ in } \mathcal{S}' \text{ with } u_j = \sum_{k \in \mathbb{Z}^n} t_{jk} a_{jk}, \quad (2)$$

and denoting  $t_j = \sum_{k \in \mathbb{Z}^n} t_{jk} \chi_{jk}$ , there is a constant  $C$  independent of  $f$  such that

$$\|\{t_j\}_0^\infty\|_{L^{p(\cdot)}(\ell^q)} = \left\| \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} |t_{jk} \chi_{jk}|^q \right)^{1/q} \right\|_{L^{p(\cdot)}} \leq C \|f\|_{F_{p(\cdot), q}^s}.$$

To prove Theorem 1, we follow the method used in the classical setting in [13, 14, 15]. Firstly, we need a preliminary.

The following lemma is the estimate for vector-valued setting in variable Lebesgue spaces, one can see Corollary 2.1 in [4].

**Lemma 1** *If  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ , then for all  $1 < q \leq \infty$ ,*

$$\|\{Mf_j\}\|_{L^{p(\cdot)}(\ell_q)} \leq C\|\{f_j\}\|_{L^{p(\cdot)}(\ell_q)},$$

where  $M$  is the Hardy-Littlewood maximal operator.

For convenience, we use the notation  $M_r(g) = (M(|g|^r))^{1/r}$ .

**Proof of Theorem 1.** From the notation in last section, we have that

$$f = \sum_{j=0}^{\infty} \theta_j * f = \sum_{j=0}^{\infty} 2^{-js} u_j,$$

where  $u_j = 2^{js} \theta_j * f$  for  $j \in \mathbb{N}_0$ .

Let  $\eta \in C_0^\infty(Q_{00})$ ,  $\int \eta(x) dx = 1$ ,  $\eta_j(x) = 2^{jn} \eta(2^j x)$ . Set  $\eta_{jk} = \eta_j * \chi_{jk}$ , so that  $\text{supp } \eta_{jk} \subset \tilde{Q}_{jk}$  and  $\sum_{k \in \mathbb{Z}^n} \eta_{jk} \equiv 1$ .

Set  $b_{jk} = \eta_{jk} u_j$ ,

$$t_{jk} = \max_{x \in \tilde{Q}_{jk}, |\gamma| \leq S} 2^{-n|\gamma|} |D^\gamma b_{jk}(x)|,$$

and

$$a_{jk} = \frac{b_{jk}}{t_{jk}},$$

of course if  $t_{jk} = 0$ , then  $a_{jk} = 0$ .

Then

$$f = \sum_{j=0}^{\infty} 2^{-js} \sum_{k \in \mathbb{Z}^n} t_{jk} a_{jk}.$$

Observe that  $\widehat{\Phi}_{j+1}(\xi) = 1$  on  $\text{supp } \widehat{u}_j$ , so that  $u_j = \Phi_{j+1} * u_j$ . Since for  $x \in \tilde{Q}_{jk}$  and  $z \in Q_{jk}$ , there exist constants  $C_1, C_2$  independent of  $j$  such that  $0 < C_1 \leq \frac{1+2^j|x-y|}{1+2^j|z-y|} \leq C_2$

for  $y \in \mathbb{R}^n$ , then for  $|\gamma| \leq S$  and  $z \in Q_{jk}$ , we have

$$\begin{aligned}
|D^\gamma u_j(x)| &\leq |D^\gamma \Phi_{j+1}| * |u_j|(x) \\
&= \int_{\mathbb{R}^n} |D^\gamma \Phi_{j+1}(x-y)| |u_j(y)| dy \\
&\leq C 2^{j|\gamma|} 2^{jn} \int_{\mathbb{R}^n} \frac{|u_j(y)|}{(1+2^j|x-y|)^L} dy \\
&\leq C 2^{j|\gamma|} 2^{jn} \int_{\mathbb{R}^n} \frac{|u_j(y)|}{(1+2^j|z-y|)^L} dy \\
&\leq C 2^{j|\gamma|} 2^{jn} \int_{\mathbb{R}^n} \frac{|u_j(y)|}{(1+2^j|z-y|)^{n/r}} \frac{1}{(1+2^j|z-y|)^{L-n/r}} dy \\
&\leq C 2^{j|\gamma|} M_r(u_j)(z) \int_{\mathbb{R}^n} \frac{2^{jn}}{(1+2^j|z-y|)^{L-n/r}} dy \\
&= C 2^{j|\gamma|} M_r(u_j)(z) \int_{\mathbb{R}^n} \frac{1}{(1+|y|)^{L-n/r}} dy \\
&\leq C 2^{j|\gamma|} M_r(u_j)(z) \int_0^\infty \frac{1}{(1+t)^{L-n/r-n+1}} dt \\
&\leq C 2^{j|\gamma|} M_r(u_j)(z) \\
&\leq C 2^{j|\gamma|} \inf_{z \in Q_{jk}} M_r(u_j)(z),
\end{aligned}$$

where we supposed  $L > n + n/r$ , and we used

$$\sup_{y \in \mathbb{R}^n} \frac{|u_j(y)|}{(1+2^j|z-y|)^{n/r}} \leq C M_r(u_j)(z) \quad (3)$$

for  $u_j \in \mathcal{S}'(\mathbb{R}^n)$  such that  $\text{supp } \hat{u}_j \subset B(0, 2^j)$ . We leave the proof of (3) to the end of the proof.

By the Leibniz's formula

$$t_{jk} \leq C \max_{x \in \bar{Q}_{jk}, |\gamma| \leq S} 2^{-j|\gamma|} |D^\gamma u_j(x)|.$$

Therefore  $t_{jk} \leq C M_r(u_j)(x)$  for all  $x \in Q_{jk}$ , and thus

$$t_j(x) = \sum_{k \in \mathbb{Z}^n} t_{jk} \chi_{jk} \leq C M_r(u_j)(x).$$

Now the inequalities (1) and (2) follow by the boundedness of the Hardy-Littlewood operator and its vector-valued operator, Lemma 1. Let us check (2). Choose  $r$  such that  $r < \min\{q, p_0\}$  (for (1), only  $0 < r < p_0$ ), by Theorem 8.1 in [8],  $p(\cdot)/r \in \mathcal{B}(\mathbb{R}^n)$ . Hence, by

Lemma 1 we have

$$\begin{aligned}
\|\{t_j\}_0^\infty\|_{L^{p(\cdot)}(\ell^q)} &= \left\| \left( \sum_{j=0}^\infty \sum_{k \in \mathbb{Z}^n} |t_{jk} \chi_{jk}|^q \right)^{1/q} \right\|_{L^{p(\cdot)}} \\
&\leq C \left\| \left( \sum_{j=0}^\infty |M_r u_j|^q \right)^{1/q} \right\|_{L^{p(\cdot)}} \\
&= C \left\| \left( \sum_{j=0}^\infty |M|u_j|^r|^{q/r} \right)^{r/q} \right\|_{L^{p(\cdot)/r}}^{1/r} \\
&\leq C \left\| \left( \sum_{j=0}^\infty |u_j|^q \right)^{1/q} \right\|_{L^{p(\cdot)}} \\
&= C \|f\|_{F_{p(\cdot),q}^s}.
\end{aligned}$$

Finally, we show (3) is true. From Theorem 1.3.1 on page 16 in [23], we know that for  $0 < r < \infty$

$$\sup_{y \in \mathbb{R}^n} \frac{|\phi(x-y)|}{(1+|y|)^{n/r}} \leq CM_r(\phi)(x)$$

holds for all  $x \in \mathbb{R}^n$  and  $\phi \in \mathcal{S}$  such that  $\text{supp } \widehat{\phi} \subset B(0,1)$ . Then with the argument in Step 1 of the proof of Theorem 1.4.1 on page 22 in [23],

$$\sup_{y \in \mathbb{R}^n} \frac{|\phi(x-y)|}{(1+|y|)^{n/r}} \leq CM_r(\phi)(x)$$

holds for all  $x \in \mathbb{R}^n$  and  $\phi \in \mathcal{S}'(\mathbb{R}^n)$  such that  $\text{supp } \widehat{\phi} \subset B(0,1)$ . Thus by scaling and translation,

$$\sup_{y \in \mathbb{R}^n} \frac{|\phi(x-y)|}{(1+2^j|y|)^{n/r}} \leq CM_r(\phi)(x)$$

holds for all  $x \in \mathbb{R}^n$  and  $\phi \in \mathcal{S}'(\mathbb{R}^n)$  such that  $\text{supp } \widehat{\phi} \subset B(0,2^j)$ . In other words,

$$\frac{|\phi(y)|}{(1+2^j|y-x|)^{n/r}} \leq CM_r(\phi)(x)$$

holds for all  $x, y \in \mathbb{R}^n$  and  $\phi \in \mathcal{S}'(\mathbb{R}^n)$  such that  $\text{supp } \widehat{\phi} \subset B(0,2^j)$ . This is (3).

This completes the proof.

Conversely, we have

**Theorem 2** Let  $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$  with  $0 < p_0 < p_-$  such that  $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$ .

(i) Let  $\{a_{mk}\}$  be a sequence  $(S, T)$  atoms for the dyadic  $\{Q_{mk}\}_{m \in \mathbb{N}_0, k \in \mathbb{Z}^n}$ , where  $S > s$ ,  $T > -1 - n - s + n/\min\{p_0, 1\}$ , and let coefficients  $\{t_{mk}\}$  such that

$$\|\{t_m\}_0^\infty\|_{\ell^q(L^{p(\cdot)})} = \left( \sum_{m=0}^\infty \left\| \sum_{k \in \mathbb{Z}^n} |t_{mk} \chi_{mk}|^q \right\|_{L^{p(\cdot)}} \right)^{1/q} < \infty,$$

where  $t_m = \sum_{k \in \mathbb{Z}^n} t_{mk} \chi_{mk}$ . Then the function  $f$  defined in  $\mathcal{S}'(\mathbb{R}^n)$  by

$$f = \sum_{m=0}^{\infty} 2^{-ms} \sum_{k \in \mathbb{Z}^n} t_{mk} a_{mk}$$

belongs to  $B_{p(\cdot),q}^s$ , and there is a constant  $C$  independent of  $f$  such that

$$\|f\|_{B_{p(\cdot),q}^s} \leq C \|\{t_m\}_0^\infty\|_{\ell^q(L^{p(\cdot)})} = C \left( \sum_{m=0}^{\infty} \left\| \sum_{k \in \mathbb{Z}^n} |t_{mk}| \chi_{mk} \right\|_{L^{p(\cdot)}}^q \right)^{1/q}.$$

(ii) Let  $\{a_{mk}\}$  be a sequence  $(S, T)$  atoms for the dyadic  $\{Q_{mk}\}_{m \in \mathbb{N}_0, k \in \mathbb{Z}^n}$ , where  $S > s$ ,  $T > -1 - n - s + n/\min\{p_0, q, 1\}$ , and let coefficients  $\{t_{mk}\}$  such that

$$\|\{t_m\}_0^\infty\|_{L^{p(\cdot)}(\ell^q)} = \left\| \left( \sum_{m=0}^{\infty} \sum_{k \in \mathbb{Z}^n} |t_{mk}|^q \chi_{mk} \right)^{1/q} \right\|_{L^{p(\cdot)}} < \infty,$$

where  $t_m = \sum_{k \in \mathbb{Z}^n} t_{mk} \chi_{mk}$ . Then the function  $f$  defined in  $\mathcal{S}'(\mathbb{R}^n)$  by

$$f = \sum_{m=0}^{\infty} 2^{-ms} \sum_{k \in \mathbb{Z}^n} t_{mk} a_{mk}$$

belongs to  $F_{p(\cdot),q}^s$ , and there is a constant  $C$  independent of  $f$  such that

$$\|f\|_{F_{p(\cdot),q}^s} \leq C \|\{t_m\}_0^\infty\|_{L^{p(\cdot)}(\ell^q)} = C \left\| \left( \sum_{m=0}^{\infty} \sum_{k \in \mathbb{Z}^n} |t_{mk}|^q \chi_{mk} \right)^{1/q} \right\|_{L^{p(\cdot)}}.$$

To prove Theorem 2, we require the following two lemmas, for them, see [13], [14].

**Lemma 2** Let  $\{\theta_j\}_0^\infty$  as in Section 1 and let  $a_{mk}$  be a  $(S, T)$  atom for  $Q_{mk}$ . Then for  $x \in \mathbb{R}^n$

$$|\theta_j * a_{mk}(x)| \leq \frac{C 2^{-(m-j)T}}{(1 + 2^j |x - x_{mk}|)^L} \quad \text{for } 0 \leq j \leq m,$$

and

$$|\theta_j * a_{mk}(x)| \leq \frac{C 2^{-(j-m)S}}{(1 + 2^m |x - x_{mk}|)^L} \quad \text{for } 0 \leq m \leq j,$$

where  $L$  is sufficiently large, for our purpose,  $L > n/\min\{1, p_-, q\}$ .

**Lemma 3** Let  $0 < r \leq 1$ ,  $L > n/r$ , and  $\eta \geq 0$ . For any sequence  $\{t_{jk}\}_{j \in \mathbb{N}_0, k \in \mathbb{Z}^n}$ , there exists a constant  $C$  such that

$$\sum_{j=\mu, k \in \mathbb{Z}^n} \frac{|t_{jk}|}{(1 + 2^{\mu-\eta} |x - x_{jk}|)^L} \leq C 2^{\eta n/r} M_r \left( \sum_{j=\mu, k \in \mathbb{Z}^n} |t_{jk}| \chi_{Q_{jk}} \right) (x), \quad x \in \mathbb{R}^n.$$

Now we show Theorem 2.

**Proof** Let  $f(x) = \sum_{m=0}^{\infty} 2^{-ms} \sum_{k \in \mathbb{Z}^n} t_{mk} a_{mk}(x)$ . By Lemma 2 and 3,

$$\begin{aligned}
2^{js} |\theta_j * f(x)| &\leq \sum_{m=0}^j 2^{(j-m)s} \sum_{k \in \mathbb{Z}^n} |t_{mk}| |\theta_j * a_{mk}| \\
&\quad + \sum_{m=j+1}^{\infty} 2^{(j-m)s} \sum_{k \in \mathbb{Z}^n} |t_{mk}| |\theta_j * a_{mk}| \\
&\leq C \sum_{m=0}^j 2^{-(j-m)(S-s)} \sum_{k \in \mathbb{Z}^n} \frac{|t_{mk}|}{(1 + 2^m |x - x_{mk}|)^L} \\
&\quad + C \sum_{m=j+1}^{\infty} 2^{-(m-j)(T+1+n+s)} \sum_{k \in \mathbb{Z}^n} \frac{|t_{mk}|}{(1 + 2^j |x - x_{mk}|)^L} \\
&\leq C \sum_{m=0}^j 2^{-(j-m)(S-s)} M_r(t_m)(x) \\
&\quad + C \sum_{m=j+1}^{\infty} 2^{-(m-j)(T+1+n+s-n/r)} M_r(t_m)(x),
\end{aligned} \tag{4}$$

where in the last inequality, we used  $\eta = 0$  in Lemma 3 for the first part and  $\eta = m - j$  for the second part.

Thus, by the assumption, if we choose  $0 < r < \min\{p_0, 1\}$  such that  $T+1+n+s-n/r > 0$ , then by Theorem 8.1 in [8],  $p(\cdot)/r \in \mathcal{B}(\mathbb{R}^n)$ . Therefore by Lemma 1 we obtain

$$\begin{aligned}
2^{js} \|\theta_j * f\|_{L^{p(\cdot)}} &\leq C \sum_{m=0}^j 2^{-(j-m)(S-s)} \|M_r(t_m)\|_{L^{p(\cdot)}} \\
&\quad + C \sum_{m=j+1}^{\infty} 2^{-(m-j)(T+1+n+s-n/r)} \|M_r(t_m)\|_{L^{p(\cdot)}} \\
&\leq C \sum_{m=0}^j 2^{-(j-m)(S-s)} \|M(|t_m|^r)\|_{L^{p(\cdot)/r}}^{1/r} \\
&\quad + C \sum_{m=j+1}^{\infty} 2^{-(m-j)(T+1+n+s-n/r)} \|M(|t_m|^r)\|_{L^{p(\cdot)/r}}^{1/r} \\
&\leq C \sum_{m=0}^j 2^{-(j-m)(S-s)} \|t_m\|_{L^{p(\cdot)}} \\
&\quad + C \sum_{m=j+1}^{\infty} 2^{-(m-j)(T+1+n+s-n/r)} \|t_m\|_{L^{p(\cdot)}} \\
&= C \sum_{m=0}^{\infty} C_{j-m} \|t_m\|_{L^{p(\cdot)}},
\end{aligned}$$

and  $\sum_{m=-\infty}^{\infty} C_m < \infty$ . Using Minkowski's inequality  $\|a * b\|_{\ell_q} \leq \|a\|_{\ell_1} \|b\|_{\ell_q}$ , we obtain that

$$\left( \sum_{j=0}^{\infty} \|2^{js} \theta_j * f\|_{L^{p(\cdot)}}^q \right)^{1/q} \leq C \left( \sum_{m=0}^{\infty} \|t_m\|_{L^{p(\cdot)}}^q \right)^{1/q}.$$

This is (i).

For (ii), let us pick  $0 < r < \min\{p_0, q, 1\}$  such that  $T + 1 + n + s - n/r > 0$ , from (4) by Minkowski's inequality again, we have

$$\left( \sum_{j=0}^{\infty} |2^{js} \theta_j * f(x)|^q \right)^{1/q} \leq C \left( \sum_{m=0}^{\infty} |M_r(t_m)(x)|^q \right)^{1/q}.$$

By Theorem 8.1 in [8] again, we have  $p(\cdot)/r \in \mathcal{B}(\mathbb{R}^n)$ . Thus, by Lemma 1 we have

$$\begin{aligned} \|\{2^{js} \theta_j * f\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)} &\leq C \|\{M_r(t_m)\}_{m=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)} \\ &= C \|\{M(|t_m|^r)\}\|_{L^{p(\cdot)/r}(\ell^{q/r})}^{1/r} \\ &\leq C \|\{t_m\}\|_{L^{p(\cdot)}(\ell^q)}. \end{aligned}$$

This finishes the proof.

**Remark** When this manuscript was finished, the author received the preprint [9] by Professor Hästö. In [9], Triebel-Lizorkin spaces with variable smoothness and integrability were introduced, which include the variable Triebel-Lizorkin spaces while  $s > 0$  in this paper as a special case, but do not include those for  $s < 0$ . In fact, atomic and molecule decompositions and applications of these new Triebel-Lizorkin spaces were given, for detail, see [9]. However, the variable Besov spaces in this paper was not considered in [9], thus the author presented these results here.

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