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SD-Groups and Embeddings

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Abstract. We show that every countable SD-group G can be subnormally embedded into a two-generator SD-group H. This embedding can have additional properties: if the group G is fully ordered then the group H can be chosen to also be fully ordered. For any non-trivial word set V this embedding can be constructed so that the image of G under the embedding lies in the verbal subgroup $V(H)$ of H.

Key words: Embeddings of groups, SD-groups, two-generator groups, ordered groups, generalized soluble groups. *Mathematics Subject Classification 2000*: 20F14, 20F19, 20E10, 20E15, 20F60

1. Results and Background Information

This survey article reflects our recent results on embedding of generalized soluble groups. See the literature cited in References for details and proof.

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In [KN_{65}] Kovács and Neumann considered some embedding properties of SN^*- and SI^*-groups (see definitions and references below). They in particular showed that every countable SN^*- or SI^*-group is embeddable into a two-generator SN^*- or SI^*-group respectively. The consideration of such embeddings of generalized soluble and generalized nilpotent groups was very natural after a series of results on embeddings into two-generator groups for soluble and nilpotent groups (see [NN_{59}, N_{68}, N_{60}, O_{89}, D_{68}] and literature cited therein). And, in general, interest in embeddings of countable groups into two-generator groups (with some additional properties or conditions) is explained by the famous theorem of Higman and Neumanns about embeddability of an arbitrary countable group into a two-generator group [HNN_{49}].

In [M_{02a}, M_{03b}] we considered embedding properties of a few other classes of generalized soluble and generalized nilpotent groups. In particular we saw that: **(a)** every countable SN-, SI-, $SN^{\grave{}}$- or $SI^{\grave{}}$-group is embeddable into a two-generator SN-, SI-, $SN^{\grave{}}$-, $SI^{\grave{}}$-group respectively; but **(b)** *not* every countable ZA- or N-group is embeddable into a two-generator ZA- or N-group respectively.

The first aim of the current paper is to consider similar problems for another popular class of generalized soluble groups: for SD-groups. That is, for groups in which the series of commutator subgroups:

$$(1) \qquad G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \cdots \geq G^{(\sigma)} \geq \cdots$$

reaches unity: $G^{(\rho)} = \{1\}$ for some finite or infinite ordinal ρ. As we will see, for any countable SD-group such an embedding into a two-generator SD-group is possible.

Moreover – and this is the second aim of this paper – the mentioned embedding can satisfy a few additional properties: the embedding can be *subnormal*, *verbal* and *fully ordered*. To have our statements in a precise form, let us first state our main theorem, and then turn to the background information about each of these three properties:

Theorem 1. (A) *Every countable SD-group G is subnormally embeddable into a two-generator SD-group H:*

$$\text{there exists } \gamma : G \to H \text{ such that } G \cong \gamma(G) \text{ and } \gamma(G) \lhd\lhd H.$$

(B) *For every non-trivial word set $V \subseteq F_\infty$ the two-generator SD-group H and the embedding γ can be chosen so that $\gamma(G)$ lies in the verbal subgroup $V(H)$ of the group H.*

(C) *Moreover, if the group G is fully ordered, then the group H can be chosen to also be fully ordered such that G is order isomorphic to its image $\gamma(G)$. Also, if the group G is torsion free, the group H can be chosen to be torsion free.*

1.1. The additional properties for embeddings. That the embedding of a general countable group into a two-generator group can be subnormal is proved by Dark in [D_{68}] (see also the paper of Hall [H_{74}]). In [M_{00}, M_{02a}, M_{03a}] we combined the subnormality of the embeddings of countable groups with other properties (see below).

The consideration of verbality of embeddings [of countable groups into two-generator groups] was initiated by the Neumanns in [NN_{59}], where they prove that every countable group G can be embedded not only into a two-generator group H but also into the second commutator subgroup $H^{(2)} = H''$ of the latter (the commutator subgroups simply are the special cases for verbal subgroups). In fact, here the second commutator subgroup could be replaced by any verbal subgroup $V(H)$ [M_{00}]. If the group G is a SN-, SI-, $SN\grave{}$- or $SI\grave{}$-group then the group H can be constructed to belong to the same class as

G [M_{02a}]. And as now Theorem 1 shows, the analog of this fact also is true for the class of SD-groups. Moreover, all these embeddings can be subnormal.

The problem whether a fully ordered countable group can be embedded into a fully ordered two-generator group was posed by Neumann, and solved by himself in [N_{60}]. In [M_{03a}] this property, too, was combined with subnormality and verbality for embeddings. Moreover, if the fully ordered countable group G is a SN-, SI-, SN^*- or SI^*-group then the fully ordered group H can be chosen in the same class [M_{02a}]. Again, Theorem 1 shows that the analog of this is true for the class of SD-groups.

1.2. An application of the argument. In [M_{03b}] we used the embedding construction of [M_{02a}] to build sets consisting of continuum of not locally soluble SI^*-groups. The reason why we devoted a paper to that topic was that so far in the literature there was only one example of the mentioned type. To be exact, there were two examples (independently built by Hall [H_{61}], and by Kovács and Neumann [KN_{65}]) presenting the *same* group. The examples built in [M_{03b}] were not only pairwise distinct groups, but, moreover, groups generating pairwise distinct varieties of groups.

Turning back to SD-groups, there is no lack of examples of SD-groups and, in particular, of not locally soluble SD-groups (consider, for example, any absolutely free group of rank greater than 1). However, we include here a scaled-down version of the construction of [M_{03b}] as an illustration of what can be obtained by means of the verbal embeddings of groups:

THEOREM 2. *There exists a continuum of torsion free, not locally soluble two-generator SD-groups, which generate pairwise different varieties of groups.*

1.3. References to the basic literature. An SN^*-group or an SI^*-group is a group possessing a soluble ascending subnormal or normal series respectively. In analogy with this, an $SN^{\grave{}}$- or $SI^{\grave{}}$-group is a group possessing a soluble descending subnormal or normal series respectively [S_{GA}]. More generally, an SN- or SI-group is a group possessing a soluble subnormal or normal (not necessarily well-ordered) system respectively. A ZA-group is a group with central ascending series. Finally, an N-group is a group in which every subgroup can be included in an ascending subnormal series. For information on the theory of generalized soluble and generalized nilpotent groups we refer to the original articles of Plotkin [P_{58}] and of Kurosh and Chernikov [KC_{47}] as well as to the books of Robinson [R_{FC}] and of Kurosh [K_{TG}]. For general information on varieties of groups we refer to the basic book of Hanna Neumann [N_{VG}]. Information on linearly (or fully) ordered groups can be found in the book of Fuchs [F_{OS}] or in the papers of Levi [L_{42}, L_{43}] and of Neumann [N_{49}, N_{60}].

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2. The Main Embedding Constructions

2.1. The subnormal embedding into a two-generator group. Let us begin with a simple but still very useful construction (see for example [NN_{59}, H_{61}]). Assume the given group G to be countable and its elements to be indexed by the set of non-negative integers:

$$G = \{g_0, g_1, ..., g_n, ...\}.$$

Then consider in the cartesian wreath product $G \,\mathrm{Wr}\, \langle f \rangle$ of the group G and of the infinite cyclic group generated by the element f the following element ω of the base subgroup $G^{\langle f \rangle}$:

$$\omega(f^i) = \begin{cases} g_k, & \text{if } \ i = 2^k, \ \ k = 0, 1, 2, ..., \\ 1, & \text{if } \ i \in \mathbb{Z} \backslash \{2^k \mid k = 0, 1, 2, ...\}. \end{cases}$$

Clearly, the element f acts on the base subgroup as a "right shift" operator. In particular: for arbitrary g_n we have $\omega^{f^{-2^n}}(1) = g_n$. Thus, for each pair $g_n, g_m \in G$ we have:

$$[\omega^{f^{-2^n}}, \omega^{f^{-2^m}}](1) = [g_n, g_m]. \tag{2}$$

Further for arbitrary $j \neq 0$:

$$[\omega^{f^{-2^n}}, \omega^{f^{-2^m}}](f^j) = 1. \tag{3}$$

Define $H = H(G) = \langle \omega, f \rangle$. Using this construction we can prove:

Lemma 1. *Let $H = H(G)$ be the above constructed two-generator group. Then there is an isomorphic embedding $\alpha : G' \to H$ of the commutator subgroup G' into H, such that $\alpha(G')$ is subnormal in H. Moreover, if the group G has any one of the properties:*

(1) *G is an SD-group,*
(2) *G is a fully ordered group,*
(3) *G is a torsion free group,*

then the group H also has the same properties (and if the group G is fully ordered, then G' is order-isomorphic to $\alpha(G')$).

Proof. For understandable reasons we can omit the case when G is a trivial group. For any pair $g_n, g_m \in G$ the embedding α can be defined as

$$\alpha([g_n, g_m]) = [\omega^{f^{-2^n}}, \omega^{f^{-2^m}}] \in H.$$

Equalities (2) and (3) show that this can be continued to an injection.

1. Assume G is an SD-group: $G^{(\delta)} = \{1\}$. Evidently H' lies in the normal closure $T = \langle\omega\rangle^H$ of $\langle\omega\rangle$ in H. As it is known (and can be calculated based on equalities (2) and (3)) the commutator T' is equal to the *direct* power $\overline{G'}^{\langle f\rangle}$ of the commutator G'. The direct power T' is an SD-group of length $\delta - 1$. Thus (taking into account the fact that the subgroups of SD-groups are SD-groups) we get that H is an SD-group of length $2 + \delta - 1 = \delta + 1$.

2. Assume the full order relation '$\prec$' is defined on G, and 'lift' it to the group H. It is easy to see that for any element $f^i\tau \in H$, where $f^i \in \langle f\rangle$ and $\tau \in H \cap G^{\langle f\rangle}$, there necessarily exists a corresponding maximal index $z_0(\tau) \in \mathbb{Z}$ such that $\tau(f^i) = 1$ for any $i \leq z_0(\tau)$. Thus we can compare any two distinct elements of H:

$$f^{i_1}\tau_1 \prec f^{i_2}\tau_2$$

if and only if $i_1 < i_2$, or if $i_1 = i_2$ and $\tau_1(f^{z_0+1}) \prec \tau_2(f^{z_0+1})$, where z_0 is the minimum of $z_0(\tau_1)$ and $z_0(\tau_2)$.

3. If the group G is torsion free, then the group H also is torsion free. □

2.2. The verbal subnormal embedding. Let $\mathfrak{A}$ be the variety of all abelian groups and $\mathfrak{V}$ be any variety different from the variety of all groups $\mathfrak{O}$ (for now the properties of this variety are immaterial, but later it will be replaced by special varieties).

LEMMA 2. *For any variety $\mathfrak{V} \neq \mathfrak{O}$ there exists a group N with the following properties:*

(1) *N is a torsion free SD-group;*
(2) *N generates a product variety $\mathfrak{V}_1\mathfrak{A}$, where $\mathfrak{V}_1 \neq \mathfrak{O}$ and $\mathfrak{V}_1$ is not properly contained in $\mathfrak{V}$ (in particular $N \notin \mathfrak{V}\mathfrak{A}$);*
(3) *N can be fully ordered.*

Proof. There are many methods to construct such a group. Let us outline one of them. Consider the relatively free nilpotent group $S = F_k(\mathfrak{N}_c)$ of some rank k and class $c \leq k$, such that $S \notin \mathfrak{V}$ (we always are able to find such a group because the set of all nilpotent groups, and even the set of all finite p-groups generates $\mathfrak{O}$). The group N we need can be constructed for S in a way rather similar to the construction of the group $H(G)$ for the group G in the previous subsection. Consider in the infinite cartesian power $S^{\langle z \rangle}$, that is, in the base subgroup of the cartesian wreath product $S \operatorname{Wr} \langle z \rangle$ (of the group S and of the infinite cyclic group $\langle z \rangle$) an element λ_s for each $s \in S$:

$$\lambda_s(z^i) = \begin{cases} s & \text{if } i \geq 0 \\ 1 & \text{if } i < 0 \end{cases}$$

and define $N = \langle \lambda_s, z | s \in S \rangle$. The group N contains the first copy S_0 of S in $S^{\langle z \rangle}$. In the sequel let us identify S with S_0 and use the same notations for their elements if no misunderstanding arises.

1. The group N clearly is a torsion free soluble group.

2. N belongs to $\mathfrak{V}_1\mathfrak{A} = \mathfrak{N}_c\mathfrak{A}$, and N contains a subgroup isomorphic to the direct wreath product $S \operatorname{wr} \langle z \rangle$. Thus, $\operatorname{var}(N) = \mathfrak{V}_1\mathfrak{A} = \mathfrak{N}_c\mathfrak{A}$.

3. Since the verbal subgroup $V(S)$ is not trivial (V is the word set corresponding to the variety $\mathfrak{V}$), we can choose an element $a \in V(S)$ of infinite order. On the free nilpotent group S a full order relation '$\prec$' can be defined as in [L_{42}, L_{43}, N_{49}]. Moreover, since in any group the full order can be replaced by its converse full order '$\prec^{-1}$' ($x \prec^{-1} y$ if and only if $y \prec x$), we can without loss of generality assume that the element a is positive: $1 \prec a$. Then according to the definition of full order, all the powers $a^2, a^3, \ldots, a^n, \ldots$ will also be positive elements. (This element a will be used later.) We 'lift' the full order relation '$\prec$'

of S to the group N. As in the proof of Lemma 1, for any element $z^i\tau \in N$, where $z^i \in \langle z\rangle$ and $\tau \in N \cap S^{\langle z\rangle}$, there necessarily exists a maximal index $z_0(\tau) \in \mathbb{Z}$ such that $\tau(z^i) = 1$ for any $i \le z_0(\tau)$. Thus we can put for any two distinct elements of N:

$$z^{i_1}\tau_1 \prec z^{i_2}\tau_2$$

if and only if $i_1 < i_2$, or if $i_1 = i_2$ and $\tau_1(z^{z_0+1}) \prec \tau_2(z^{z_0+1})$, where z_0 is the minimum of $z_0(\tau_1)$ and $z_0(\tau_2)$. $\square$

Now take a group G, a non-trivial word set V, and the group N constructed as above for the given V. Consider the cartesian wreath product $G \operatorname{Wr} N$ and select the following elements χ_g in the base group G^N of this wreath product:

$$\chi_g(n) = \begin{cases} g, & \text{if } n = a^i, \text{ for some positive integer } i \in \mathbb{N}, \\ 1, & \text{if } n \notin \{a^i \mid i \in \mathbb{N}\}, \end{cases}$$

where a is the element chosen above. Denote by $K = K(G, V)$ the following subgroup of $G \operatorname{Wr} N$:

$$K = \langle \chi_g, N \mid g \in G\rangle. \tag{4}$$

Denote by U the word set corresponding to the variety $\mathfrak{VA}$. In this notations the following holds:

Lemma 3. *Let $K = K(G, V)$ be the above constructed group for the group G and for the non-trivial word set V. Then there is an isomorphic embedding $\beta : G \to K$ of the group G into K, such that $\beta(G)$ is subnormal in K and lies in the verbal subgroup $U(K)$. Moreover, if the group G has any one of the properties:*

(1) *G is an SD-group,*
(2) *G is a fully ordered group,*
(3) *G is a torsion free group,*

then the group K also has the same properties (and if the group G is fully ordered, then it is order-isomorphic to $\beta(G)$).

Proof. Let π_g be the element of the first copy of G in G^N corresponding to $g \in G$:

$$\pi_g(n) = \begin{cases} g, & \text{if } n = 1 \\ 1, & \text{if } n \in N \backslash \{1\}. \end{cases}$$

The embedding β can be defined as

$$\beta(g) = \pi_g \text{ for all } g \in G.$$

Then $(\chi_g^{-1})^a \, \chi_g = \pi_g$, because it is easy to calculate that:

$$\left[\left(\chi_g^{-1}\right)^a \chi_g\right](n) = \begin{cases} 1, & \text{if } \ n \in N \backslash \{a^i \mid \ i = 0, 1, 2, ...\}, \\ g, & \text{if } \ n = 1 = a^0, \\ 1, & \text{if } \ n = a, a^2, a^3, ... \end{cases}$$

Since

$$a \in V(S) \subseteq U(N) \subseteq U(K),$$

we have $\pi_g = a^{-1}a^{\chi_g} \in U(K)$. Thus the mapping $g \to \pi_g$ defines an isomorphic embedding of G onto the first copy $\beta(G)$ of G in G^N, and in $U(K)$. Clearly, $\beta(G)$ is subnormal in K (and, in fact, even in $G \,\mathrm{Wr}\, N$).

1. If G is an SD-group: $G^{(\nu)} = \{1\}$, then $K^{(\nu+c+1)} = \{1\}$, where c is the nilpotency class of S (in fact, c could be replaced by a smaller integer, but it is immaterial for our purposes). For, clearly, $K' \subseteq \langle G^N, S^{\langle z \rangle} \rangle$, $K^{(c+1)} \subseteq G^N$, and $(G^N)^{(\nu)} \leq (G^{(\nu)})^N = \{1\}$. Notice that we could use this argument in the point 1 of the proof of Lemma 1. However, there we used a somewhat different argument to stress that in that case we deal with a direct (not cartesian) product.

2. Assume a full order relation '$\prec$' is defined on G. From the definition of the elements χ_g (and of the operation of elements of N on χ_g), it is clear that for any nontrivial element $n\theta \in K$, where $n \in N$ and $\theta \in K \cap G^N$, there necessarily exists an element $n_0(\theta) \in N$ with the following property:

$$\theta(n) = 1 \text{ for all } n \prec n_0(\theta), \text{ and } \theta(n_0(\theta)) \neq 1. \tag{5}$$

Now the full orders available on the groups G and N can be 'continued' to the group K. Let $n_1\theta_1$ and $n_2\theta_2$ be any two distinct elements of K. Then

$$n_1\theta_1 \prec n_2\theta_2$$

if and only if $n_1 \prec n_2$, or if $n_1 = n_2$ and $\theta_1(n_0) \prec \theta_2(n_0)$, where n_0 is the minimum of $n_0(\theta_1)$ and $n_0(\theta_2)$.

3. It is easy to see that if the group G is torsion free, then the group $G \operatorname{Wr} N$ and its subgroup K also are torsion free. $\square$

2.3. The proof of Theorem 1. Lemmas 1 and 3 already allow us to prove the statements of Theorem 1.

The proof of statement (A). Assume $G = G_0$ is a countable SD-group and $\mathfrak{V} = \mathfrak{E}$ is the trivial variety consisting of the group $\{1\}$ only. By Lemma 3 the group G can be subnormally embedded into an SD-group G_1 such that $\beta(G) \subseteq U(G_1) = G_1'$, where this time the word set U corresponds to the variety of abelian groups $\mathfrak{U} = \mathfrak{V}\mathfrak{A} = \mathfrak{A}$. It is easy to see that the group G_1 also is countable. Thus, by Lemma 1 the commutator subgroup G_1' can be subnormally embedded into a two-generator SD-group $G_2 = H(G_1)$, $\alpha : G_1' \to G_2$. The subnormal embedding we are looking for can be defined as the composition $\beta \cdot \alpha$. $\square$

The proof of statement (B). Assume $V \subseteq F_\infty$ is any non-trivial word set corresponding to the variety $\mathfrak{V}$. Again, by Lemma 3 the group G can be subnormally embedded into an SD-group $G_1 = K(G, V)$ such that $\beta(G) \subseteq U(G_1)$, where U corresponds to $\mathfrak{V}\mathfrak{A}$. By Lemma 1 the commutator subgroup G_1' can be subnormally embedded into some two-generator SD-group $G_2 = H(G_1)$, $\alpha : G_1' \to G_2$.

If we now show that

$$\beta(G) \subseteq V(G_1'), \tag{6}$$

the statement will be proved because:

a. $\alpha(\beta(G))$ is subnormal in G_2 (for, $\beta(G)$ is subnormal in G_1', and the latter is subnormal in G_2);

b. $\alpha(\beta(G))$ lies in $V(G_2)$ (for, $\alpha(\beta(G)) \subseteq \alpha(V(G_1')) \subseteq V(G_2)$).

To prove (6) we first notice that:

$$S_0 \subseteq N' \ \ (\subseteq G_1'),$$

where under S_0 we understand the first copy of S in $S^{\langle z \rangle}$. Indeed, we should simply apply the argument of the proof of Lemma 3 to see that S_0 lies in N'. We have:

$$a \in V(S_0) \subseteq V(G_1').$$

Therefore for any $g \in G$:

$$\pi_g = \beta(g) = a^{-1} a^{\chi_g} \in V(G_1'),$$

and we can again take: $\gamma = \beta\alpha$. $\square$

The proof of statement (C). This statement directly follows from Lemma 1, Lemma 3 and from the fact that a subgroup – in this case the commutator subgroup G_1' – of an SD-group (of a fully ordered group) is an SD-group (a fully ordered group). Also, if G is torsion free, the groups G_1 and G_2, evidently, also have that property. Theorem 1 is proved. $\square$

3. Continuum two-Generator SD-Groups

3.1. Bisections. We need a special set of groups constructed by Ol'shanskii in [O_{70}]. Namely, let $\{L_n \,|\, n \in \mathbb{N}\}$ be a countable set of finite groups with the following properties:

(1) $L_n \in \mathfrak{L}$, where $\mathfrak{L} \neq \mathfrak{O}$ is a soluble variety of finite exponent.
(2) $L_n \notin \mathrm{var}\,(L_1, \ldots, L_{n-1}, L_{n+1}, \ldots)$ for an arbitrary $n \in \mathbb{N}$,

In fact, as $\mathfrak{L}$ one can take the variety $\mathfrak{G}_5 \cap \mathfrak{B}_{8qr}$, $n = 1, 2, \ldots$, where $\mathfrak{G}_5$ is the variety of soluble groups of length at most 5; q, r are distinct primes; and $\mathfrak{B}_{8qr}$ is the Burnside variety of groups of exponents dividing $8qr$ [O_{70}].

Let us define a *bisection* (B) as follows. The set $\mathbb{N}$ of positive integers can be split as:

(B) $\qquad \mathbb{N}' \cup \mathbb{N}'' = \mathbb{N}, \quad \text{where } \mathbb{N}' \cap \mathbb{N}'' = \emptyset \text{ and } \mathbb{N}', \mathbb{N}'' \neq \emptyset.$

Then denote the group $L_{(\mathrm{B})}$ to be the direct product of the groups L_n, $n \in \mathbb{N}'$, the variety $\mathfrak{V}_{(\mathrm{B})}$ to be the variety generated by the groups L_n, $n \in \mathbb{N}'' = \mathbb{N}\backslash\mathbb{N}'$, and the word set $V_{(\mathrm{B})}$ to be that corresponding to $\mathfrak{V}_{(\mathrm{B})}$.

3.2. Construction of SD-groups with bisections (B)**.** Take any torsion free insoluble SD-group G not generating the variety $\mathfrak{O}$, and put:

$$G_{(\mathrm{B})} = F_\infty \left(\mathrm{var}\,(G)\ \mathrm{var}\left(L_{(\mathrm{B})}\right)\right). \tag{7}$$

It is easy to see that $G_{(\mathrm{B})}$ is an SD-group. Also, it is a torsion free group by theorem of Kovács about torsion free relatively free groups of product varieties [K_{79}]. Since $V_{(\mathrm{B})}$ clearly is a non-trivial word set, we are in a position to apply Theorem 1 to subnormally embed $G_{(\mathrm{B})}$ into an appropriate two-generator SD-group

$$H_{(\mathrm{B})} = H\big(G_{(\mathrm{B})}, V_{(\mathrm{B})}\big).$$

Since the set of all the bisections (B) is of continuum cardinality, to prove Theorem 2 it is sufficient to find continuum many bisections which determine groups $H_{(\mathrm{B})}$ generating pairwise distinct varieties of groups.

Consider another bisection

$$(\tilde{\mathrm{B}}) \qquad \tilde{\mathbb{N}}' \cup \tilde{\mathbb{N}}'' = \mathbb{N}, \quad \text{where } \tilde{\mathbb{N}}' \cap \tilde{\mathbb{N}}'' = \emptyset \text{ and } \tilde{\mathbb{N}}', \tilde{\mathbb{N}}'' \neq \emptyset.$$

different from (B) and consider the groups $G_{(\tilde{\mathrm{B}})}$ and $H_{(\tilde{\mathrm{B}})} = H\big(G_{(\tilde{\mathrm{B}})}, V_{(\tilde{\mathrm{B}})}\big)$ corresponding to this bisection $(\tilde{\mathrm{B}})$. Here the inequality $(\mathrm{B}) \neq (\tilde{\mathrm{B}})$, of course, simply means that $\mathbb{N}' \neq \tilde{\mathbb{N}}'$ (or, equivalently, $\mathbb{N}'' \neq \tilde{\mathbb{N}}''$). Clearly, $\mathrm{var}(L_{(\mathrm{B})}) \neq \mathrm{var}(L_{(\tilde{\mathrm{B}})})$. Moreover:

Lemma 4. *If* $(\mathrm{B}) \neq (\tilde{\mathrm{B}})$ *then* $\mathrm{var}(G_{(\mathrm{B})}) \neq \mathrm{var}(G_{(\tilde{\mathrm{B}})})$.

Proof. We have

$$\mathrm{var}(G_{(\mathrm{B})}) = \mathrm{var}\,(G)\ \mathrm{var}(L_{(\mathrm{B})})$$

and

$$\mathrm{var}(G_{(\tilde{\mathrm{B}})}) = \mathrm{var}\,(G)\ \mathrm{var}(L_{(\tilde{\mathrm{B}})}).$$

Then by [N_{VG}, Theorm 23.23], if $\mathrm{var}(G_{(\mathrm{B})}) = \mathrm{var}(G_{(\tilde{\mathrm{B}})})$, we have $\mathrm{var}(L_{(\mathrm{B})}) = \mathrm{var}(L_{(\tilde{\mathrm{B}})})$. This contradicts to the fact that $(\mathrm{B}) \neq (\tilde{\mathrm{B}})$ and to the selection of the group set $\{L_n \,|\, n \in \mathbb{N}\}$. $\square$

The next step of our construction is the group $K_{(\mathrm{B})} = K(G_{(\mathrm{B})}, V_{(\mathrm{B})})$ that we built for the group $G_{(\mathrm{B})}$ and the word set $V_{(\mathrm{B})}$. The set of all possible word sets (as well as the set of all varieties of groups) is of continuum cardinality. In fact, the continuum of distinct bisections (B) already provide us with continuum of distinct word sets $V_{(\mathrm{B})}$. This, however, does not mean that building the groups $K_{(\mathrm{B})}$ for continuum distinct words sets $V_{(\mathrm{B})}$ we will get *continuum* examples of distinct SD-groups $H_{(\mathrm{B})}$. The point is that the role of the word set V in K (and, thus, in H) is in determination of the class c and of the rank k of the free nilpotent group $S = F_k(\mathfrak{N}_c)$ that we are using to build the appropriate group N. The set of such integers is *countable*. This means that there exist many (in fact, continuum) word sets $V_{(\mathrm{B})}$ for which the same nilpotent group $S_{(\mathrm{B})}$ should be chosen. This also means that the construction, that we have at our disposal at the current moment, does not yet allow us to build a continuum of SD-groups using the fact about continuum set of words $V_{(\mathrm{B})}$ only.

But this observation also allows us to modify and shorten one of the segments of our proof. Namely, let us restrict ourselves to such a set (of continuum cardinality) of word sets $V_{(\mathrm{B})}$ which correspond to a fixed pair of integers c and k. It will be sufficient to prove that this set already can give rise to continuum two-generator (torsion free) SD-groups.

Lemma 5. *Assume* (B) *and* $(\tilde{\mathrm{B}})$ *are two distinct bisections of the type mentioned:* $S_{(\mathrm{B})} = S_{(\tilde{\mathrm{B}})}$. *If* $\mathrm{var}(G_{(\mathrm{B})}) \neq \mathrm{var}(G_{(\tilde{\mathrm{B}})})$ *then* $\mathrm{var}(K_{(\mathrm{B})}) \neq \mathrm{var}(K_{(\tilde{\mathrm{B}})})$.

Proof. As we saw in the proof of Lemma 3, for the given bisection (B) the group $K_{(\mathrm{B})}$ contains the first copy of $G_{(\mathrm{B})}$. That copy together with $N_{(\mathrm{B})}$ generates the direct wreath product $G_{(\mathrm{B})} \,\mathrm{wr}\, N_{(\mathrm{B})}$. Since the group $N_{(\mathrm{B})}$ discriminates the variety $\mathfrak{N}_c\mathfrak{A}$, the group $G_{(\mathrm{B})} \operatorname{wr} N_{(\mathrm{B})}$ generates $\operatorname{var}\left(G_{(\mathrm{B})}\right) \operatorname{var}\left(N_{(\mathrm{B})}\right) = \operatorname{var}\left(G_{(\mathrm{B})}\right) \mathfrak{N}_c\mathfrak{A} = \operatorname{var}\left(K_{(\mathrm{B})}\right)$. Thus, taking another bisection $(\tilde{\mathrm{B}})$ we would get $\operatorname{var}(K_{(\tilde{\mathrm{B}})}) = \operatorname{var}(G_{(\tilde{\mathrm{B}})})\mathfrak{N}_c\mathfrak{A}$ (recall that according to the remark proceeding this lemma, we can use the same variety $\mathfrak{N}_c$ for both bisections). The latter is distinct from $\operatorname{var}\left(G_{(\mathrm{B})}\right) \mathfrak{N}_c\mathfrak{A}$ whenever the bisections are distinct. $\square$

The final step of our argument is the construction of the two-generator group $H_{(\mathrm{B})} = H(K_{(\mathrm{B})})$.

Lemma 6. *If* $\operatorname{var}(K_{(\mathrm{B})}) \neq \operatorname{var}(K_{(\tilde{\mathrm{B}})})$ *then* $\operatorname{var}(H_{(\mathrm{B})}) \neq \operatorname{var}(H_{(\tilde{\mathrm{B}})})$.

Proof. The proof will immediately follow from the fact that

$$\operatorname{var}(H_{(\mathrm{B})}) = \operatorname{var}(K_{(\mathrm{B})} \operatorname{wr} \langle f \rangle) = \operatorname{var}(K_{(\mathrm{B})})\mathfrak{A}.$$

Let us take any non-identity $w = w(x_1, \ldots, x_n)$ for the variety $\operatorname{var}(K_{(\mathrm{B})})\mathfrak{A}$ and show that w can be falsified on some elements of $H_{(\mathrm{B})}$, as well. This will prove the point because $H_{(\mathrm{B})}$ evidently belongs to $\operatorname{var}(K_{(\mathrm{B})})\mathfrak{A}$. Take $c_1, \ldots, c_n \in K_{(\mathrm{B})} \operatorname{wr} \langle f \rangle$ such that $w(c_1, \ldots, c_n) = 1$. Clearly: $c_i = f^{m_i}\rho_i$, where ρ_i belongs to the base subgroup $K_{(\mathrm{B})}^{\langle f \rangle}$; $i = 1, \ldots, n$. Finitely many elements ρ_i in this direct wreath product have only finitely many non-trivial "coordinates" $\rho_i(f^j)$, $i = 1, \ldots, n$; $j \in \mathbb{Z}$. This means that there is a big enough positive integer n^* such that if we replace (trivial) "coordinates" $\rho_i(f^j)$, $|j| > n^*$ of each ρ_i by arbitrarily chosen values from the group $K_{(\mathrm{B})}$, and denote these new strings by ρ_i' correspondingly, then we will still have:

$$w(f^{m_1}\rho_1', \ldots, f^{m_n}\rho_n',) \neq 1.$$

Since all the powers $f^{m_1}, \ldots, f^{m_n}$ already belong to $H(K_{(\mathrm{B})})$, the proof will be completed if we show the following rather more general fact: for arbitrary positive integer n_0 and arbitrary pregiven values $d_j \in K_{(\mathrm{B})}$,

$j = -n_0, \ldots, n_0$ the group $H(K_{(\mathrm{B})})$ contains such an element $\rho'' \in H(K_{(\mathrm{B})}) \cap K_{(\mathrm{B})}^{\langle f \rangle}$ for which: $\rho''(f^j) = d_j$; $j = -n_0, \ldots, n_0$.

Taking into account the "shifting" effect of the element f, it will be sufficient to show that for any pregiven $d \in K_{(\mathrm{B})}$ there is an element $\rho'''_d \in H(K_{(\mathrm{B})}) \cap K_{(\mathrm{B})}^{\langle f \rangle}$ such that:

$$\rho'''_d(f^j) = \begin{cases} d & \text{if } j = 0 \\ 1 & \text{if } -2n_0 \leq j \leq 2n_0 \text{ and } j \neq 0. \end{cases}$$

(Notice that we did not put any requirements on $\rho'''(f^j)$ for $j > 2n_0$ or for $j < -2n_0$.) The elements ρ'' will then be products of elements of type ρ'''_d (for various d's) and of their conjugates by powers of f. It remains to construct elements ρ'''_d (for any d and n_0) by means of two generators $\omega_{(\mathrm{B})}$ and f. We have:

$$\omega_{(\mathrm{B})}(f^i) = \begin{cases} g_k, & \text{if } \ i = 2^k, \ \ k = 0, 1, 2, ..., \\ 1, & \text{if } \ i \in \mathbb{Z} \backslash \{2^k \mid k = 0, 1, 2, ...\}, \end{cases}$$

where this time the countable group $K_{(\mathrm{B})}$ is presented as $K_{(\mathrm{B})} = \{g_0, g_1, \ldots\}$. The element d can be presented as a product $g_i \cdot g_j$ for infinitely many pairs $g_i, g_j \in K_{(\mathrm{B})}$. On the other hand, the number of all possible pairs g_i, g_j with a common upper bound on $|i|$ and $|j|$ clearly is finite. Thus, there necessarily exists a pair g_i, g_j such that $d = g_i \cdot g_j$ and $2^i, 2^j > 2n_0$. Then:

$$\rho'''_d = \omega^{f^{-2^i}} \omega^{f^{-2^j}}.$$

$\square$

Lemmas 4, 5, 6 prove Theorem 2 because the continuum of two-generator torsion free SD-groups we constructed do generate pairwise distinct varieties of groups. And none of these groups is locally soluble because it would then be a soluble group, and, thus, it could not contain the initial group $G_{(\mathrm{B})}$ that was chosen to be insoluble.

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