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About groups with property (U)

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Abstract. In this article we prove that for any odd $n \geq 1003$ there exist continuum non isomorphic (simple) groups with property (U), whose U- constant satisfies the inequality $u(G) < \frac{3n^4}{2}$. The question of finding infinite groups with property (U) was proposed in the joint work by D.Osin and D.Sonkin.

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Let G be an arbitrary group with finite set S of generators.

Definition 1. *We will say that a group G has property (U) (with respect to S) if there exists a number $M = M(G)$ such that for any generating set P of group G there exists an element $t \in G$ for which holds the inequality*

$$\max_{x \in S} |t^{-1}xt|_P \leq M. \tag{1.1}$$

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The lowest boundary of such numbers M that satisfy the inequality (1) relatively to some generating set S is called the U-constant of the given group G and is denoted by $u(G)$.

It is not hard to show that if the group has property (U) with respect to some S then it has property (U) relatively to any finite generating set. It is obvious that finite groups have property (U).

In the joint work [1] by D.Osin and D.Sonkin is proposed a question about finding infinite groups with property (U) and is proved that any non elementary hyperbolic torsion-free group has infinite quotient group with property (U) (by this result it follows the existence of finite generated group with non zero uniform Kazhdan constant).

In the paper [2], G.Arzhantseva proved that for given $k > 0$ elements in a word hyperbolic group G, there exists a number $M = M(G, k) > 0$ such that at least one of the assertions is true: (i) these elements generate a free and quasiconvex subgroup of G; (ii) they are Nielsen equivalent to a system of k elements containing an element of length at most M up to conjugation in G. The constant $M = M(G, k)$ can be calculated explicitly.

We are interested in the question of estimation of constant $u(G)$ of groups with property (U).

It is well known that for any odd $n \geq 1003$ in [3] there is constructed infinite 2-generated group any maximal subgroup of which is cyclic group of order n (such groups are called Tarskii "monsters").In [4] V.Atabekian proved that Adian-Lysenok groups from [3] have property (U).

Our goal is the following

Theorem 1. *For any odd $n \geq 1003$ there exists an n-periodic group with property U, whose U-constant satisfies the following inequality*

$$u(\Gamma) < \frac{3n^4}{2}. \tag{1.2}$$

As a corollary from this result and theorem 2 of article [4] we obtain

Corollary 1. *For any odd $n \geq 1003$ in the variety of groups that satisfy the identity $x^n = 1$ there exist continuum non isomorphic simple groups with property (U), for any one of which holds the inequality (2).*

In connection with Corollary 1, note that the above mentioned Osin-Sonin result implies the existence of at most a countable set of groups with the property (U), since the set of hyperbolic groups is countable. Corollary 1 also immediately implies the existence of a countable set of variety (variety of groups of simple exponent $n \geq 1003$) pairwise intersecting by the trivial variety, each containing a continuum set of non-isomorphic groups endowed with the property (U).

1. The proof of Theorem 1

In the further description we will use designations of works [3] and [4] without any references.

Consider the group

$$\Gamma \rightleftharpoons \left\langle a, b \mid E^n = 1, F = 1, R \in \bigcup_{\beta>0} \mathcal{E}_\beta,\ F \in \bigcup_{\beta>2} \Phi_\beta \right\rangle$$

(see definition (2.5)[4] and §1[3]). The main result of [3] implies that Γ is an infinite group and its any proper subgroup is contained in some cyclic subgroup of the order n.

Lemma 1. *If $X^\delta \overset{\Gamma}{=} T\, X^\varepsilon T^{-1}$, then the group $\langle X^\varepsilon, X^\delta, T\rangle_\Gamma$ is contained in some cyclic subgroup of the order n.*

Proof. Let X be an arbitrary element of the group Γ. It follows from the equality $X^\delta \overset{\Gamma}{=} T X^\varepsilon T^{-1}$ that $|X^\delta| = |X^\varepsilon| \le n$. Hence $\langle X^\delta \rangle_\Gamma = \langle X^\varepsilon \rangle_\Gamma$ and $|\left\langle X^\varepsilon,\, X^\delta,\, T \right\rangle_\Gamma| = |\langle X^\varepsilon, T\rangle| \leqslant n^2$. It follows from the inequality $|\langle X^\varepsilon, T\rangle| \leqslant n^2$ that $\langle X^\varepsilon, T\rangle$ is a proper subgroup of the group Γ, and consequently $\langle X^\varepsilon, T\rangle$ is a subgroup of a cyclic group of the order n.

Lemma 2. *If P is a generating set of the group Γ, then it contains a pair of non-commutable elements X and Y.*

Proof. Since Γ is a finitely generated group, it suffices to prove our statement only for a finite set P. If the elements of the set P happen to be pairwise commutable, then they would generate a finite subgroup $|\langle P\rangle|_\Gamma \le |P|^n$ (the identity $x^n = 1$ is true in Γ). This contradicts the fact that Γ is an infinite group.

Lemma 3. *Let $n \geqslant 1003$ be an arbitrary odd number. Then any noncyclic subgroup $\Delta \rightleftharpoons \langle X, Y\rangle$ of group Γ of work [3] contains such a noncyclic subgroup of type $U\,\langle A\,,\ C\rangle\, U^{-1}$ that $C \in \Psi_\alpha$ and*

$$UCU^{-1} \overset{\alpha-2}{=} [(UA^dU^{-1})\,,\, X^{-1}(UA^dU^{-1})X],$$

where A - minimized elementary period of some rank γ, $\gamma \leqslant \alpha - 2$ and the length of word UAU^{-1} relatively to generators X and Y satisfies to the inequality

$$|UAU^{-1}|_{\{X,Y\}} \le 2(n+1).$$

Proof. Let $\Delta \rightleftharpoons \langle X,\, Y\rangle_\Gamma$ - be an arbitrary noncyclic subgroup of group Γ. From VI.2.4[5], VI.1.2 [5] and Lemma 1 we have $X \overset{\Gamma}{=} T\,E^i\,T^{-1}\quad T^{-1} Y\, T \overset{\Gamma}{=} Z$ for some words T, Z and minimized elementary period E that has rank β. According to VI.2.4[5] and IV.1.13[5] we can take into consideration that $Z \in \mathcal{M}_\xi \cap \mathcal{A}_{\xi+1}$ for some $\xi \geqslant \beta$. Let g.c.d.$(i, n) = k$ and r is such an integer that $|r| < n\quad E^{i\,r} = E^k$. By choosing the number $s \rightleftharpoons [n/3k]$, we will obtain $n/5 < sk < n/3$. Thus $X^{rs} = T\,E^{irs}\,T^{-1} = T\,E^{ks}\,T^{-1}$ $186 < ks < \frac{n+1}{2} - 148$, as far as $n \geqslant 1003$. Thus, for the word $X_1 \rightleftharpoons X^{rs}$ we have $X_1 = T\,E^{ks}\,T^{-1}$

$$|X_1|_{\{X,Y\}} \leqslant \frac{n-1}{2}|X|_{\{X,Y\}} < \frac{n-1}{2},$$

because their is a number p such that $|p| \leq \frac{n-1}{2}$ and $X^{rs} = X^p$.

By taking into account the last inequality, then we repeat the proof of the lemma 3 of work [4] and instead of inequality (6)[4] we obtain (1.3)

$$|UAU^{-1}|_{\{X,Y\}} = |\left[X_1,\, Y^{-1}X_1Y\right]|_{\{X,Y\}} \leq \frac{4(n-1)}{2} + 4 = 2(n+1).$$

The lemma is proved.

Now lets turn on to the proof of theorem 1.

Choose $\{a,b\}$ as a generating set S and assume that P is an arbitrary generating set of the group Γ. By lemma 2, in P there exists pair of non-commuting elements X and Y. By lemma 3, for some U in subgroup $\langle X, Y\rangle$, one can choose elements UAU^{-1} and $UCU^{-1} \overset{\alpha-2}{=} [(UA^dU^{-1})\,,\, X^{-1}(UA^dU^{-1})X] \overset{0}{=} U[A^d\,,\; Z^{-1}A^dZ]U^{-1}$ where $Z = U^{-1}XU$, C is an elementary period of some rank α, A is elementary period of rank $\gamma \leq \alpha - 2$ $\;C \in \Psi_\alpha$. By point 2. of definition of the set $\overline{\Psi}_\alpha$[4] there exists a period $D \in \overline{\Psi}_\alpha$ which is conjugate whether with C or with C^{-1} and

$$D \overset{\alpha-2}{=} \left[E^d, Z_1^{-1}E^dZ_1\right],$$

where E is minimized elementary period of some rank β, $Z_1 \in \mathcal{M}_{\,\alpha-2}$ $\beta \leqslant \alpha - 2$. By lemma 4 the elementary period A is conjugate whether with E or with E^{-1}. Suppose $A = VE^\tau V^{-1}$ where $\tau = \pm 1$.

By (2), (3) [4] we have

$$a = D^{200}ED^{200}E^2 \cdot\ \cdot\ \cdot\ E^{n-1}D^{200},\ \ b = D^{300}ED^{300}E^2 \cdot\ \cdot\ \cdot\ E^{n-1}D^{300}.$$

Direct calculations show that

$$a^\tau = D^{\tau 200}ED^{\tau 200}E^2 \cdot \cdot \cdot\ E^{(n-1)}D^{\tau 200},\ \ b^\tau = D^{\tau 300}ED^{\tau 300}E^2 \cdot \cdot \cdot\ E^{(n-1)}D^{\tau 300}.$$

In consideration of the equality $A = VE^\tau V^{-1}$ we obtain

$$Va^\tau V^{-1} = (VDV^{-1})^{\tau 200}A^\tau(VDV^{-1})^{\tau 200}A^{\tau 2} \cdot\ \cdot\ \cdot\ A^{\tau(n-1)}(VDV^{-1})^{\tau 200}.$$

and therefor

$$(1.4)\ \ VaV^{-1} = (VDV^{-1})^{200}A(VDV^{-1})^{200}A^2 \cdot \cdot \cdot\ A^{(n-1)}(VDV^{-1})^{200}.$$

So, when $\tau = \pm 1$, the element

$$VDV^{-1} = V\left[E^d, Z_1^{-1}E^dZ_1\right]V^{-1} = \left[A^{\tau d}, (VZ_1V^{-1})^{-1}A^{\tau d}(VZ_1V^{-1})\right]$$

is conjugate with $\left[A^d, (VZ_1V^{-1})^{-1}A^d(VZ_1V^{-1})\right]$ and on the other hand with $[A^d,\ Z^{-1}A^dZ]$ or $[A^d,\ Z^{-1}A^dZ]^{-1}$ where $Z = U^{-1}XU$. At the first case by lemma 5[4] for some integers u and v we have $0 \le u, v \le n-1$ or

$$(1.5) \qquad (VZ_1V^{-1})^{-1}A^d(VZ_1V^{-1}) = A^uZ^{-1}A^dZA^v$$

or

$$(1.6) \qquad (VZ_1V^{-1})A^d(VZ_1V^{-1})^{-1} = A^uZ^{-1}A^{-d}ZA^v.$$

At the second case in (5) and (6) we just need to change Z by Z^{-1} and conversely because it is obvious that

$$\left[A^d,\ Z\,A^dZ^{-1}\right] \overset{0}{=} Z\left[A^d,\ Z^{-1}\,A^dZ\right]^{-1}Z^{-1}.$$

We will consider only the first case (in the second case in the proof we change the places of Z and Z^{-1}).

Suppose that (5) holds. Because

$$(UV)D(UV)^{-1} = U\left[A^{\tau d}, A^{u}Z^{-1}A^{\tau d}ZA^{v}\right]U^{-1} =$$
$$= UA^{-v}\left[A^{\tau d}, Z^{-1}A^{\tau d}Z\right]A^{v}U^{-1}$$
$$= UA^{-v}U^{-1}\left[(UA^{\tau d}U^{-1}), X^{-1}(UA^{\tau d}U^{-1})X\right]UA^{v}U^{-1}$$

(in the case of $\tau = -1$ we need to change places $\pm u \quad \mp v$) therefore from (3) we have

$$|UVDV^{-1}U^{-1}|_{\{X,Y\}} \le |v|(2n+2) + 4d\cdot(2n+2) + 4 + |v|(2n+2) =$$

$$= (2n+2)(|v|+4d)+4 \le 2n^2-2+4+8d(n+1) = 2n^2+1528n+1530.$$

Denote $M_1 \rightleftharpoons 2n^2 + 1528n + 1530$.

Therefore from (4) we obtain

$$|UVaV^{-1}U^{-1}|_{\{X,Y\}} \le 200n\cdot M_1 + (2n+2)\cdot\frac{n(n-1)}{2}.$$

By the same way $|UVbV^{-1}U^{-1}|_{\{X,Y\}} \le 200n\cdot M_1 + (2n+2)\cdot\frac{n(n-1)}{2}$.

Now lets suppose that (6) holds. Then

$$(VZ_1V^{-1})VDV^{-1}(VZ_1V^{-1})^{-1} = \left[(VZ_1V^{-1})A^{\tau d}(VZ_1V^{-1})^{-1}, A^{\tau d}\right] =$$
$$= \left[A^{u}Z^{-1}A^{-\tau d}ZA^{v}, A^{\tau d}\right]$$

(in the case $\tau = -1$ we need to change the places of $\pm u$ and $\mp v$) and

$$(UVZ_1)D(Z_1V^{-1})^{-1} = UA^{v}U^{-1}\left[X^{-1}UA^{-\tau d}U^{-1}X, UA^{\tau d}U^{-1}\right]UA^{-v}U^{-1}.$$

From the last equation by (3) we obtain

$$(UVZ_1)D((Z_1^{-1}V^{-1}U^{-1})|_{\{X,Y\}} \le 2n^2 + 1528n + 1530 = M_1.$$

Since g.c.d.$(d, n) = 1$ then for some integer s where $|s| < n$ we have

$$(VZ_1V^{-1})A(VZ_1V^{-1})^{-1} = (A^uZ^{-1}A^{-d}ZA^v)^s$$

and

(1.7)

$$U(VZ_1V^{-1})A(VZ_1V^{-1})^{-1}U^{-1} = ((UA^uU^{-1})X^{-1}(UA^{-d}U^{-1})X(UA^vU^{-1}))^s.$$

From inequality (3) and equality (7) it follows that for any natural number r, $1 \le r \le n-1$ holds the inequality

$$|U(VZ_1V^{-1})A^r(VZ_1V^{-1})^{-1}U^{-1}|_{\{X,Y\}} \le$$

(1.8)

$$\le \frac{n-1}{2}((2n+2)(|u|+1+|v|)+2) \le \frac{n-1}{2}((2n+2)n+2) = n(n^2-1)+1,$$

because for every natural number r their is an integer p such that $|p| \le \frac{n-1}{2}$ and for which holds equality $A^{-rds} = A^p$.

Conjugating both sides of equality (4) by UVZ_1V^{-1} and using (7), (8) we will finally obtain

$$|UVZ_1aZ_1^{-1}V^{-1}U^{-1}|_{\{X,Y\}} < 200n \cdot M_1 + n \cdot (n(n^2-1)+1).$$

In the same way we can show that

$$|UVZ_1bZ_1^{-1}V^{-1}U^{-1}|_{\{X,Y\}} < 200n \cdot M_1 + n \cdot (n(n^2-1)+1).$$

Note that for any element R of group Γ is valid $|R|_P \le |R|_{\{X,Y\}}$ because $\{X,Y\} \subseteq P$.

It is not difficult to show that if $n \geq 1003$ and $M_1 = 2n^2 + 1528n + 1530$, then

$$200n \cdot M_1 + n \cdot (n(n^2 - 1) + 1) < \frac{3n^4}{2}.$$

The theorem is proved.

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