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Existence of a solution in the Holder space for a nonlinear functional integral equation

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Abstract. This article is entirely devoted to the application of the measure of noncompactness defined in the Holder space. Here the emphasis is on the study of the nonlinear functional integral equation with changed arguments. Precisely, the existence of a solution is obtained by employing the Darbo fixed point theorem under certain hypotheses. Finally, we provide a tangible example which supports our results.

Key Words: Functional integral equation, Fixed point theory, Holder space
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Introduction

Nonlinear integral equations in abstract spaces have become an important topic of research. Specifically, solutions of nonlinear integral equations in various Banach spaces have been studied extensively by many researchers, e.g., [1, 5, 8, 9, 11], and so on. Recently, a few works appeared where the analysis has been done in Holder spaces (see, for example, [6]). In this work, we have attempted to study the following nonlinear functional integrodifferential equation

$$y(t) = f(t) + \int_a^b p(t,\zeta)h\Big(\zeta, y(\tilde{a}(\zeta)), y'(\tilde{b}(\zeta)), y''(\tilde{c}(\zeta)), y'''(\tilde{d}(\zeta))\Big)d\zeta, \qquad (1)$$

where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ are continuous nondecreasing functions from $[a,b]$ to $[a,b]$ in the Holder space $C^{k,\gamma}(\Omega)$ where Ω is a compact subset of R^N $(N \geq 1)$, $k \in \mathbb{N}$ and $\gamma \in (0,1]$.

$C^{k,\gamma}(\Omega)$ is a Banach space with the norm

$$\|v\|_{k,\gamma} := \|v\|_k + \Sigma_{|\alpha|=k}\|D^\alpha v\|_{H^\gamma} < \infty,$$

where

$$\|v\|_{H^\gamma} = \sup_{x,y\in\Omega, x\neq y} \frac{|v(x)-v(y)|}{|x-y|^\gamma} + |v(x_0)|$$

and x_0 is an arbitrary point in Ω. More details of this space can be found in [10].

1 Preliminaries

Suppose Y is a Banach space, M_Y is the collection of all nonempty and bounded subsets of Y, and N_Y is its subcollection comprising of relatively compact sets.

Definition 1 *[3] A function $\nu : M_Y \to R_+$ is said to be a measure of noncompactness in Y if it satisfies the following conditions:*
(i) $\ker \nu = \{A \in \mathcal{M}_Y : \mu(A) = 0\}$ is nonempty and $\ker \nu \subset \mathcal{N}_Y$;
(ii) From $A \subset B$ it follows that $\nu(A) \leq \nu(B)$;
(iii) $\nu(\bar{A}) = \nu(A)$;
(iv) $\nu(\mathrm{Conv}\, A) = \nu(A)$;
(v) $\nu(\lambda A + (1-\lambda)A) \leq \lambda\nu(A) + (1-\lambda)\nu(A)$ for $\lambda \in [0,1]$;
(vi) If the sequence $(Y_n) \in \mathcal{M}_Y$ of closed sets is such that $Y_{n+1} \subset Y_n$ for all $n \geq 1$ and $\lim_{n\to\infty} \nu(Y_n) = 0$, then $Y_\infty = \bigcap_{n=1}^\infty Y_n \neq \emptyset$.

Theorem 1 *[7] Suppose S is a nonempty, bounded, convex and closed subset of a Banach space Y, and let $\phi : S \to S$ is a continuous mapping such that $\nu(\phi X) \leq \lambda\nu(X)$ where $\lambda \in [0,1)$ and X is an arbitrary nonempty subset of S. Then ϕ has a fixed point in the set X.*

Now, we recall the notion of the measure of noncompactness on $C^{k,\gamma}(\Omega)$ given in [10]. To define such a measure, the author used the help of papers [2] and [4].

Let S be a bounded subset of $C^{k,\gamma}(\Omega)$ and let $\epsilon > 0$. For $g \in S$ we have

$$w_k(g,\epsilon) := \sup\{|D^\alpha g(x) - D^\alpha g(y)|; x,y \in \Omega, x \neq y, |x-y| \leq \epsilon, |\alpha| \leq k\};$$

$$w_{k,\gamma}(g,\epsilon) := \sup\left\{\frac{|D^\alpha g(x) - D^\alpha g(y)|}{|x-y|^\gamma}; x,y \in \Omega, x \neq y, |x-y| \leq \epsilon, |\alpha| = k\right\};$$

$$w_k(G,\epsilon) := \sup\nolimits_{g\in G} w_k(g,\epsilon), \qquad w_{k,\gamma}(G,\epsilon) := \sup\nolimits_{g\in G} w_{k,\gamma}(g,\epsilon).$$

Let us put

$$\eta_{k,\gamma}(G,\epsilon) := w_k(G,\epsilon) + w_{k,\gamma}(G,\epsilon).$$

The function $\epsilon \to \eta_{k,\gamma}(G,\epsilon)$ is nondecreasing on $(0,\infty)$, and thus, the limit $\lim\limits_{\epsilon\to 0} \eta_{k,\gamma}(G,\epsilon)$ exists.

Theorem 2 *[10] The function* $\eta_0^{k,\gamma} : M_{C^{k,\gamma}(\Omega)} \to [0,\infty)$ *given by* $\eta_0^{k,\gamma}(G) = \lim_{\epsilon\to 0} \eta^{k,\gamma}(G,\epsilon)$ *is a measure of noncompactness on* $C^{k,\gamma}(\Omega)$*.*

2 Main Results

In order to derive the existence results, we have to consider certain assumptions which are as follows:
(a1) For some $\theta \in (0,1]$, $f \in C^{3,\theta}([a,b])$.
(a2) For every $\zeta \in [a,b]$, $p(\cdot,\zeta) \in C^{3,\theta}([a,b])$.
(a3) h is a continuous function and for every $(\zeta,u,v,w,x) \in [a,b]\times R^4$, $h(\zeta,u,v,w,x) \le l(\zeta)N(|u|+|v|+|w|+|x|)$ where $l \in L^\infty([a,b])$ and N is a nondecreasing and continuous function on R.
(a4) There exists $r_0 > 0$ such that

$$\|f\|_{3,\theta} + l^+\overline{p}N(r_0) \le r_0$$

where

$$\overline{p} := \int_a^b |p(\cdot,\zeta)\|_{3,\theta}d\zeta \qquad \text{and} \qquad l^+ := \sup_{\zeta\in[a,b]} l(\zeta).$$

Theorem 3 *Under the assumptions* $(a1)-(a4)$*, equation (1) has at least one solution in* $C^{3,\gamma}([a,b])$ *for any* $\gamma \in (0,\theta)$*.*

Proof. For a fixed $\gamma \in (0,\theta)$ and for any $y \in C^{3,\gamma}([a,b])$, define Φy by:

$$(\Phi y)(t) = f(t) + \int_a^b p(t,\zeta)h\left(\zeta, y(\tilde{a}(\zeta)), y'(\tilde{b}(\zeta)), y''(\tilde{c}(\zeta)), y'''(\tilde{d}(\zeta))\right)d\zeta.$$

Step 1: Let us show that $\Phi y \in C^{3,\gamma}([a,b])$. We have

$$(\Phi y)'(t) = f'(t) + \int_a^b \frac{\partial p(t,\zeta)}{\partial t}h\left(\zeta, y(\tilde{a}(\zeta)), y'(\tilde{a}(\zeta)), y''(\tilde{c}(\zeta)), y'''(\tilde{d}(\zeta))\right)d\zeta,$$

$$(\Phi y)''(t) = f''(t) + \int_a^b \frac{\partial^2 p(t,\zeta)}{\partial t^2}h\left(\zeta, y(\tilde{a}(\zeta)), y'(\tilde{b}(\zeta)), y''(\tilde{c}(\zeta)), y'''(\tilde{d}(\zeta))\right)d\zeta,$$

and

$$(\Phi y)'''(t) = f'''(t) + \int_a^b \frac{\partial^3 p(t,\zeta)}{\partial t^3}h\left(\zeta, y(\tilde{a}(\zeta)), y'(\tilde{b}(\zeta)), y''\tilde{c}((\zeta)), y'''(\tilde{d}(\zeta))\right)d\zeta.$$

Thus,

$$
\begin{aligned}
&|(\Phi y)(t)| + |(\Phi y)'(t)| + |(\Phi y)''(t)| + |(\Phi y)'''(t)| \\
&\leq |f(t)| + |f'(t)| + |f''(t)| + |f'''(t)| \\
&+ \int_0^1 \left(|p(t,\zeta)| + \left|\frac{\partial p(t,\zeta)}{\partial t}\right| + \left|\frac{\partial^2 p(t,\zeta)}{\partial t^2}\right| + \left|\frac{\partial^3 p(t,\zeta)}{\partial t^3}\right| \right) \\
&\quad \times h\Big(\zeta, y(\tilde{a}(\zeta)), y'(\tilde{b}(\zeta)), y''(\tilde{c}(\zeta)), y'''(\tilde{d}(\zeta))\Big) d\zeta.
\end{aligned}
$$

Using $(a3)$, we get

$$\|\Phi y\|_{3,\gamma} \leq \|f\|_{3,\theta} + N(\|y\|_{3,\gamma}) \int_a^b \|p(\cdot,\zeta)\|_{3,\theta} l(\zeta) d\zeta < \infty.$$

Applying $(a4)$ for $\|y_0\|_{3,\gamma} = r_0$, we have $\|\Phi y\|_{3,\gamma} \leq r_0$, which implicates that Φ maps $Br_0 := \{y \in C^{3,\gamma}([a,b]); \|y\|_{3,\gamma} < r_0\}$ in Br_0.

Step 2: Now we show that $\Phi : Br_0 \to Br_0$ is a continuous map.

Suppose $u, v \in Br_0$ and $\epsilon > 0$. For $\|u - v\|_{3,\gamma} \leq \epsilon$, we have

$$
\begin{aligned}
\|\Phi u - \Phi v\|_{3,\gamma} &= \|\Phi u - \Phi v\|_\infty + \|(\Phi u - \Phi v)'\|_\infty \\
&\quad + \|(\Phi u - \Phi v)''\|_\infty + \|(\Phi u - \Phi v)'''\|_\infty + \|(\Phi u - \Phi v)'''\|_{H^\gamma} \\
&\leq \int_a^b \Big[\sup_{t\in[a,b]} \big(|p(t,\zeta)| + |\frac{\partial p(t,\zeta)}{\partial t}| + |\frac{\partial^2 p(t,\zeta)}{\partial t^2}| + |\frac{\partial^3 p(t,\zeta)}{\partial t^3}| \big) \\
&\quad + \sup_{t,s\in[a,b];t\neq s} \frac{|\frac{\partial^3 p(t,\zeta)}{\partial t^3} - \frac{\partial^3 p(s,\zeta)}{\partial t^3}|}{|t-s|^\theta} |t-s|^{\theta-\gamma} \Big] \\
&\quad \times |h(\zeta, u(\zeta), u'(\zeta), u''(\zeta), u'''(\zeta)) \\
&\qquad - h(\zeta, v(\zeta), v'(\zeta), v''(\zeta), v'''(\zeta))| d\zeta \\
&\leq \int_a^b \|p(\cdot,\zeta)\|_{3,\theta} |h(\zeta, u(\zeta), u'(\zeta), u''(\zeta), u'''(\zeta)) \\
&\quad - h(\zeta, v(\zeta), v'(\zeta), v''(\zeta), v'''(\zeta))| d\zeta.
\end{aligned}
$$

Hence,

$$\|\Phi u - \Phi v\|_{3,\gamma} \leq \bar{p}\sigma(\epsilon), \tag{2}$$

where $\sigma(\epsilon) = \sup\{|h(\zeta, u_1, u_2, u_3, u_4) - h(\zeta, v_1, v_2, v_3, v_4)|; \zeta \in [a,b], u_i, v_i \in [-r_0, r_0], |u_i - v_i| \leq \epsilon\}$.

Since $\|u-v\|_{3,\gamma} \leq \epsilon$, we have $\|u-v\|_\infty \leq \epsilon$, $\|u'-v'\|_\infty \leq \epsilon$, $\|u''-v''\|_\infty \leq \epsilon$ and $\|u''' - v'''\|_\infty \leq \epsilon$. Thus, by continuity of h it follows that $\sigma(\epsilon) \to 0$ as $\epsilon \to 0$. This implies the continuity of ϕ.

Step 3: Further, we show that Φ satisfies Darbo contraction.

Let G be a bounded nonempty subset of $C^{3,\gamma}([a,b])$ and let $\epsilon > 0$. Suppose $|t-s| \leq \epsilon$. For every $y \in G$ and $t, s \in [a,b]$, we have

$$|(\Phi y)(t) - (\Phi y)(s)| \leq |f(t) - f(s)| + \int_a^b |p(t,\zeta) - p(s,\zeta)| \times \left| h\left(\zeta, y(\tilde{a}(\zeta)), y'(\tilde{b}(\zeta)), y''(\tilde{c}(\zeta)), y'''(\tilde{d}(\zeta))\right) \right| d\zeta,$$

$$|(\Phi y)'(t) - (\Phi y)'(s)| \leq |f'(t) - f'(s)| + \int_a^b \left| \frac{\partial p(t,\zeta)}{\partial t} - \frac{\partial p(s,\zeta)}{\partial t} \right| \times \left| h\left(\zeta, y(\tilde{a}(\zeta)), y'(\tilde{b}(\zeta)), y''(\tilde{c}(\zeta)), y'''(\tilde{d}(\zeta))\right) \right| d\zeta,$$

$$|(\Phi y)''(t) - (\Phi y)''(s)| \leq |f''(t) - f''(s)| + \int_a^b \left| \frac{\partial^2 p(t,\zeta)}{\partial t^2} - \frac{\partial^2 p(s,\zeta)}{\partial t^2} \right| \times \left| h\left(\zeta, y(\tilde{a}(\zeta)), y'(\tilde{b}(\zeta)), y''(\tilde{c}(\zeta)), y'''(\tilde{d}(\zeta))\right) \right| d\zeta$$

and

$$|(\Phi y)'''(t) - (\Phi y)'''(s)| \leq |f'''(t) - f'''(s)| + \int_a^b \left| \frac{\partial^3 p(t,\zeta)}{\partial t^3} - \frac{\partial^3 p(s,\zeta)}{\partial t^3} \right| \times \left| h\left(\zeta, y(\tilde{a}(\zeta)), y'(\tilde{b}(\zeta)), y''(\tilde{c}(\zeta)), y'''(\tilde{d}(\zeta))\right) \right| d\zeta.$$

Hence,

$$w_3(\Phi y, \epsilon) \leq w_3(f, \epsilon) + \int_a^b w_3(p(\cdot,\zeta), \epsilon) |h(\zeta, y(\tilde{a}(\zeta)), y'(\tilde{b}(\zeta)), y''(\tilde{c}(\zeta)), y'''(\tilde{d}(\zeta)))| d\zeta. \quad (3)$$

Further,

$$w_{3,\gamma}(\Phi y, \epsilon) \leq \epsilon^{\theta-\gamma} \Big(w_{3,\theta}(f, \epsilon) + \int_a^b w_{3,\theta}(p(\cdot,\zeta)) \left| h\left(\zeta, y(\tilde{a}(\zeta)), y'(\tilde{b}(\zeta)), y''(\tilde{c}(\zeta)), y'''(\tilde{d}(\zeta))\right) \right| d\zeta \Big). \quad (4)$$

Therefore, using (3) and (4), we get

$$\eta_{3,\gamma}(\Phi G, \epsilon) \leq w_3(f, \epsilon) + \epsilon^{\theta-\gamma} w_{3,\theta}(f, \epsilon) + \overline{h}\Big[\int_a^b \Big(w_3(p(\cdot,\zeta), \epsilon) + \epsilon^{\theta-\gamma} w_{3,\theta}(\frac{\partial^3 p}{\partial t^3}(\cdot,\zeta) \Big) d\zeta \Big],$$

where $\overline{h} = \sup\{h(\zeta, y_1, y_2, y_3, y_4); \zeta \in [a,b], |y_i| \leq r_0\}$.

Now, let $\epsilon \to 0$. Since the functions f, f', f'' and f''' are uniformly continuous on the compact set $[a,b]$, we have $w_3(f,\epsilon) \to 0$. Analogously, since $p(t,\zeta)$, $\frac{\partial p}{\partial t}(t,\zeta)$, $\frac{\partial^2 p}{\partial t^2}(t,\zeta)$ and $\frac{\partial^3 p}{\partial t^3}(t,\zeta)$ are uniformly continuous on the compact set $[a,b] \times [a,b]$, we have $w_3(p(\cdot,\zeta),\epsilon) \to 0$; and, finally, since $\theta > \gamma$, we get $\eta^0_{3,\gamma}(\phi G) = 0$.

Hence, all the assumptions are satisfied, and by Theorem 1 we conclude that for every $\gamma < \theta$, equation (1) has a solution in $C^{3,\gamma}([a,b])$. $\square$

3 Numerical Example

In order to support the efficiency of our approach, let us consider the following functional integral equation:

$$y(t) = \frac{1}{8}\log(1+t) + \frac{1}{25}\int_0^1 \cos(t+\zeta^2)\frac{\zeta y(\zeta) + \zeta^2 y''(\zeta)e^{\zeta\cos(y'(\zeta))} + \zeta^3 y'''(\zeta)}{1 + y'^2(\zeta) + y'''^2(\zeta)}d\zeta.$$

Comparing it with Eq.(1), we have

$$f(t) = \frac{1}{8}\log(1+t), \qquad p(t,\zeta) = \cos(t+\zeta^2)$$

and

$$h(\zeta,u,v,w,x) = \frac{\zeta u + \zeta^2 w e^{\zeta\cos v} + \zeta^3 x}{25(1+v^2+x^2)}.$$

Here $\tilde{a}(\zeta) = \tilde{b}(\zeta) = \tilde{c}(\zeta) = \tilde{d}(\zeta) = \zeta$.

We see that $f, f', f'' \in C([0,1])$ and, moreover, $f''' \in H^1([0,1])$. Also, $f'''(t) = -0.75(1+t)^{-4}$, since $\max f^{(4)}(t) = 0.75 = m_0$. By the mean value theorem, for every $\tau_1, \tau_2 \in [0,1]$, we have

$$|f'''(\tau_1) - f'''(\tau_2)| \leq m_0|\tau_1 - \tau_2|,$$

and thus, $f'''(t)$ is a Holder continuous function. Now $\left|\frac{\partial^4 p}{\partial t^4}\right| \leq 1$, and from the mean value theorem, it follows that $\frac{\partial^3 p}{\partial t^3} \in H^1([0,1])$.

Moreover,

$$\begin{aligned} |h(\zeta,u,v,w,x)| &\leq \frac{1}{25}(\zeta + \zeta^2 e^{\zeta} + \zeta^3)\max\{|x|,|z|,|w|\} \\ &\leq \frac{1}{25}(\zeta + \zeta^2 e^{\zeta} + \zeta^3)(|u| + |w| + |x|) \\ &\leq \frac{1}{25}(\zeta + \zeta^2 e^{\zeta} + \zeta^3)(|u| + |v| + |w| + |x|). \end{aligned} \tag{5}$$

Thus, $l(\zeta) = 0.04(\zeta + \zeta^2 e^\zeta + \zeta^3)$ and $N(r) = r$, and, therefore, the hypothesis (a3) is fulfilled.

Further,

$$\begin{aligned} \|f\|_{3,1} &= \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty + \|f'''\|_\infty + \|f'''\|_{H^1} \\ &= \frac{1}{8}\log 2 + \frac{1}{8} + \frac{1}{8} + \frac{1}{4} + \frac{3}{4} \simeq 1.2876. \end{aligned}$$

Since $\|p\|_{3,1} = 5$, we have $\overline{p} = 5$. In addition, $l^+ = \dfrac{e+2}{25} \simeq 0.18873$. The inequality in assumption $(a4)$ takes the form $1.2876 + 5 \times 0.1887 r_0 \leq r_0$, which holds for $r_0 \geq 0.63$. Hence, by Theorem 3 the equation (4) in the Holder space $C^{3,\gamma}([0,1])$ has a solution for every $\gamma \in (0,1]$.

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