A Note on Holomorphic Spaces Defined with the Help of Luzin Cone and Area Integral

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Abstract

The goal of this note is to introduce a new approach to a known problem of finding estimates for "analytic quazinorms" with Luzin area integral or Luzin cone $\Gamma_\alpha(\xi)$ in them, namely we use new connections between Luzin Area integral and p-Carleson measures and apply classical interpolation theorems of real analysis to get various new inequalities for analytic functions in the unit disk and polydisk.

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1 Introduction and notations

Let $n \in \mathbb{N}$ and $\mathbb{C}^n = \{z = (z_1, ..., z_n) \mid z_k \in \mathbb{C}, 1 \le k \le n\}$ be the n-dimensional space of complex coordinates. We denote the unit polydisk by

$$D^n = \{z \in \mathbb{C}^n : \ |z_k| < 1, 1 \le k \le n\}$$

and the distinguished boundary of D^n by

$$T^n = \{z \in \mathbb{C}^n : \ |z_k| = 1, 1 \le k \le n\}.$$

We use dm_{2n} to denote the volume measure on D^n and dm_n to denote the normalized Lebesgue measure on T^n. If $z \in D^n, \beta = (\beta_1, \cdots, \beta_n)$, we denote $(1 - |z|^2)^\beta$ by $\prod_{k=1}^n (1 - |z_k|^2)^{\beta_k}$. Let $H(D^n)$ be the space of all holomorphic functions on D^n. When $n = 1$, we simply denote D^1 by D, T^1 by T. We note that in polydisk the following fractional derivative is well studied(see, e.g.[5]),

$$(\mathcal{D}^\alpha f)(z) = \sum_{k_1,...,k_n \ge 0} (k_1 + 1)^\alpha \cdots (k_n + 1)^\alpha a_{k_1,...,k_n} z_1^{k_1} z_n^{k_n},$$

where $\alpha \in \mathbb{R}, f \in H(D^n)$. We refer to [12] for further details for the knowledge of polydisk.

Let $\beta(z, w)$ denote the Bergman metric between two points z and w in D. It is well known that

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|},$$

where $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ is the Möbius map of D. For $z \in D$ and $r > 0$ we use

$$D(z, r) = \{w \in D : \beta(z, w) < r\}$$

to denote the Bergman metric ball at z with radius r. If r is fixed, then it can be checked that the area of $D(z, r)$, denoted by $|D(z, r)|$, is comparable to $(1 - |z|^2)^2$ as z approaches the unit circle (see, e.g. [8, 18]). For a subarc $I \in T$, let

$$S(I) = \{r\zeta \in D : 1 - |I| < r < 1, \zeta \in I\}.$$

If $|I| \geq 1$, then we set $S(I) = D$. Let μ denote a positive Borel measure on D. For $0 < p < \infty$, we say that μ is a p-Carleson measure on D if

$$\sup_{I \subset T} \mu(S(I))/|I|^p < \infty.$$

Here and henceforth $\sup_{I \subset T}$ indicates the supremum taken over all subarcs I of T. Note that $p = 1$ gives the classical Carleson measure. The p-Carleson measures in the unit disk was studied intensively recently, see [15, 16] and references there.

Let $L^0(dm)$ be the space of complex-valued measure functions on T. For $s \geq 0$ and $f \in L^0(dm)$, denote the distribution function by

$$\lambda_f(s) = m(\{\xi \in T : |f(\xi)| > s\})$$

and the decreasing rearrangement of $|f|$ by

$$f^*(s) = \inf(\{t \geq 0 : \lambda_f(t) \leq s\}).$$

Let $0 < p, q \leq \infty$ and $f \in L^0(dm)$. Then Lorentz functional $\| \cdot \|_{p,q}$ is defined at f by

$$\|f\|_{p,q} = \left(\int_0^1 [f^*(s)s^{\frac{1}{p}}]^q \frac{ds}{s} \right)^{1/q}$$

and

$$\|f\|_{p,\infty} = \sup_{s \geq 0} f^*(s)s^{\frac{1}{p}}.$$

Let $0 < p, q \leq \infty$. Denote as usual by $L^{p,q}(dm)$ the Lorentz space(see, e.g. [1, 9]), i.e.

$$L^{p,q}(dm) = \left\{ f \in L^0(dm) : \|f\|_{p,q} < \infty \right\}.$$

We denote as usual by H^p the Hardy space(see [6]), where

$$H^p = \left\{ f \in H(D) : \|f\|_{H^p}^p = \sup_{0<r\leq 1} \int_T |f(r\xi)|^p dm(\xi) < \infty \right\}.$$

For $0 < p < \infty$, $\alpha > -1$, the weighted Lebesgue space $L^p(dm_\alpha)$ on D, consists of those measure functions f for which

$$\|f\|_{L^p(dm_\alpha)}^p = \int_D |f(z)|^p (1 - |z|^2)^\alpha dm_2(z) < \infty.$$

We denote by A_α^p the weighted Bergman space, where $A_\alpha^p = L^p(dm_\alpha) \bigcap H(D)$.

The Hardy spaces on the polydisk, denoted by $H^p(D^n)(0 < p \leq \infty)$, are defined by

$$H^p(D^n) = \{f \in H(D^n) : \sup_{0\leq r<1} M_p(f,r) < \infty\},$$

where

$$M_p^p(f,r) = \int_{T^n} |f(r\xi)|^p dm_n(\xi), \ M_\infty(f,r) = \max_{\xi \in T^n} |f(r\xi)|, \ r \in (0,1), \ f \in H(D^n).$$

For $\alpha_j > -1, j = 1, \cdots, n, 0 < p < \infty$, recall that the weighted Bergman space $A_{\vec{\alpha}}^p(D^n)$ consists of all holomorphic functions on the polydisk satisfying the condition

$$\|f\|_{A_{\vec{\alpha}}^p}^p = \int_{D^n} |f(z)|^p \prod_{i=1}^n (1 - |z_i|^2)^{\alpha_i} dm_{2n}(z) < \infty.$$

For $\sigma > 0$, $\xi \in T$, set

$$\Gamma_\sigma(\xi) = \{z \in D : |\xi - z| < (\sigma + 1)(1 - |z|)\}.$$

When $\sigma = 1$, we simply denote $\Gamma_\sigma(\xi)$ by $\Gamma(\xi)$. The well-known maximal theorem of classical Hardy spaces in the unit disk assert(see [8])

$$\| \sup_{z\in\Gamma(\xi)} |f(z)|^p \|_{L^1(dm)} \leq \|f\|_{H^p}^p \tag{1}$$

where $0 < p < \infty$, $L^p(dm)$ is a usual Lebesgue space on T. Various generalizations of this result to the case of unit ball are well-known(see [11, 18]).

The goal of this note to propose a completely new promising approach to such type estimates using connections of so-called Luzin area integral with so called p-Carleson measures for $p < 1$. Let us note that ideas of this note can be developed further.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $C^{-1}B \leq A \leq CB$.

2 Preliminary

In this section, we state some preliminary result which will be needed for the proof of main results. The following lemma can be found in [6] or [10].

Lemma 1. *Let $f \in H(D)$, $0 < q < \infty$, $\xi \in T$, $\alpha > -1$. Then*

$$\int_0^1 |f(\rho\xi)|^q (1-\rho)^\alpha d\rho \leq C \int_{\Gamma_t(\xi)} |f(z)|^q (1-|z|)^{\alpha-1} dm_2(z).$$

From Lemma 1 we easily get the following corollary.

Corollary 1. *Let $f \in H(D)$, $\xi \in T$, $0 < q < \infty$, $\alpha > -1$, $s \in (0, \infty)$. Then*

$$\sup_{I \subset T} \frac{1}{|I|^s} \int_{S(I)} |f(z)|^q (1-|z|)^\alpha dm_2(z)$$

$$\leq C \sup_{I \subset T} \frac{1}{|I|^s} \int_I \int_{\Gamma_t(\xi)} |f(z)|^q (1-|z|)^{\alpha-1} dm_2(z) dm(\xi).$$

From the above corollary and a result in [2], we get the following lemma.

Lemma 2. *Let $f \in H(D)$, $0 < q < \infty$, $\alpha > -1$, $s \in (0, 1)$. Then*

$$\sup_{I \subset T} \frac{1}{|I|^s} \int_I \int_{\Gamma_t(\xi)} |f(z)|^q (1-|z|)^{\alpha-1} dm_2(z) dm(\xi)$$

$$\asymp \sup_{I \subset T} \frac{1}{|I|^s} \int_{S(I)} |f(z)|^q (1-|z|)^\alpha dm_2(z).$$

The following result can be founded in [14]. For the completeness of the exposition we will give the short proof of the following lemma 3 from [14].

Lemma 3. *Let $\sigma_t = (\sigma + 2)e^{2t} - 2$, $\sigma > 0$, $t > 0$, $\xi \in T$. Then*

$$D(z, t) \subset \Gamma_{\sigma_t}(\xi) \quad \text{for all } z \in \Gamma_\sigma(\xi).$$

Proof. From [17] we see that the Bergman metric ball $D(w, t)$ is Euclidean disk on $\mathbb{C}$ with center $P = \frac{1-\tau^2}{1-|w|^2\tau^2} w$ and radius $R = \frac{\tau(1-|w|^2)}{1-|w|^2\tau^2}$, where $\tau = \frac{e^{2t}-1}{e^{2t}+1}$. We must show if $w \in D(z, t)$ and $z \in \Gamma_\sigma(\xi)$, then $w \in \Gamma_{\sigma_t}(\xi)$.

Obviously

$$|\xi - w| \leq |\xi - z| + |z - w| \leq (\sigma + 1)(1 - |w|) + (\sigma + 2)|z - w|. \tag{2}$$

It remains to show that $|z - w| \leq C(1 - |w|)$. Since $z \in D(w, t)$ is the same as $w \in D(z, t)$, we have

$$|z - w| \leq \frac{\tau(1-|w|^2)}{1-|w|^2\tau^2}.$$

From (2) it is not difficult to see

$$|\xi - w| \le ((\sigma + 2)e^{2t} - 1)(1 - |w|).$$

That is exactly what we need.□

Lemma 4. [9] *Let f be measurable in T^n, $0 < q < \infty$. Then*

$$\|f\|_{L^{q,\infty}} \asymp \left(\sup_{\substack{I_1 \subset T, \cdots, \\ I_n \subset T}} \frac{1}{\prod_{k=1}^n |I_k|^{1-\tau/q}} \int_I \cdots \int_I |f(\xi_1, \cdots, \xi_n)|^\tau dm_n(\xi) \right)^{1/\tau}$$

for any $\tau \in (0, q)$.

The following result can be founded in [15].

Lemma 5. *Let $f \in H(D)$, $0 < p < \infty$, $\alpha > p - 1$. Then*

$$\sup_{I \subset T} \left(\frac{1}{|I|} \int_{S(I)} |f'(z)|^p (1 - |z|)^\alpha dm_2(z) \right)^{1/p}$$

$$\asymp \sup_{I \subset T} \left(\frac{1}{|I|} \int_{S(I)} |f(z)|^p (1 - |z|)^{\alpha - p} dm_2(z) \right)^{1/p}.$$

Various interpolation theorems has proved to be an important useful and powerful tool in many applications in complex and real analysis. We will formulate the classical Marcinkiewicz interpolation theorem that will be used by us(see, e.g. [9]).

Lemma 6. *Let (X, μ) and (Y, ν) be two measure spaces. Let $0 < p_0 < p_1 \le \infty$, $0 < q_0, q_1 < \infty$. If T is a sublinear operator defined on the space $L^{p_0}(X)$ and $L^{p_1}(X)$ taking values in Y and*

$$\|Tf\|_{L^{q_0,\infty}(Y)} \le C\|f\|_{L^{p_0}(X)}, \quad f \in L^{p_0}(X)$$

$$\|Tf\|_{L^{q_1,\infty}(Y)} \le C\|f\|_{L^{p_1}(X)}, \quad f \in L^{p_1}(X)$$

and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}; \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}; \quad p \le q, 0 < q_z ero, q_1 \le \infty. \tag{3}$$

Then for all $f \in L^p(X)$,

$$\|Tf\|_{L^q(X)} \le C\|f\|_{L^p(X)}.$$

3 Main results and proofs

Now we formulate the main results of this paper.

Theorem 1 *Let $f \in H(D)$ and $0 < p < \infty$. Then the following statements hold.*
1) If $\tau \in (0, \infty]$, $0 < q < \infty$ and $\alpha > -1$, then

$$\left\| \sup_{w \in \Gamma_\sigma(\xi)} |f(w)|^p (1 - |w|)^{\alpha+2} \right\|_{L^{q,\tau}(dm)}$$

$$\leq C \left\| \int_{\Gamma_{\sigma_t}(\xi)} |f(z)|^p (1 - |z|)^\alpha dm_2(z) \right\|_{L^{q,\tau}(dm)}.$$

2) If $0 < s < 1$ and $\alpha > -1$, then

$$\sup_{I \subset T} \frac{1}{|I|^s} \int_I \sup_{w \in \Gamma_\sigma(\xi)} |f(z)|^p (1 - |z|)^{\alpha+2} dm(\xi)$$

$$\leq C \sup_{I \subset T} \frac{1}{|I|^s} \int_{S(I)} |f(z)|^p (1 - |z|)^{\alpha+1} dm_2(z).$$

Proof. Since $f \in H(D)$, for all $p \in (0, \infty)$, $\alpha > -1$ and $z \in \Gamma_\sigma(\xi)$, we have(see, e.g. [17])

$$|f(z)|^p \leq \frac{C}{(1 - |z|)^{2+\alpha}} \int_{D(z,t)} |f(w)|^p (1 - |w|^2)^\alpha dm_2(w).$$

Since $D(z, t) \subset \Gamma_{\sigma_t}(\xi)$ for all $z \in \Gamma_\sigma(\xi)$, we have

$$\sup_{w \in \Gamma_\sigma(\xi)} |f(w)|^p (1 - |w|^2)^{\alpha+2} \leq C \int_{\Gamma_{\sigma_t}(\xi)} |f(w)|^p (1 - |w|^2)^\alpha dm_2(w).$$

Then the first estimate follows.

Obviously 2) follows from 1)(case $\tau = \infty$) and Lemma 2. The proof is finished. $\square$

Remark 1. Putting formally in 1) $\alpha = -2$ and $q = \tau = 1$, we get a classical maximal theorem for Hardy classes. Let us also note that we actually obtain a maximal theorem (that is (1) type inequality) for Q_p type space(see, e.g. [15, 16]) and holomorphic Hardy-Lorentz classes.

Remark 2. Let $q > 1$. Then according to Lemma 4 from preliminaries

$$\|f\|_{L^{q,\infty}} \asymp \sup_{I \subset T} \frac{1}{|I|^{1-1/q}} \int_I |f(\xi)| dm(\xi).$$

Hence putting in the second estimates of the the theorem $s = 1 - 1/q$, we get

$$\left\| \sup_{w \in \Gamma_\sigma(\xi)} |f(w)|^p (1 - |w|)^{\alpha+2} \right\|_{L^{q,\infty}}$$

$$\leq C \sup_{I \subset T} \frac{1}{|I|^{1-1/q}} \int_{S(I)} |f(z)|^p (1 - |z|)^{\alpha+1} dm_2(z).$$

Let now $q \to \infty$. Then by Lemma 5

$$\sup_{z \in D} |f(z)|(1 - |z|)^{\frac{\alpha+2}{p}} \asymp \sup_{z \in D} |f'(z)|(1 - |z|)^{\frac{\alpha+2}{p}+1}$$

$$\leq C \Big(\sup_{I \subset T} \frac{1}{|I|} \int_{S(I)} |f'(z)|^p (1 - |z|)^{\alpha+1+p} dm_2(z) \Big)^{1/p}.$$

Putting formally in last estimate $p = 2$ and $\alpha = -2$, we get $BMOA \subset \mathcal{B}$(see, e.g.[6, 8]), a classical embedding between two well studied classes. So the second estimate in Theorem 1 can be seen as an extension of classical result $BMOA \subset \mathcal{B}$.

Remark 3. Using more general estimate

$$|f(z)|^p \leq \frac{C}{(1 - |z|)^{2+\alpha-\beta}} \int_{D(z,t)} |\mathcal{D}^\beta f(w)|^p (1 - |w|^2)^\alpha dm_2(w).$$

where $0 < p < \infty, \beta > 0$. We have a more general formulation of theorem 1

$$\Big\| \sup_{z \in \Gamma_\sigma(\xi)} |f(z)|^p (1 - |z|)^{\alpha+2-\beta} \Big\|_{L^{q,\tau}(dm)}$$

$$\leq C \Big\| \int_{\Gamma_{\sigma_t}(\xi)} |\mathcal{D}^\beta f(z)|^p (1 - |z|^2)^\alpha dm_2(z) \Big\|_{L^{q,\tau}(dm)},$$

where $\alpha > \beta - 2$.

The same with part 2) of Theorem 1

$$\sup_{I \subset T} \frac{1}{|I|^s} \int_I \sup_{w \in \Gamma_\sigma(\xi)} |f(z)|^p (1 - |z|)^{\alpha+2-\beta} dm(\xi)$$

$$\leq C \sup_{I \subset T} \frac{1}{|I|^s} \int_{S(I)} |\mathcal{D}^\beta f(z)|^p (1 - |z|)^{\alpha+1} dm_2(z),$$

where $\alpha > \beta - 2$.

Remark 4. Adding the amount of variables and repeating the same arguments we get a polydisk version of Theorem 1.

$$\Big\| \sup_{w_1 \in \Gamma_{\sigma_1}(\xi_1), \cdots, w_n \in \Gamma_{\sigma_n}(\xi_n)} |f(w_1, \cdots, w_n)|^p (1 - |w_1|)^{\alpha_1+2} \cdots (1 - |w_n|)^{\alpha_n+2} \Big\|_{L^{q,\tau}(dm_n)}$$

$$\leq C \Big\| \int_{\Gamma_{\sigma_{t_1}}(\xi_1)} \cdots \cdot \int_{\Gamma_{\sigma_{t_1}}(\xi_1)} |f(z)|^p (1 - |z_1|)^{\alpha_1} \cdots (1 - |z_n|)^{\alpha_n} dm_{2n}(z) \Big\|_{L^{q,\tau}(dm_n)},$$

where $\alpha_j > -2, 0 < p, q < \infty, \tau \in (0, \infty]$.

In the following Theorem we apply the off-diagonal interpolation theorem (Lemma 6) apply for $q = p$. The general case can be applied similarly.

Theorem 2 *1) Let $p \geq 1 + \frac{2}{\min \beta_k}$, $\beta_k > 0$, $k = 1, 2 \cdots, n$, $f \in H(D^n)$. Then*

$$\int_{T^n} \sup_{z_1 \in \Gamma_{t_1}(\xi_1), \cdots z_n \in \Gamma_{t_n}(\xi_n)} |f(z_1, \cdots, z_n)|^p (1 - |z_1|^2)^{\beta_1 p} \cdots (1 - |z_n|^2)^{\beta_n p} dm_n(\xi)$$

$$\leq C \int_{D^n} |f(z_1, \cdots, z_n)|^p (1 - |z_1|^2)^{\beta_1} \cdots (1 - |z_n|^2)^{\beta_n} dm_{2n}(z).$$

2) Let $\tilde{p} > p$, $\gamma_k > p(\frac{\beta_k + 2}{\tilde{p}} - \beta_k)$, $\gamma_k > 0$, $\beta_k > -1$, $f \in H(D^n)$. Then

$$\int_{T^n} \left(\int_{\Gamma_{t_1}(\xi_1)} \cdots \int_{\Gamma_{t_n}(\xi_n)} |f(z)|^p \prod_{k=1}^n (1 - |z|^2)^{\beta_k p + \gamma_k - 2} dm_{2n}(z) \right)^{\tilde{p}/p} dm_n(\xi)$$

$$\leq C \int_{D^n} |f(z)|^{\tilde{p}} (1 - |z|^2)^\beta dm_{2n}(z).$$

Proof. 1) Our intention is to apply the interpolation theorem we formulated in previous section. Let

$$(X, \mu) = (D^n, \prod_{k=1}^n (1 - |z_k|^2)^\beta dm_{2n}(z)); \qquad (Y, \nu) = (T^n, dm_n(\xi)).$$

Then if

$$(T_1 f)(\xi_1, \cdots, \xi_n) = \sup_{z_1 \in \Gamma_{t_1}(\xi_1), \cdots, z_n \in \Gamma_{t_n}(\xi_n)} |f(z)|(1 - |z|^2)^\beta, \quad \beta > 0, \quad f \in H(D^n),$$

we have

$$\|T_1 f\|_{L^\infty(T^n)} \leq C \|f(z)(1 - |z|^2)^\beta\|_{L^\infty(D^n)} \tag{4}$$

So we must prove that for some $p_0 > 1$

$$\|T_1 f\|_{L^{p_0, \infty}(dm_n)} \leq C \|f(z)\|_{L^{p_0}(dm_\beta)}, \tag{5}$$

then we get the desired result by applying Lemma 6.

Using Lemma 4 we have that (5) is equivalent to

$$\left(\sup_{\substack{I_1 \subset T, \cdots, \\ I_n \subset T}} \frac{1}{\prod_{k=1}^n |I_k|^t} \int_I \cdots \int_I \left(\sup_{\substack{z_1 \in \Gamma_{t_1}(\xi_1), \cdots \\ z_n \in \Gamma_{t_n}(\xi_n)}} |f(z)|(1 - |z|^2)^\beta \right)^\tau dm_n(\xi) \right)^{1/\tau}$$

$$\leq C \left(\int_{D^n} |f(z)|^{p_0} (1 - |z|^2)^\beta dm_{2n}(z) \right)^{1/p_0}, \tag{6}$$

where $t = 1 - \tau/p_0$, $0 < \tau < p_0$, $\beta_k > 0$.

Using the first inequality of Theorem 1, it is enough to show that

$$M = \left(\sup_{\substack{I_1 \subset T, \cdots, \\ I_n \subset T}} \frac{1}{\prod_{k=1}^n |I_k|^t} \int_I \cdots \int_I \left(\int_{\Gamma_{t_1}(\xi_1)} \cdots \int_{\Gamma_{t_n}(\xi_n)} |f(z)|(1 - |z|^2)^{\beta-2} dm_{2n}(z) \right)^\tau dm_n(\xi) \right)^{1/\tau}$$

$$\leq C \|f\|_{A_\beta^{p_0}}.$$

Using Lemma 2 we have($\tau_k = \beta_k - 1 - \frac{\beta_k+2}{p_0}$)

$$
\begin{aligned}
M &\leq \||f(z)|(1-|z|^2)^{\frac{\beta+2}{p_0}}\|_{L^\infty(D^n)} \times \prod_{k=1}^{n} \sup_{I_k \subset T} \frac{1}{|I_k|^t} \int_{S(I_k)} (1-|z|^2)^{\tau_k} dm_2(z_k) \\
&\leq C \prod_{k=1}^{n} \sup_{I_k \subset T} \frac{1}{|I_k|^t} \int_{S(I_k)} (1-|z_k|^2)^{\tau_k} dm_2(z_k),
\end{aligned}
$$

where $p_0 \in (0, \infty)$, $\beta_k > 0$. Hence

$$
M \leq C\|f\|_{A_\beta^{p_0}}
$$

which is true if $\beta_k > 0$ and $p_0 > 1 + 2/\min \beta_k, k = 1, \cdots, n$. The first part of Theorem 3 is proved.

2). To apply the Lemma 6 to get the second estimate we introduce the operator

$$
T_{\beta,p}^\gamma f(\xi_1, \cdots, \xi_n) = \left(\int_{\Gamma_{t_1}(\xi_1)} \cdots \int_{\Gamma_{t_n}(\xi_n)} |f(z)|^p (1-|z|^2)^{\beta p + \gamma - 2} dm_{2n}(z) \right)^{1/p},
$$

where $\beta_k p + \gamma_k > 0$, $k = 1, \cdots, n$, $0 < p < \infty$. Using similar argument as for the first part we obtain

$$
\|T_{\beta,p}^\gamma f\|_{L^\infty(T^n)} \leq C\|f(z)(1-|z|^2)^\beta\|_{L^\infty(D^n)}
$$

And it remains to show

$$
\|T_{\beta,p}^\gamma f\|_{L^{p_0,\infty}(T^n)} \leq C\|f\|_{L^{p_0}(D^n)}
$$

for some $p_0 \in (0, \infty)$. The last estimate is equivalent to

$$
\begin{aligned}
&\left(\sup_{\substack{I_1 \subset T, \cdots, \\ I_n \subset T}} \frac{1}{\prod_{k=1}^{n} |I_k|^t} \int_{I_1} \cdots \int_{I_n} \left(\int_{\Gamma_{t_1}(\xi_1)} \cdots \int_{\Gamma_{t_n}(\xi_n)} |f(z)|^p (1-|z|^2)^{\beta p + \gamma - 2} dm_{2n}(z) \right)^{\tau/p} \right. \\
&\leq C\|f\|_{A_\beta^{p_0}}^\tau, \tag{7}
\end{aligned}
$$

where $t = 1 - \tau/p_0$ for some $\tau < p_0$. Let us put in the last estimate $\tau = p$. Then using Lemmas 2 and 3 we have

$$
\begin{aligned}
&\left(\sup_{\substack{I_1 \subset T, \cdots, \\ I_n \subset T}} \frac{1}{\prod_{k=1}^{n} |I_k|^t} \int_{I_1} \cdots \int_{I_n} \left(\int_{\Gamma_{t_1}(\xi_1)} \cdots \int_{\Gamma_{t_n}(\xi_n)} |f(z)|^p (1-|z|^2)^{\beta p + \gamma - 2} dm_{2n}(z) \right)^{\tau/p} \right. \\
&\leq \sup_{z \in D^n} |f(z)| \prod_{i=1}^{n} (1-|z_i|)^{\frac{\beta+2}{p_0}} \prod_{k=1}^{n} \left(\sup_{I_k \subset T} \frac{1}{|I_k|^t} \int_{S(I_k)} (1-|z|)^{\alpha_k} dm_2(z) \right)^{\frac{1}{p}} \\
&\leq C\|f\|_{A_\beta^{p_0}}, \tag{8}
\end{aligned}
$$

where $\alpha_k = \beta_k p + \gamma_k - 1 - \frac{\beta+2}{p_0} p > -1$, which is equivalent to $\gamma_k > p(\frac{\beta_k+2}{p_0} - \beta_k)$, $\beta_k > -1$. The proof is completed.$\square$

Remark 5. Estimates of Theorem 2 are known in disk, see [5, 11].

We give below some new results using the the same technique. It is known that various estimates for so-called $C_{q,\alpha}^s(f)(\xi)$ function

$$C_{q,\alpha}^s(f)(\xi) = \left(\sup_{\xi \in I} \frac{1}{|I|^s} \int_{S(I)} |f(z)|^q (1 - |z|^2)^{\alpha-1} dm_2(z) \right)^{1/q}$$

where $s \in (0, \infty), q \in (0, \infty), \alpha > 0, f \in H(D)$, are played crucial role in complex and Harmonic analysis(see, e.g. [3, 4, 7, 8]).

We apply again the classical interpolation theorem(to be more precise it is vector valued version, see [9]) with Lemmas 3 and 4. Put

$$(X, \mu) = (T, dm); \quad (Y, \nu) = (T, dm).$$

Consider the operator

$$S_s f(\xi) = \sup_{z \in \Gamma_t(\xi)} |\mathcal{D}^k f(z)|(1 - |z|)^{k-s}, k > s.$$

We have(see[11])

$$\|S_s f\|_{L^\infty(dm)} \le C \|C_{q,(k-s)q}^1(\mathcal{D}^k f)\|_{L^\infty(dm)}, 0 < q < \infty, s < k, s \in (-\infty, \infty).$$

To complete the proof we need to show that for some $p_0 > 1$,

$$\|S_s f\|_{L^{p_0,\infty}(dm(\xi))} \le C \|C_{q,k-s}^1(\mathcal{D}^k f)\|_{L^{p_0}(T)}. \tag{9}$$

Using Lemmas 3 and 4,

$$\|S_s f\|_{L^{p_0,\infty}(dm)}$$
$$\le C \sup_{I \subset T} \frac{1}{|I|^{1-1/p_0}} \int_I \left(\int_{\Gamma_t(\xi)} |\mathcal{D}^k f(z)|^q (1 - |z|^2)^{(k-s)q-2} dm_2(z) \right)^{1/q} dm(\xi)$$
$$\le C \left\| \left(\int_{\Gamma_t(\xi)} |\mathcal{D}^k f(z)|^q (1 - |z|^2)^{(k-s)q-2} dm_2(z) \right)^{1/q} \right\|_{L^{p_0}(dm)}, \tag{10}$$

where we used the fact that $\|f\|_{L^{p_0,\infty}} \le C \|f\|_{L^{p_0}}$ for $1 < p_0 < \infty$. Here $p \in (1, \infty), s \in \mathbb{R}, k > s, p_0 \in (1, \infty), q \in (0, \infty)$. Applying theorem 3 of [4] we get (9). Applying vector valued version of Lemma 6(see [9]) we get

$$\|S_s f\|_{L^p} \le C \|C_{q,(k-s)q}^1(D^k f)\|_{L^p}, 1 < p < \infty, 0 < q < \infty. \tag{11}$$

Remark 6. The estimate (11) for $p = \infty$ was given in [11] for the case of unit ball.

Remark 7. Theorem 2 can be extended to more general form with

$$|\mathcal{D}^s f(z)|^p (1 - |z|^2)^{sp}, s > 0 \tag{12}$$

instead of $|f(z)|^p$ in estimates of Theorem 2.

Remark 8. Some other applications of characterizations of p-Carleson measures via Luzin area integral were noticed and announced at the conference by the author(see [13]).

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References

[1] J. Bergh and J. Lofstrom, *Interpolation Spaces*, Springer, 1976.

[2] W. Cohn, Generalized area operators on Hardy spaces, *J. Math. Anal. Appl.* **216** (1) (1997), 112-121.

[3] W. Cohn, Factorization of tent spaces and Hankel operators, *J. Func. Anal.* **175** (2000), 308-329.

[4] R. Coifman, Y. Meyer and E. Stein, Some new function spaces and their use to harmonic analysis, *J. Func. Anal.* **62** (1985), 304-335.

[5] A. Djrbashian and F. Shamoian, *Topics in the Theory of A_α^p Spaces*, Leipzig, Teubner, 1988.

[6] P. Duren, *Theory of H^p Space*, Academic Press, New York, 1970.

[7] E. Dynkin, Methods of the theory of singular integrals(Hilbert transform and Calderon-Zygmund theory), *Itogi Nauki Tekh., Sovr. Probl. Matem., Fund. Napravl.*, **15** (1987), 197-292 [Russian].

[8] J. Garnett, *Bounded Analytic Functions*, Academic Press, New York/London, 1981.

[9] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson/Prentice Hall, New Jersey, 2004.

[10] S. Li and R. Shamoyan, On some properties of a differential operator on the polydisk, *Submitted*, 2008.

[11] J. Ortega and J. Fàbrega, Hardy's inequality and embeddings in holomorphic Triebel-Lizorkin spaces. *Illinois J. Math.* **43** (4) (1999), 733-751.

[12] W. Rudin, *Function Theory in the Polydisk*, Benjamin, New York, 1969.

[13] R. Shamoyan, On some inequalities for holomorphic Hardy-Lorentz spaces in the unit disk and characterizations of BMOA, Confrence abstracts, CMFT 2005, Finland.

[14] Z. Wu, Area operator on Bergman spaces. *Sci. China Ser. A*, **49** (7) (2006), 987-1008.

[15] J. Xiao, *Holomorphic Q Classes*, Lecture Notes in Math. 1767, Springer-Verlag , 2001.

[16] J. Xiao, *Geometric Q_p Functions*, Frontiers in Math. Birkhauser-Verlag, 2006.

[17] K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, Inc. Pure and Applied Mathematics 139, New York and Basel, 1990.

[18] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer-Verlag, New York, 2005.