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Weighted L_P- Norm Inequalities for Various Convolution Type Transformations and their Applications

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Abstract

In this paper we introduce several iterated convolutions and establish weighted $L_p, p > 1, -$ norm inequalities for them by using the Hölder's inequality and the transform of integrals. Applications to wave and heat equations, Klein-Gordon equation are also considered. Especially, we will see their applications to inhomogeneous differential equations.

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1 Introduction

Many boundary value problems and initial value problems in applied mathematics, mathematical physics, and engineering sciences can be effectively solved by the use of the integral transforms. In order to give physical interpretations of the solutions, various convolution type transformations were introduced. The transform method combined with the convolution theorem provides an elegant representation of the solution for many boundary value and initial value problems. Therefore, the estimates for the solutions of many equations usually base on the convolution inequalities. As we know, the Fourier and Laplace convolutions are certainly the best known of the convolution type transformations. We restate the Young's inequality, which is a fundamental tool to estimate the Fourier convolution here.

Let $f \in L_p(\mathbb{R}^n)$, $g \in L_q(\mathbb{R}^n)$, and $p^{-1} + q^{-1} > 1$. The Young's inequality (see [15]) says that the Fourier convolution

$$f * g := \int_{\mathbb{R}^n} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y})\, d\boldsymbol{y}$$

belongs to $L_r(\mathbb{R}^n)$, where $r^{-1} = p^{-1} + q^{-1} - 1$, and moreover,

$$\|f * g\|_r \leq \|f\|_p \|g\|_q. \tag{1}$$

Note that for the typical case of $f, g \in L_2(\mathbb{R}^n)$, the inequality (1) does not hold. In [14], by considering the L_p - norms in weighted spaces, S. Saitoh gave convolution norm inequalities in the form

$$\|f * g\|_{L_p(\mathbb{R}^n, \rho)} \leq \|f\|_{L_p(\mathbb{R}^n, \rho)} \|g\|_{L_p(\mathbb{R}^n, \rho)}. \tag{2}$$

Unlike the Young's inequality, inequality (2) holds also in case $p = 2$.

In recent publications ([5, 11]), the first two authors have given several new type of convolution inequalities in weighted $L_p(\mathbb{R}^n, \rho)$ $(p > 1)$ spaces in the following form

$$\left\| ((F_1\rho_1) * (F_2\rho_2))(\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n)} \leq \|F_1\|_{L_p(\mathbb{R}^n, \rho_1)} \|F_2\|_{L_p(\mathbb{R}^n, \rho_2)}. \tag{3}$$

and derived its applications (see [5, 6, 11]).

Furthermore, in [10] we generalized inequality (3) to the iterated Fourier convolution, namely

$$\left\| \left(\prod_{j=1}^{m} * (F_j \rho_j) \right) \left(\prod_{j=1}^{m} * \rho_j \right)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n)} \leq \prod_{j=1}^{m} \|F_j\|_{L_p(\mathbb{R}^n, \rho_j)}. \tag{4}$$

From the above inequality, we not only derived an important extension of Saitoh's inequality ([13, Theorem 1.2]) but also obtained several inequalities for the Fourier transform which have been successfully implemented in many L_2- weighted integral estimates for the solutions of partial differential equations. As we see from the proof the inequality (4), our basic idea comes from the Hölder's inequality and the transform of integrals. So, for various type iterated convolutions, we can also obtain similar type iterated convolution inequalities. Indeed, several weighted $L_p, p > 1$, norm inequalities in iterated Laplace convolutions ([9]) were derived by a simple and general principle whose L_2 version was given by S. Saitoh ([12]) and obtained by using the theory of reproducing kernels. These inequalities are very convenient for various applications to fractional differential equations in weighted L_p- spaces (see [9] for more details).

For our specific purpose, we will introduce various iterated convolutions inequalities in $L_p(\mathbb{R}^n, \rho)$ spaces.

2 Preliminaries

Throughout this paper, by $\boldsymbol{x} = (x_1, ..., x_n),\ x_j \in \mathbb{R},\ j = 1, 2, ..., n$, we denote a vector in $\mathbb{R}^n$. In particular,

$$\mathbf{1} = (1, 1, ..., 1),\ \mathbf{2} = (2, 2, ..., 2), ... \tag{5}$$

We shall write $\boldsymbol{x} > \boldsymbol{y}$ to denote $x_j > y_j,\ j = 1, 2, ..., n$. Anologously one has to understand $\boldsymbol{x} \geq \boldsymbol{y},\ \boldsymbol{x} < \boldsymbol{y},\ \boldsymbol{x} \leq \boldsymbol{y}$.

We always assume that ρ and $\rho_j,\ j = 1, \cdots, q$ -the weight functions, to be nonnegative, and $1 < p < \infty$. When we write $A \leq B$, we understand that if B is finite, then A is also finite, and bounded above by B.

We shall denote some subsets of $\mathbb{R}^n$

$$\mathbb{R}^n_+(\boldsymbol{t}) = \{\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n, \mathbf{0} < \boldsymbol{x} < \boldsymbol{t}\} \tag{6}$$

$$\mathbb{R}^n_{\boldsymbol{t}} = \{\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{t} < \boldsymbol{x} < \boldsymbol{\infty}\}. \tag{7}$$

Let D be a domain in $\mathbb{R}^n$, $F_i(.) : D \to \mathbb{R},\ i = 1, \cdots, q$, and $\varphi(.,.), \psi(.,.) : D \times D \to D$.

Then, we introduce a convolution type integral, called the $\varphi-$ convolution

Definition 2.1 *([14]) The $\varphi-$ convolution of F_1 and F_2, denoted by $F_1 *_\varphi F_2$, is defined by*

$$(F_1 *_\varphi F_2)(\boldsymbol{\eta}) = \int_D F_1(\boldsymbol{\xi}) F_2(\varphi(\boldsymbol{\xi}, \boldsymbol{\eta})) |\varphi_{\boldsymbol{\eta}}(\boldsymbol{\xi}, \boldsymbol{\eta})| d\boldsymbol{\xi}, \tag{8}$$

when this integral exists. Here,

$$|\varphi_{\boldsymbol{\eta}}(\boldsymbol{\xi}, \boldsymbol{\eta})| := \det \left(\frac{\partial}{\partial \boldsymbol{\eta}} \varphi(\boldsymbol{\xi}, \boldsymbol{\eta}) \right)$$

is the Jacobian of the transformation $\boldsymbol{\eta} \to \varphi(., \boldsymbol{\eta})$.

Example 2.2 *The following convolutions are particular cases of the $\varphi-$convolution ([1], [4]):*

- *The Fourier convolution*

$$F_1 *_{\mathfrak{F}} F_2 := \int_{\mathbb{R}^n} F_1(\boldsymbol{y}) F_2(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{y}, \quad \boldsymbol{x} \in \mathbb{R}^n. \tag{9}$$

- *The Laplace convolution*

$$F_1 *_{\mathfrak{L}} F_2 := \int_{\mathbb{R}^n_+(\boldsymbol{t})} F_1(\boldsymbol{\tau}) F_2(\boldsymbol{t} - \boldsymbol{\tau}) d\boldsymbol{\tau}, \quad \boldsymbol{t} \in \mathbb{R}^n_+. \tag{10}$$

- *The Mellin convolution*

$$F_1 *_{\mathfrak{M}} F_2 := \int_{\mathbb{R}^n_+} F_1(\boldsymbol{x}) F_2(\boldsymbol{t}/\boldsymbol{x}) \boldsymbol{x}^{-1} d\boldsymbol{x}, \quad \boldsymbol{t} \in \mathbb{R}^n_+. \tag{11}$$

Definition 2.3 *The Fourier-Laplace convolution of* $G_1(\boldsymbol{\tau},\boldsymbol{\zeta})$ *and* $G_2(\boldsymbol{\tau},\boldsymbol{\zeta})$ *is defined by*

$$(G_1 *_{\mathfrak{F},\mathfrak{L}} G_2)(\boldsymbol{\xi},\boldsymbol{\eta}) := \int_{\mathbb{R}^n_+(\boldsymbol{\xi})} d\boldsymbol{\tau} \int_{\mathbb{R}^n} G_1(\boldsymbol{\tau},\boldsymbol{\zeta}) G_2(\boldsymbol{\xi}-\boldsymbol{\tau},\boldsymbol{\eta}-\boldsymbol{\zeta}) d\boldsymbol{\zeta}. \tag{12}$$

Remark 2.4 *We observe that the Fourier-Laplace convolution is a special case of the* $\varphi-$*convolution.*

Then, by using the Hölder's inequality ([8, p. 106]) we have

Lemma 2.5 *For non-negative functions* ρ_j *on* $D(j=1,2)$ *such that the convolution* $\rho_1 *_\varphi \rho_2$ *exists and for functions* $F_j \in L_p(D, \rho_j d\boldsymbol{x}_j),\ j=1,2;\ p>1,$ *we have the inequality*

$$|((F_1\rho_1) *_\varphi (F_2\rho_2))(\boldsymbol{\eta})|^p \leq ((\rho_1 *_\varphi \rho_2)(\boldsymbol{\eta}))^{p-1}((|F_1|^p\rho_1) *_\varphi (|F_2|^p\rho_2)(\boldsymbol{\eta})) \tag{13}$$

for all $\boldsymbol{\eta} \in D$*. Equality holds here if and only if there exists a function* $k(\boldsymbol{\eta})$ *in* $\boldsymbol{\eta}$ *such that*

$$F_1(\boldsymbol{\xi}) F_2(\varphi(\boldsymbol{\xi},\boldsymbol{\eta})) = k(\boldsymbol{\eta}) \quad a.e. \quad \text{on } D. \tag{14}$$

Definition 2.6 *Define the* $\varphi-$ *convolution product (iterated* $\varphi-$*convolution)* $\prod_{j=1}^q *_\varphi F_j$ *by*

$$\prod_{j=1}^q *_\varphi F_j(\boldsymbol{\xi}_q) = \left[\prod_{j=1}^{q-1} *_\varphi F_j\right] *_\varphi F_q(\boldsymbol{\xi}_q). \tag{15}$$

Under the stated assumption, we have

Lemma 2.7 *For non-negative functions* ρ_j *on* D *such that the convolution* $\prod_{j=1}^q *_\varphi \rho_j$ *exists and for functions* $F_j \in L_p(D, \rho_j d\boldsymbol{x}_j),\ j=1,...,q;\ p>1,$ *we have the inequality*

$$\left|\prod_{j=1}^q *_\varphi (F_j\rho_j)(\boldsymbol{\xi}_q)\right|^p \leq \left(\prod_{j=1}^q *_\varphi \rho_j(\boldsymbol{\xi}_q)\right)^{p-1} \prod_{j=1}^q *_\varphi (|F_j|^p\rho_j)(\boldsymbol{\xi}_q) \tag{16}$$

for all $\boldsymbol{\xi}_q \in D$*. Equality holds here if and only if there exist some functions* $k_l(\boldsymbol{\xi}_l)$*,* $\boldsymbol{\xi}_l \in D$*, such that*

$$F_1(\boldsymbol{\xi}_1) \prod_{j=2}^l F_j(\varphi(\boldsymbol{\xi}_{j-1},\boldsymbol{\xi}_j)) = k_l(\boldsymbol{\xi}_l), \quad l=2,...,q. \tag{17}$$

Proof. We use induction on q. When $q=2$, the inequality (16) and the equality statement (17) are reduced to Lemma 2.5.

Now suppose (16) and (17) hold for some integer $q \geq 2$. We claim that they also hold for $q+1$.

For each $\boldsymbol{\xi}_{q+1}$, put

$$f_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q) = \left\{\prod_{j=1}^q *_\varphi \rho_j(\boldsymbol{\xi}_q) \rho_{q+1}(\varphi(\boldsymbol{\xi}_q,\boldsymbol{\xi}_{q+1})) |\varphi_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q,\boldsymbol{\xi}_{q+1})|\right\}^{(p-1)/p}$$

and

$$g_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q) = \left\{\prod_{j=1}^{q} *_\varphi(|F_j|^p\rho_j)(\boldsymbol{\xi}_q)\rho_{q+1}(\varphi(\boldsymbol{\xi}_q,\boldsymbol{\xi}_{q+1}))|\varphi_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q,\boldsymbol{\xi}_{q+1})|\right\}^{1/p} \times |F_{q+1}(\varphi(\boldsymbol{\xi}_q,\boldsymbol{\xi}_{q+1}))|.$$

By induction hypothesis, we arrive at

$$\left|\prod_{j=1}^{q+1} *_\varphi(F_j\rho_j)(\boldsymbol{\xi}_{q+1})\right| \leq \left(\prod_{j=1}^{q} *_\varphi|F_j\rho_j|\right) *_\varphi (|F_{q+1}\rho_{q+1}|)(\boldsymbol{\xi}_{q+1}) \leq \int_D f_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q)g_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q)d\boldsymbol{\xi}_q.$$

Application of the Hölder's inequality to $f_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q)$ and $g_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q)$ yields

$$\left(\int_D f_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q)g_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q)d\boldsymbol{\xi}_q\right)^p \leq \left\{\int_D [f_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q)]^{p/(p-1)}d\boldsymbol{\xi}_q\right\}^{p-1}\int_D [g_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q)]^p d\boldsymbol{\xi}_q = \left\{\prod_{j=1}^{q+1} *_\varphi\rho_j(\boldsymbol{\xi}_{q+1})\right\}^{p-1}\prod_{j=1}^{q+1} *_\varphi(|F_j|^p\rho_j)(\boldsymbol{\xi}_{q+1}).$$

Hence, we have

$$\left|\prod_{j=1}^{q+1} *_\varphi(F_j\rho_j)(\boldsymbol{\xi}_{q+1})\right|^p \leq \left(\prod_{j=1}^{q+1} *_\varphi\rho_j(\boldsymbol{\xi}_{q+1})\right)^{p-1}\prod_{j=1}^{q+1} *_\varphi(|F_j|^p\rho_j)(\boldsymbol{\xi}_{q+1}). \tag{18}$$

Equality holds in (18) if and only if

$$\left|\prod_{j=1}^{q} *_\varphi(F_j\rho_j)(\boldsymbol{\xi}_q)\right|^p = \left(\prod_{j=1}^{q} *_\varphi\rho_j(\boldsymbol{\xi}_q)\right)^{p-1}\prod_{j=1}^{q} *_\varphi(|F_j|^p\rho_j)(\boldsymbol{\xi}_q), \tag{19}$$

and

$$[f_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q)]^{p/(p-1)}\int_D [g_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q)]^p d\boldsymbol{\xi}_q = [g_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q)]^p\int_D [f_{\boldsymbol{\xi}_{q+1}}(\boldsymbol{\xi}_q)]^{p/(p-1)}d\boldsymbol{\xi}_q. \tag{20}$$

By induction hypothesis, we have (19) and (20), and therefore the assertion follows. □

From the $\varphi-$ convolution and the $\psi-$ convolution, we get the following definition

Definition 2.8 *Under suitable hypotheses for the $\varphi-$ convolution $F_1 *_\varphi F_2$ and the $\psi-$ convolution $F_1 *_\psi F_2$, the $(\varphi+\psi)-$ convolution, denoted by $F_1 *_{\varphi+\psi} F_2$, is defined by*

$$(F_1 *_{\varphi+\psi} F_2)(\boldsymbol{\xi}) := \int_D F_1(\boldsymbol{\tau})\left[F_2(\varphi(\boldsymbol{\tau},\boldsymbol{\xi}))|\varphi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})| + F_2(\psi(\boldsymbol{\tau},\boldsymbol{\xi}))|\psi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})|\right]d\boldsymbol{\tau}. \tag{21}$$

We also define the iterated $(\varphi+\psi)-$convolution by

$$\prod_{j=1}^{q} *_{\varphi+\psi}F_j(\boldsymbol{\xi}_q) = \left[\prod_{j=1}^{q-1} *_{\varphi+\psi}F_j\right] *_{\varphi+\psi} F_q(\boldsymbol{\xi}_q). \tag{22}$$

Example 2.9 *The Fourier cosine convolution (see [16, 17])*

$$F_1 *_{\mathfrak{F}_c} F_2(\boldsymbol{x}) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n_+} F_1(\boldsymbol{y})[F_2(\boldsymbol{x}+\boldsymbol{y}) + F_2(|\boldsymbol{x}-\boldsymbol{y}|)]d\boldsymbol{y} \tag{23}$$

is a special case of the $(\varphi+\psi)-$*convolution.*

Lemma 2.10 *For non-negative functions* $\rho_j,\ j=1,2,$ *on* D *such that the convolution* $\rho_1 *_{\varphi+\psi} \rho_2$ *exists, and for functions* $F_j \in L_p(D, \rho_j(\boldsymbol{\xi})d\boldsymbol{\xi}),\ j=1,2;\ p>1,$ *we have the inequality*

$$\begin{aligned} &|(F_1\rho_1) *_{\varphi+\psi} (F_2\rho_2)(\boldsymbol{\xi})|^p \\ &\le (\rho_1 *_{\varphi+\psi} \rho_2(\boldsymbol{\xi}))^{p-1}(|F_1|^p\rho_1) *_{\varphi+\psi} (|F_2|^p\rho_2)(\boldsymbol{\xi}) \end{aligned} \tag{24}$$

for all $\boldsymbol{\xi} \in D$*. Equality holds here if and only if*

$$\begin{aligned} |F_2(\varphi(\boldsymbol{\tau},\boldsymbol{\xi}))| = |F_2(\psi(\boldsymbol{\tau},\boldsymbol{\xi}))|, \quad \forall\ (\boldsymbol{\tau},\boldsymbol{\xi}) \in D\times D \\ \textit{and} \quad F_1(\boldsymbol{\tau})F_2(\varphi(\boldsymbol{\tau},\boldsymbol{\xi})) = k(\boldsymbol{\xi}) \quad \textit{a.e.} \quad \text{on } D, \end{aligned} \tag{25}$$

for a function $k(\boldsymbol{\xi})$ *in* $\boldsymbol{\xi}$*.*

Moreover, we also have the following generalized result:

Lemma 2.11 *For non-negative functions* $\rho_j,\ j=1,2,...,q,$ *on* D *such that there exists the convolution* $\prod_{j=1}^q *_{\varphi+\psi}\rho_j$*, and for functions* $F_j \in L_p(D, \rho_j(\boldsymbol{\xi})d\boldsymbol{\xi}),\ j=1,2,...,q;\ p>1,$ *we have the inequality*

$$\left|\prod_{j=1}^q *_{\varphi+\psi}(F_j\rho_j)(\boldsymbol{\xi}_q)\right|^p \le \left(\prod_{j=1}^q *_{\varphi+\psi}\rho_j(\boldsymbol{\xi}_q)\right)^{p-1} \prod_{j=1}^q *_{\varphi+\psi}(|F_j|^p\rho_j)(\boldsymbol{\xi}_q) \tag{26}$$

for all $\boldsymbol{\xi}_q \in D$*. Equality holds here if and only if*

$$|F_j(\varphi(\boldsymbol{\tau},\boldsymbol{\xi}))| = |F_j(\psi(\boldsymbol{\tau},\boldsymbol{\xi}))|, \quad \forall\ (\boldsymbol{\tau},\boldsymbol{\xi}) \in D\times D, \quad j=2,...,q, \tag{27}$$

and for some functions $k_l(\boldsymbol{\xi}_l)$*,* $\boldsymbol{\xi}_l \in D$*, such that*

$$F_1(\boldsymbol{\xi}_1)\prod_{j=2}^l F_j(\varphi(\boldsymbol{\xi}_{j-1},\boldsymbol{\xi}_j)) = k_l(\boldsymbol{\xi}_l), \quad l=2,...,q. \tag{28}$$

Proof of Lemma 2.10 By the Hölder's inequality, we have

$$\begin{aligned} &|F_2(\varphi(\boldsymbol{\tau},\boldsymbol{\xi}))\rho_2(\varphi(\boldsymbol{\tau},\boldsymbol{\xi}))|\varphi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})| + F_2(\psi(\boldsymbol{\tau},\boldsymbol{\xi}))\rho_2(\psi(\boldsymbol{\tau},\boldsymbol{\xi}))|\psi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})|| \\ &\le \left(|F_2(\varphi(\boldsymbol{\tau},\boldsymbol{\xi}))|^p\rho_2(\varphi(\boldsymbol{\tau},\boldsymbol{\xi}))|\varphi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})| + |F_2(\psi(\boldsymbol{\tau},\boldsymbol{\xi}))|^p\rho_2(\psi(\boldsymbol{\tau},\boldsymbol{\xi}))|\psi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})|\right)^{1/p} \\ &\quad \times \left(\rho_2(\varphi(\boldsymbol{\tau},\boldsymbol{\xi}))|\varphi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})| + \rho_2(\psi(\boldsymbol{\tau},\boldsymbol{\xi}))|\psi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})|\right)^{(p-1)/p}. \end{aligned} \tag{29}$$

For all $\boldsymbol{\xi} \in D$, let

$$f_{\boldsymbol{\xi}}(\boldsymbol{\tau}) = (\rho_1(\boldsymbol{\tau}))^{(p-1)/p} \left(\rho_2(\varphi(\boldsymbol{\tau},\boldsymbol{\xi}))|\varphi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})| + \rho_2(\psi(\boldsymbol{\tau},\boldsymbol{\xi}))|\psi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})|\right)^{(p-1)/p}$$

and

$$\begin{aligned} g_{\boldsymbol{\xi}}(\boldsymbol{\tau}) =& (|F_2(\varphi(\boldsymbol{\tau},\boldsymbol{\xi}))|^p \rho_2(\varphi(\boldsymbol{\tau},\boldsymbol{\xi}))|\varphi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})| + |F_2(\psi(\boldsymbol{\tau},\boldsymbol{\xi}))|^p \rho_2(\psi(\boldsymbol{\tau},\boldsymbol{\xi}))|\psi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})|)^{1/p} \\ &\times |F_1(\boldsymbol{\tau})|(\rho_1(\boldsymbol{\tau}))^{(p-1)/p}. \end{aligned}$$

Application of the Hölder's inequality to $f_{\boldsymbol{\xi}}(\boldsymbol{\tau})$ and $g_{\boldsymbol{\xi}}(\boldsymbol{\tau})$ yields

$$|(F_1\rho_1) *_{\varphi+\psi} (F_2\rho_2)(\boldsymbol{\xi})|^p \leq (\rho_1 *_{\varphi+\psi} \rho_2(\boldsymbol{\xi}))^{p-1}(|F_1|^p\rho_1) *_{\varphi+\psi} (|F_2|^p\rho_2)(\boldsymbol{\xi}). \tag{30}$$

The equality holds if and only if equality holds in (29) and in (30). So we have (25). □

Remark 2.12 *Similarly, for $\varphi^j(.,.) : D \times D \to D, j = 1, 2, ..., q,$ we can introduce the $\sum_{j=1}^m \varphi^j-$ convolution*

$$(F_1 *_{\sum_{j=1}^m \varphi^j} F_2)(\boldsymbol{\xi}) := \int_D F_1(\boldsymbol{\tau}) \sum_{j=1}^m \left\{F_2(\varphi^j(\boldsymbol{\tau},\boldsymbol{\xi}))|\varphi^j_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})|\right\} d\boldsymbol{\tau} \tag{31}$$

and for all $\boldsymbol{\xi} \in D$ we also have the inequality

$$\begin{aligned} &|(F_1\rho_1) *_{\sum_{j=1}^m \varphi^j} (F_2\rho_2)(\boldsymbol{\xi})|^p \\ &\leq (\rho_1 *_{\sum_{j=1}^m \varphi^j} \rho_2(\boldsymbol{\xi}))^{p-1}(|F_1|^p\rho_1) *_{\sum_{j=1}^m \varphi^j} (|F_2|^p\rho_2)(\boldsymbol{\xi}). \end{aligned} \tag{32}$$

For the sake of convenience, we restate the inequality of Hardy, Littlewood and Pólya [7, pp. 148-150] (see also [8, p. 106]) here.

Lemma 2.13 *If $p > 1$, then*

$$\left[\int \left|\sum f_m(\boldsymbol{x})\right|^p d\boldsymbol{x}\right]^{1/p} \leq \sum \left[\int |f_m(\boldsymbol{x})|^p d\boldsymbol{x}\right]^{1/p} \tag{33}$$

with equality if and only if $f_m(\boldsymbol{x}) = C_m g(\boldsymbol{x})$, where C_m are constants and $g(\boldsymbol{x})$ is an integrable function.

3 The main results

Our main result is the following:

Theorem 3.1 *For non-negative functions $\rho_j,\ j = 1, 2, ..., q,$ on D such that the convolution $\prod_{j=1}^q *_\varphi \rho_j$ exists, and for functions $F_j \in L_p(D, \rho_j d\boldsymbol{x}_j),\ j = 1, ..., q;\ p > 1,$ we have the inequality*

$$\left\| \left(\prod_{j=1}^q *_\varphi (F_j\rho_j)\right) \left(\prod_{j=1}^q *_\varphi \rho_j\right)^{\frac{1}{p}-1} \right\|_{L_p(D)} \leq \prod_{j=1}^q \|F_j\|_{L_p(D,\rho_j)}. \tag{34}$$

Equality holds here if and only if there exist some functions $k_l(\boldsymbol{\xi}_l)$, $\boldsymbol{\xi}_l \in D$, such that

$$F_1(\boldsymbol{\xi}_1)\prod_{j=2}^{l} F_j(\varphi(\boldsymbol{\xi}_{j-1},\boldsymbol{\xi}_j)) = k_l(\boldsymbol{\xi}_l), \quad l = 2, ..., q. \tag{35}$$

Remark 3.2 *From the above theorem, we can derive inequalities for iterated convolutions obtained previously:*

- *For the iterated Fourier convolution $\prod_{j=1}^q *_{\mathfrak{F}}$, we have ([9])*

$$\left\| \left(\prod_{j=1}^{q} *_{\mathfrak{F}}(F_j\rho_j)\right)\left(\prod_{j=1}^{q} *_{\mathfrak{F}}\rho_j\right)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n)} \le \prod_{j=1}^{q} \|F_j\|_{L_p(\mathbb{R}^n,\rho_j)}. \tag{36}$$

- *For the iterated Laplace convolution $\prod_{j=1}^q *_{\mathfrak{L}}$, we have ([10])*

$$\left\| \left(\prod_{j=1}^{q} *_{\mathfrak{L}}(F_j\rho_j)\right)\left(\prod_{j=1}^{q} *_{\mathfrak{L}}\rho_j\right)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n_+)} \le \prod_{j=1}^{q} \|F_j\|_{L_p(\mathbb{R}^n_+,\rho_j)}. \tag{37}$$

- *For the iterated Mellin convolution $\prod_{j=1}^q *_{\mathfrak{M}}$, we have ([10])*

$$\left\| \left(\prod_{j=1}^{q} *_{\mathfrak{M}}(F_j\rho_j)\right)\left(\prod_{j=1}^{q} *_{\mathfrak{M}}\rho_j\right)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n_+)} \le \prod_{j=1}^{q} \|F_j\|_{L_p(\mathbb{R}^n_+,\rho_j)}. \tag{38}$$

Proof of Theorem 3.1 From Lemma 2.7, we have

$$\left|\prod_{j=1}^{q} *_{\varphi}(F_j\rho_j)(\boldsymbol{\xi}_q)\right|^p \left(\prod_{j=1}^{q} *_{\varphi}\rho_j(\boldsymbol{\xi}_q)\right)^{1-p} \le \prod_{j=1}^{q} *_{\varphi}(|F_j|^p\rho_j)(\boldsymbol{\xi}_q). \tag{39}$$

Now, by taking integration of both sides of (39) with respect to $\boldsymbol{\xi}_q$ on D we obtain the inequality

$$\int_D \left|\prod_{j=1}^{q} *_{\varphi}(F_j\rho_j)(\boldsymbol{\xi}_q)\right|^p \left(\prod_{j=1}^{q} *_{\varphi}\rho_j(\boldsymbol{\xi}_q)\right)^{1-p} d\boldsymbol{\xi}_q \le \int_D \prod_{j=1}^{q} *_{\varphi}(|F_j|^p\rho_j)(\boldsymbol{\xi}_q)d\boldsymbol{\xi}_q.$$

Here we have, by definition,

$$\int_D \prod_{j=1}^{q} *_{\varphi}(|F_j|^p\rho_j)(\boldsymbol{\xi}_q)d\boldsymbol{\xi}_q = \int_D\int_D \left[\prod_{j=1}^{q-1} *_{\varphi}(|F_j|^p\rho_j)(\boldsymbol{\xi}_{q-1})\right] |F_q(\varphi(\boldsymbol{\xi}_{q-1},\boldsymbol{\xi}_q))|^p$$
$$\rho_q(\varphi(\boldsymbol{\xi}_{q-1},\boldsymbol{\xi}_q))|\varphi_{\boldsymbol{\xi}_q}(\boldsymbol{\xi}_{q-1},\boldsymbol{\xi}_q)|d\boldsymbol{\xi}_{q-1}d\boldsymbol{\xi}_q,$$

which is, by the Fubini's theorem and the change of variables $\boldsymbol{x}_q = \varphi(\boldsymbol{\xi}_{q-1},\boldsymbol{\xi}_q)$,

$$= \int_D \left[\prod_{j=1}^{q-1} *_{\varphi}(|F_j|^p\rho_j)(\boldsymbol{\xi}_{q-1})\right] d\boldsymbol{\xi}_{q-1} \int_D |F_q(\boldsymbol{x}_q)|^p\rho_q(\boldsymbol{x}_q)d\boldsymbol{x}_q.$$

Therefore,

$$\int_D \left|\prod_{j=1}^{q} *_\varphi (F_j\rho_j)(\boldsymbol{\xi}_q)\right|^p \left(\prod_{j=1}^{q} *_\varphi \rho_j(\boldsymbol{\xi}_q)\right)^{1-p} d\boldsymbol{\xi}_q \le \prod_{j=1}^{q} \int_D |F_j(\boldsymbol{x})|^p \rho_j(\boldsymbol{x}) d\boldsymbol{x}. \tag{40}$$

Raising both sides of the inequality (40) to power $1/p$ yields the inequality (34).

Equality holds here if and only if equality holds in Lemma 2.7, and we have (35). □

We now show the inequality for a sum of iterated $\varphi-$ convolution

Corollary 3.3 *Let functions $\rho_{j,m}$ be non-negative on D such that the convolution $\prod_{j=1}^{q} *_\varphi \rho_{j,m}(\boldsymbol{\xi}_{q,m})$ exists. Then, we have the inequality for the sum of iterated $\varphi-$ convolution*

$$\begin{aligned} &\left\| \sum_{m=1}^{n} \left(\prod_{j=1}^{q} *_\varphi (F_{j,m}\rho_{j,m})\right) \left(\prod_{j=1}^{q} *_\varphi \rho_{j,m}\right)^{\frac{1}{p}-1} \right\|_{L_p(D)} \\ &\qquad \le \sum_{m=1}^{n} \prod_{j=1}^{q} \|F_{j,m}\|_{L_p(D,\rho_{j,m})} \end{aligned} \tag{41}$$

for functions $F_{j,m} \in L_p(D, \rho_{j,m} d\boldsymbol{\xi}_{j,m})$, $j = 1, ..., q$; $m = 1, ..., n$, $p > 1$. Equality holds here if and only if there exist some functions $k_{l,m}(\boldsymbol{\xi}_{l,m})$, $\boldsymbol{\xi}_{l,m} \in D$, such that

$$F_{1,m}(\boldsymbol{\xi}_{1,m}) \prod_{j=2}^{l} F_{j,m}(\varphi(\boldsymbol{\xi}_{j-1,m}, \boldsymbol{\xi}_{j,m})) = k_l(\boldsymbol{\xi}_{l,m}), \quad l = 2, ..., q,\ m = 1, ..., n, \tag{42}$$

$$k_{q,m}(\boldsymbol{x}) \prod_{j=1}^{q} *_\varphi \rho_{j,m}(\boldsymbol{x}) = C_m h(\boldsymbol{x}), \quad \boldsymbol{x} \in D, \quad C_m: \text{ constants}, m = 1, ..., n, \tag{43}$$

where $h(\boldsymbol{x})$ is an integrable function.

Proof. By Lemma 2.13, we first observe that

$$\begin{aligned} &\left\| \sum_{m=1}^{n} \left(\prod_{j=1}^{q} *_\varphi (F_{j,m}\rho_{j,m})\right) \left(\prod_{j=1}^{q} *_\varphi \rho_{j,m}\right)^{\frac{1}{p}-1} \right\|_{L_p(D)} \\ &\le \sum_{m=1}^{n} \left\| \left(\prod_{j=1}^{q} *_\varphi (F_{j,m}\rho_{j,m})\right) \left(\prod_{j=1}^{q} *_\varphi \rho_{j,m}\right)^{\frac{1}{p}-1} \right\|_{L_p(D)}, \end{aligned}$$

which is, by Theorem 3.1,

$$\le \sum_{m=1}^{n} \prod_{j=1}^{q} \|F_{j,m}\|_{L_p(D,\rho_{j,m})}.$$

The equality holds if and only if for an integrable function $h(\boldsymbol{x})$

$$\left|\prod_{j=1}^{q} *_\varphi (F_{j,m}\rho_{j,m})(\boldsymbol{x})\right|^p \left(\prod_{j=1}^{q} *_\varphi \rho_{j,m}(\boldsymbol{x})\right)^{1-p} = \prod_{j=1}^{q} *_\varphi (|F_j|^p \rho_j)(\boldsymbol{x}) = C_m h(\boldsymbol{x}),$$

where C_m are constants. Thus the Corollary is proved. □

Corollary 3.4 *For non-negative functions ρ_j on $\mathbb{R}^n \times \mathbb{R}_+^m$ such that the convolution $\prod_{j=1}^q *_{\mathfrak{F},\mathfrak{L}} \rho_j$ exists, and for functions $F_j \in L_p(\mathbb{R}^n \times \mathbb{R}_+^m, \rho_j d\boldsymbol{\xi}_j d\boldsymbol{\eta}_j)$, $j = 1, ..., q$; $p > 1$, we have the inequality*

$$\left\| \left(\prod_{j=1}^q *_{\mathfrak{F},\mathfrak{L}} (F_j \rho_j) \right) \left(\prod_{j=1}^q *_{\mathfrak{F},\mathfrak{L}} \rho_j \right)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n \times \mathbb{R}_+^m)} \leq \prod_{j=1}^q \|F_j\|_{L_p(\mathbb{R}^n \times \mathbb{R}_+^m, \rho_j)}. \tag{44}$$

Equality holds here if and only if

$$F_j(\boldsymbol{\xi}, \boldsymbol{\eta}) = C_j e^{\boldsymbol{\alpha\xi} + \boldsymbol{\beta\eta}}, \quad (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^n \times \mathbb{R}_+^m, \quad C_j : \text{ constants}, \tag{45}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^n$, $\boldsymbol{\beta} \in \mathbb{R}^m$ are two constants such that $F_j \in L_p(\mathbb{R}^n \times \mathbb{R}_+^m, \rho_j d\boldsymbol{\xi}_j d\boldsymbol{\eta}_j)$.

Remark 3.5 *In Corollary 3.4, let $q = 2$, $m = 1$,*

$$\rho_1 \equiv 1 \quad \text{and } F_2(\boldsymbol{\xi} - \boldsymbol{x}, t - \tau) = G(\boldsymbol{\xi} - \boldsymbol{x}, t - \tau),$$

for some Green's function $G(\boldsymbol{\xi} - \boldsymbol{x}, t - \tau)$. Then, we have the inequality

$$\begin{aligned} &\int_0^\infty dt \int_{\mathbb{R}^n} \frac{\left| \int_0^t \int_{\mathbb{R}^n} F(\boldsymbol{x}, \tau) \rho(\boldsymbol{x}, \tau) G(\boldsymbol{\xi} - \boldsymbol{x}, t - \tau) d\boldsymbol{x} d\tau \right|^p}{\left(\int_0^t \int_{\mathbb{R}^n} |\rho(\boldsymbol{\xi}, \tau)| d\boldsymbol{\xi} d\tau \right)^{p-1}} d\boldsymbol{x} \\ &\qquad \leq \int_0^\infty d\tau \int_{\mathbb{R}^n} |F(\boldsymbol{x}, \tau)|^p |\rho(\boldsymbol{x}, \tau)| d\boldsymbol{x} \int_0^\infty d\tau \int_{\mathbb{R}^n} |G(\boldsymbol{x}, \tau)|^p d\boldsymbol{x}, \end{aligned} \tag{46}$$

for $\rho(\boldsymbol{x}, t) \in L_1(\mathbb{R}^n \times \mathbb{R}_+, d\boldsymbol{x} dt)$.

Remark 3.6 *Inequality (46) can be generalized further as follows*

$$\begin{aligned} &\int_0^T dt \int_{\boldsymbol{a}}^{\boldsymbol{b}} \left| \int_0^t \int_{\mathbb{R}^n} F(\boldsymbol{x}, \tau) \rho(\boldsymbol{x}, \tau) G(\boldsymbol{\xi} - \boldsymbol{x}, t - \tau) d\boldsymbol{x} d\tau \right|^p d\boldsymbol{x} \\ &\qquad \leq \left(\int_0^T d\tau \int_{\mathbb{R}^n} |\rho(\boldsymbol{\xi}, \tau)| d\boldsymbol{\xi} \right)^{p-1} \\ &\qquad\quad \times \int_0^T d\tau \int_{\mathbb{R}^n} |F(\boldsymbol{\xi}, \tau)|^p |\rho(\boldsymbol{\xi}, \tau)| \int_0^T d\tau \int_{\boldsymbol{a}-\boldsymbol{\xi}}^{\boldsymbol{b}-\boldsymbol{\xi}} |G(\boldsymbol{x}, \tau)|^p d\boldsymbol{x} d\boldsymbol{\xi}. \end{aligned} \tag{47}$$

From Lemma 2.10 we obtain

Theorem 3.7 *For non-negative functions ρ_j, $j = 1, ..., q$, on D such that the convolution $\prod_{j=1}^q *_{\varphi+\psi} \rho_j$ exists, and for functions $F_j \in L_p(D, \rho_j d\boldsymbol{x}_j)$, $j = 1, ..., q$; $p > 1$, we have the inequality*

$$\left\| \left(\prod_{j=1}^q *_{\varphi+\psi} (F_j \rho_j) \right) \left(\prod_{j=1}^q *_{\varphi+\psi} \rho_j \right)^{\frac{1}{p}-1} \right\|_{L_p(D)}^p \leq 2^{q-1} \prod_{j=1}^q \|F_j\|_{L_p(D,\rho_j)}^p. \tag{48}$$

The equality holds in (48) *if and only if*

$$|F_j(\varphi(\boldsymbol{\tau},\boldsymbol{\xi}))| = |F_j(\psi(\boldsymbol{\tau},\boldsymbol{\xi}))|, \quad \forall\ (\boldsymbol{\tau},\boldsymbol{\xi}) \in D\times D, \quad j=2,...,q, \tag{49}$$

and for some functions $k_l(\boldsymbol{\xi}_l)$, $\boldsymbol{\xi}_l \in D$, *such that*

$$F_1(\boldsymbol{\xi}_1)\prod_{j=2}^{l} F_j(\varphi(\boldsymbol{\xi}_{j-1},\boldsymbol{\xi}_j)) = k_l(\boldsymbol{\xi}_l), \quad l=2,...,q. \tag{50}$$

Corollary 3.8 *For non-negative functions* ρ_j, $j=1,...,q$, *on* $\mathbb{R}^n_+$ *such that the convolution* $\prod_{j=1}^q *_{\mathfrak{F}_c}\rho_j$ *exists, and for functions* $F_j \in L_p(\mathbb{R}^n_+,\rho_j d\boldsymbol{x}_j)$, $p>1$, *we have the inequality*

$$\begin{aligned}\left\|\left(\prod_{j=1}^{q} *_{\mathfrak{F}_c}(F_j\rho_j)\right)\left(\prod_{j=1}^{q} *_{\mathfrak{F}_c}\rho_j\right)^{\frac{1}{p}-1}\right\|^p_{L_p(\mathbb{R}^n_+)} \\ \le \left[\frac{2}{\sqrt{2\pi}^n}\right]^{q-1}\prod_{j=1}^{q}\|F_j\|^p_{L_p(\mathbb{R}^n_+,\rho_j)}.\end{aligned} \tag{51}$$

Remark 3.9 *In Corollary 3.8, for* $\rho_q\equiv 1$ *and* $F_q \equiv G$, *we have*

$$\begin{aligned}&\left\|\left(\prod_{j=1}^{q-1} *_{\mathfrak{F}_c}(F_j\rho_j)\right) *_{\mathfrak{F}_c} G\right\|^p_{L_p(\mathbb{R}^n_+)} \\ &\le \left[\frac{2}{\sqrt{2\pi}^n}\right]^{2q-3}\|G\|^p_{L_p(\mathbb{R}^n_+)}\prod_{j=1}^{q-1}\|\rho_j\|^{p-1}_{L_1(\mathbb{R}^n_+)}\prod_{j=1}^{q-1}\|F_j\|_{L_p(\mathbb{R}^n_+,\rho_j)}\end{aligned} \tag{52}$$

for $\rho_j \in L_1(\mathbb{R}^n_+)$ *and for functions* F_j *and* G *such that the right hand side of* (52) *is finite.*

Proof of Corollary 3.8 We have

$$\begin{aligned}\int_{\mathbb{R}^n_+}\left|\prod_{j=1}^{q} *_{\mathfrak{F}_c}(F_j\rho_j)(\boldsymbol{\xi}_q)\right|^p\left(\prod_{j=1}^{q} *_{\mathfrak{F}_c}\rho_j(\boldsymbol{\xi}_q)\right)^{1-p} d\boldsymbol{\xi}_q \\ \le \int_{\mathbb{R}^n_+}\prod_{j=1}^{q} *_{\mathfrak{F}_c}(|F_j|^p\rho_j)(\boldsymbol{\xi}_q)d\boldsymbol{\xi}_q.\end{aligned} \tag{53}$$

Since

$$\begin{aligned}&\int_{\mathbb{R}^n_+}\left[|F_q(\boldsymbol{\xi}_q+\boldsymbol{\xi}_{q-1})|^p\rho_q(\boldsymbol{\xi}_q+\boldsymbol{\xi}_{q-1}) + |F_q(|\boldsymbol{\xi}_q-\boldsymbol{\xi}_{q-1}|)|^p\rho_q(|\boldsymbol{\xi}_q-\boldsymbol{\xi}_{q-1}|)\right]d\boldsymbol{\xi}_q \\ &= \int_{\mathbb{R}^n_{\boldsymbol{\xi}_{q-1}}}|F_q(\boldsymbol{z})|^p\rho_q(\boldsymbol{z})d\boldsymbol{z} + \int_{\mathbb{R}^n(\boldsymbol{\xi}_{q-1})}|F_q(\boldsymbol{t})|^p\rho_q(\boldsymbol{t})d\boldsymbol{t} + \int_{\mathbb{R}^n_+}|F_q(\boldsymbol{t})|^p\rho_q(\boldsymbol{t})d\boldsymbol{t} \\ &= 2\|F_q\|^p_{L_p(\mathbb{R}^n_+,\rho_q)},\end{aligned}$$

and from the Fubini's theorem, its follows that

$$\int_{\mathbb{R}^n_+} \prod_{j=1}^{q} *_{\mathfrak{F}_c}(|F_j|^p \rho_j)(\boldsymbol{\xi}_q) d\boldsymbol{\xi}_q$$

$$= \frac{2}{\sqrt{2\pi}^n} \|F_q\|^p_{L_p(\mathbb{R}^n_+, \rho_q)} \int_{\mathbb{R}^n_+} \prod_{j=1}^{q-1} *_{\mathfrak{F}_c}(|F_j|^p \rho_j)(\boldsymbol{\xi}_{q-1}) d\boldsymbol{\xi}_{q-1}.$$

Therefore,

$$\int_{\mathbb{R}^n_+} \prod_{j=1}^{q} *_{\mathfrak{F}_c}(|F_j|^p \rho_j)(\boldsymbol{\xi}_q) d\boldsymbol{\xi}_q = \left[\frac{2}{\sqrt{2\pi}^n}\right]^{q-1} \prod_{j=1}^{q} \|F_j\|^p_{L_p(\mathbb{R}^n_+, \rho_j)}. \tag{54}$$

Combining (53) and (54) yields (51). □

Now, from Theorem 3.7 and Lemma 2.13 we have

Corollary 3.10 *For non-negative functions $\rho_{j,m}$ on D such that the convolution $\prod_{j=1}^{q} *_{\varphi+\psi}\rho_{j,m}$ exists we have the inequality for the sum of iterated $(\varphi+\psi)-$ convolution*

$$\left\| \sum_{m=1}^{n} \left(\prod_{j=1}^{q} *_{\varphi+\psi}(F_{j,m}\rho_{j,m}) \right) \left(\prod_{j=1}^{q} *_{\varphi+\psi}\rho_j \right)^{\frac{1}{p}-1} \right\|_{L_p(D)} \le 2^{q-1} \sum_{m=1}^{n} \prod_{j=1}^{q} \|F_{j,m}\|_{L_p(D, \rho_{j,m})}. \tag{55}$$

The equality holds here if and only if

$$|F_{j,m}(\varphi(\boldsymbol{\tau}, \boldsymbol{\xi}))| = |F_{j,m}(\psi(\boldsymbol{\tau}, \boldsymbol{\xi}))|, \quad \forall\ (\boldsymbol{\tau}, \boldsymbol{\xi}) \in D \times D, \quad j = 2, ..., q,\ m = 1, ..., n, \tag{56}$$

and for some functions $k_{l,m}(\boldsymbol{\xi}_{l,m})$, $\boldsymbol{\xi}_{l,m} \in D$, such that

$$F_{1,m}(\boldsymbol{\xi}_{1,m}) \prod_{j=2}^{l} F_{j,m}(\varphi(\boldsymbol{\xi}_{j-1,m}, \boldsymbol{\xi}_{j,m})) = k_l(\boldsymbol{\xi}_{l,m}), \qquad l = 2, ..., q,\ m = 1, ..., n, \tag{57}$$

$$k_{q,m}(\boldsymbol{x}) \prod_{j=1}^{q} *_{\varphi+\psi}\rho_{j,m}(\boldsymbol{x}) = C_m h(\boldsymbol{x}), \quad \boldsymbol{x} \in D, \quad C_m:\ \textit{constants},\ m = 1, ..., n, \tag{58}$$

where $h(\boldsymbol{x})$ is an integrable function.

4 Applications

In this section we will get L_p integral estimates for the solutions of the inhomogeneous Cauchy problems for the wave equations, the linear Klein-Gordon equation, the Cauchy problems for the inhomogeneous heat equations ([1], [3], [4]).

4.1 The Inhomogeneous Cauchy Problem for the Wave Equation

Let

$$\theta(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Consider the integral transform

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\rho(\xi,\tau)d\xi \\ &= \frac{1}{2c}\int_0^t d\tau \int_{-\infty}^{\infty} \theta\left(c(t-\tau) - |x-\xi|\right) F(\xi,\tau)\rho(\xi,\tau)d\xi, \end{aligned} \tag{59}$$

which gives the formal solution $u(x,t)$ of the inhomogeneous wave equation ([3, pp. 54-55], see also [4], [1])

$$u_{tt} = c^2 u_{xx} + F(x,t)\rho(x,t), \quad x \in \mathbb{R}, \quad t > 0, \quad (c > 0 \text{ is a constant}), \quad \rho \geq 0, \tag{60}$$

satisfying the homogeneous initial conditions

$$u(x,0) = u_t(x,0) = 0, \quad \text{on } \mathbb{R}. \tag{61}$$

Then, from (47) we have the inequality

$$\begin{aligned} \int_0^T dt \int_{-\infty}^{\infty} |u(x,t)|^p dx \leq \frac{cT^2}{(2c)^p} &\left(\int_0^T dt \int_{-\infty}^{\infty} \rho(x,t)\, dx\right)^{p-1} \\ &\times \int_0^T dt \int_{-\infty}^{\infty} |F(x,t)|^p \rho(x,t)\, dx, \quad \forall T > 0, \end{aligned} \tag{62}$$

for $\rho \in L_1(\mathbb{R}\times[0,T], dxdt)$ and $F \in L_p(\mathbb{R}\times[0,T], \rho(x,t)\, dxdt)$.

We consider the non-perfectly elastic string equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r(x)}\left[\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) + qu\right] + F(t)\rho(t), \quad a \leq x \leq b,\ t > 0,\ \rho \geq 0, \tag{63}$$

with the boundary conditions

$$a_1 u(a,t) + a_2 u_x(a,t) = 0; \quad b_1 u(b,t) + b_2 u_x(b,t) = 0, \quad t > 0, \tag{64}$$

and the initial conditions

$$u(x,0) = 0 = u_t(x,0), \quad a < x < b, \tag{65}$$

where p, q and r are assumed to be continuous functions of x in $a \leq x \leq b$ and a_1, a_2, b_1, b_2 are real constants such that

$$a_1^2 + a_2^2 > 0 \quad \text{and} \quad b_1^2 + b_2^2 > 0.$$

Following the method of separation of variables, we obtain the solution of the string equation (63) in the form (see [3, pp. 94-96])

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{1}{\alpha_n} \phi_n(x) \int_a^b \phi_n(\xi) d\xi \int_0^t \sin \alpha_n (t-\tau) F(\tau) \rho(\tau) d\tau \right]. \tag{66}$$

Here, $\lambda_n = -\alpha_n^2$ and $\phi_n, n = 1, 2, ...$, are the eigenvalues and the orthonormal eigenfunctions respectively of the equation

$$\frac{1}{r(x)} \left[\frac{\partial}{\partial x} \left(p \frac{\partial \phi}{\partial x} \right) + q\phi \right] = \lambda \phi, \tag{67}$$

where λ is a separation constant, with the associated boundary conditions for $\phi(x)$

$$a_1 \phi(a) + a_2 \phi'(a) = 0, \quad b_1 \phi(b) + b_2 \phi'(b) = 0. \tag{68}$$

From Corollary 3.3, the formal solution $u(x,t)$ satisfies the inequality

$$\begin{aligned} \int_0^T |u(x,t)|^p dt \le \left(\int_0^T \rho(t) dt \right)^{p-1} \int_0^T |F(t)|^p \rho(t) dt \\ \times T \left[\sum_{n=1}^{\infty} \left| \frac{1}{\alpha_n} \phi_n(x) \int_a^b \phi_n(\xi) d\xi \right| \right]^p \end{aligned} \tag{69}$$

for all $T > 0$ and for $\rho \in L_1([0,T]), F \in L_p([0,T], \rho(t)dt)$.

4.2 The Klein-Gordon Equation

The one-dimensional inhomogeneous Klein-Gordon equation is given by

$$u_{tt} - c^2 u_{xx} + d^2 u = F(x,t)\rho(x,t), \quad x \in \mathbb{R}, \quad t > 0, \quad \rho \ge 0, \tag{70}$$

where c and d are constants, with the initial boundary conditions

$$u(x,0) = 0 = u_t(x,0), \quad \text{for} \quad x \in \mathbb{R}, \tag{71}$$

$$u(x,t) \to 0 \quad \text{as} \quad |x| \to \infty, \quad t > 0. \tag{72}$$

Application of the joint Laplace and Fourier transform gives the solution as (see [3, pp. 558-560])

$$u(x,t) = \frac{1}{2c} \int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} J_0 \left[\frac{d}{c} \sqrt{c^2(t-\tau)^2 - (x-\xi)^2} \right] F(\xi,\tau)\rho(\xi,\tau) d\xi. \tag{73}$$

Here, $J_0(x)$ is the Bessel function of the first kind of order zero, given by ([4, pp. 592-598])

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{x}{2} \right)^{2r}.$$

For all $T > 0$, application of Lemma 2.13 gives

$$\begin{aligned}
&\int_0^T d\tau \int_0^{c\tau} \left| J_0 \left[\frac{d}{c}\sqrt{c^2\tau^2 - \xi^2} \right] \right|^p d\xi \\
&\le \int_0^T \left[\sum_{r=0}^{\infty} \frac{1}{(r!)^2} \left(\frac{d}{2c}\right)^{2r} [c\tau]^{2r+1/p} \left[\frac{1}{2} B\left(\frac{1}{2}, rp+1\right) \right]^{1/p} \right]^p d\tau \\
&\le \left[\sum_{r=0}^{\infty} \frac{1}{(r!)^2} \left(\frac{d}{2}\right)^{2r} c^{1/p} \frac{T^{2r+2/p}}{(2rp+2)^{1/p}} \left[\frac{1}{2} B\left(\frac{1}{2}, rp+1\right) \right]^{1/p} \right]^p,
\end{aligned}$$

which is, by using the inequalities $\frac{1}{2}B\left(\frac{1}{2}, rp+1\right) \le 1, 2rp+2 \ge 2$, and the representation of the modified Bessel function of the first kind of order zero $I_0(x)$,

$$\le \frac{cT^2}{2} I_0^p(dT).$$

By the inequality (47), the formal solution $u(x,t)$ satisfies the following estimate

$$\begin{aligned}
\int_0^T dt \int_{-\infty}^{\infty} |u(x,t)|^p dx \le & \frac{cT^2}{(2c)^p} \frac{I_0^p(dT)}{2} \left(\int_0^T dt \int_{-\infty}^{\infty} \rho(x,t) dx \right)^{p-1} \\
& \times \int_0^T dt \int_{-\infty}^{\infty} |F(x,t)|^p \rho(x,t) dx, \quad \forall T > 0,
\end{aligned} \tag{74}$$

for $\rho \in L_1(\mathbb{R} \times [0,T], dxdt)$ and $F \in L_p(\mathbb{R} \times [0,T], \rho(x,t)dxdt)$.

The two-dimensional linear inhomogeneous Klein-Gordon equation is

$$u_{tt} - c^2(u_{xx} + u_{yy}) + d^2 u = F(x,y,t)\rho(x,y,t), \quad x, y \in \mathbb{R}, \quad t > 0, \quad \rho \ge 0. \tag{75}$$

The initial boundary conditions are

$$u(x,y,0) = 0 = u_t(x,y,0), \quad \text{for all } x \text{ and } y, \tag{76}$$

$$u(x,y,t) \to 0 \quad \text{as} \quad r = \sqrt{x^2+y^2} \to \infty, \quad t > 0. \tag{77}$$

Application of the joint Laplace and Hankel transform of order zero gives the solution as

$$u(x,y,t) = \int_0^t d\tau \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x,y,t;\xi,\eta,\tau) d\xi d\eta. \tag{78}$$

Here, the Green function $G(x,y,t;\xi,\eta,\tau)$ assumes the form

$$G(x,y,t;\xi,\eta,\tau) = \frac{1}{2\pi c^2} \left(A^2 - \frac{B^2}{c^2}\right)^{-1/2} \cos\left(d\sqrt{A^2 - \frac{B^2}{c^2}}\right) \theta\left(A - \frac{B}{c}\right), \tag{79}$$

where $A = t - \tau$ and $B^2 = (x-\xi)^2 + (y-\eta)^2$.

For all $T > 0$ and for $1 < p < 2$, we have

$$\begin{aligned}&\int_0^T dt \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,y,t;\xi,\eta,\tau)|^p dxdy\\ &\le \frac{4}{(2\pi c^2)^p}\int_0^T dt\int_0^{ct}\int_0^{\sqrt{c^2t^2-y^2}} \left(t^2-\frac{x^2+y^2}{c^2}\right)^{-p/2} dxdy\\ &= \frac{2}{(2\pi c)^p} B\left(\frac{1}{2},\frac{2-p}{2}\right)\int_0^T dt\int_0^{ct}(c^2t^2-y^2)^{(1-p)/2}dy\\ &= \frac{c^2T^{3-p}}{(3-p)(2\pi c^2)^p} B\left(\frac{1}{2},\frac{2-p}{2}\right) B\left(\frac{1}{2},\frac{3-p}{2}\right).\end{aligned}$$

So, for all $T > 0$ and for $1 < p < 2$ we obtain the following estimate

$$\begin{aligned}&\int_0^T dt \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x,y,t)|^p dxdy\\ &\le \frac{c^2T^{3-p}}{(3-p)(2\pi c^2)^p} B\left(\frac{1}{2},\frac{2-p}{2}\right) B\left(\frac{1}{2},\frac{3-p}{2}\right)\left(\int_0^T dt\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho(x,y,t)dxdy\right)^{p-1} \qquad (80)\\ &\quad\times \int_0^T dt\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(x,y,t)|^p\rho(x,y,t)dxdy,\end{aligned}$$

where $\rho \in L_1(\mathbb{R}^2 \times [0,T], dxdydt)$ and $F \in L_p(\mathbb{R}^2 \times [0,T], \rho(x,y,t)dxdydt)$.

4.3 The Cauchy Problem for the Inhomogeneous Heat Equation

The equation of heat conduction with sources is given by

$$u_t(t,\boldsymbol{x}) - c^2\Delta_n u(t,\boldsymbol{x}) = F(t,\boldsymbol{x})\rho(t,\boldsymbol{x}), \quad (t,\boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad \rho \ge 0, \qquad (81)$$

where $F\rho \in L_1$ for every $t \in \mathbb{R}_+$, under the initial value condition

$$u(0,\boldsymbol{x}) = 0. \qquad (82)$$

Its solution has the form (see [1, pp. 58-59])

$$u(t,\boldsymbol{x}) = \frac{1}{(4\pi c^2)^{n/2}}\int_0^t d\tau\int_{\mathbb{R}^n}\frac{F(\tau,\boldsymbol{\xi})\rho(\tau,\boldsymbol{\xi})}{(t-\tau)^{n/2}}\exp\left\{-\frac{|\boldsymbol{\xi}-\boldsymbol{x}|^2}{4c^2(t-\tau)}\right\}d\boldsymbol{\xi}. \qquad (83)$$

It is easy to check that

$$\int_0^T d\tau\int_{\mathbb{R}^n}\frac{1}{\tau^{pn/2}}\exp\left\{-\frac{p|\boldsymbol{\xi}|^2}{4c^2\tau}\right\}d\boldsymbol{\xi} = \left(\frac{2c}{\sqrt{\pi p}}\right)^n\frac{2T^{1-n(p-1)/2}}{2-n(p-1)},$$

for all $T > 0$ and for $1 < p < 1 + 2/n$.

Then, by using (47), for all $T > 0$ we obtain the inequality

$$\begin{aligned}&\int_0^T dt\int_{\mathbb{R}^n}|u(t,\boldsymbol{x})|^p d\boldsymbol{x}\\ &\le \frac{1}{(\pi p^2)^{n/2}}\frac{2T^{1-n(p-1)/2}}{2-n(p-1)}\left(\int_0^T dt\int_{\mathbb{R}^n}\rho(t,\boldsymbol{x})d\boldsymbol{x}\right)^{p-1} \qquad (84)\\ &\quad\times\int_0^T dt\int_{\mathbb{R}^n}|F(t,\boldsymbol{x})|^p\rho(t,\boldsymbol{x})d\boldsymbol{x},\end{aligned}$$

for $\rho \in L_1([0,T]\times\mathbb{R}^n, dtd\boldsymbol{x})$ and $F \in L_p([0,T]\times\mathbb{R}^n, \rho(t,\boldsymbol{x})dtd\boldsymbol{x}), 1 < p < 1 + 2/n$.

Finally, let us consider the one-dimensional heat equation with variable conductivity of material in the form

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial v}{\partial x}\right] + q(x)v + F(t)\rho(t), \quad a \le x \le b, \ t > 0, \quad \rho \ge 0, \tag{85}$$

with the boundary conditions for $t > 0$

$$a_1 v(a,t) + a_2 v_x(a,t) = 0; \quad b_1 v(b,t) + b_2 v_x(b,t) = 0 \tag{86}$$

and the initial condition

$$v(x,0) = 0, \quad a < x < b. \tag{87}$$

We use the method of separation of variables to seek a solution of the equation (85) in the form (see [3, pp. 96-99])

$$v(x,t) = \sum_{n=1}^{\infty}\left[\phi_n(x)\int_a^b \phi(\xi)d\xi \int_0^t \exp\{\lambda_n(t-\tau)\}F(\tau)\rho(\tau)d\tau\right]. \tag{88}$$

Then, we have the inequality

$$\begin{aligned}\int_0^T |v(x,t)|^p dt \le \left(\int_0^T \rho(t)dt\right)^{p-1} \int_0^T |F(t)|^p \rho(t)dt \\ \times \left[\sum_{n=1}^{\infty}\left|\phi_n(x)\left(\frac{\exp\{p\lambda_n T\}-1}{p\lambda_n}\right)^{1/p}\int_a^b \phi_n(\xi)d\xi\right|\right]^p\end{aligned} \tag{89}$$

for all $T > 0$ and $\rho \in L_1([0,T]), F \in L_p([0,T], \rho(t)dt)$.

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