Armenian Journal of Mathematics
Volume 1, Number 3, 2008, 7–13

Interlaced, modular, distributive and boolean bilattices

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Abstract. We will discuss the concepts of interlaced, modular, distributive , Boolean (DeMorgan) and Ginsberg bilattices with their characterizations by hyperidentities and superproducts.

Key words: hyperidentity , superproduct, variety, quasigroup, lattice, Boolean and DeMorgan algebras, bilattice, interlaced, modular, distributive , Boolean (DeMorgan) and Ginsberg bilattices.

Mathematics Subject Classification 2000: 08A70, 03C05, 03C85, 06D30, 06F99, 08C05, 08A62, 68N17, 20N05.

1.. Introduction

Let's recall that a hyperidentity is a second-order formula of the form:

$$\forall X_1, \ldots, X_m \,\forall x_1, \ldots, x_n \quad (w_1 = w_2),$$

where w_1, w_2 are words (terms) in the alphabet of functional variables $X_1, \ldots, X_m$ and objective variables $x_1, \ldots, x_n$. However hyperidentities

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*** Presented at the First International Algebra and Geometry Conference in Armenia, 16–20 May, 2007, Yerevan**

are usually presented without universal quantifiers. The hyperidentity $w_1 = w_2$ is said to be satisfied in the algebra $(Q; \Sigma)$ if this equality holds whenever every functional variable X_i is replaced by an arbitrary operation of the corresponding arity from Σ and every objective variable x_j is replaced by an arbitrary element from Q. The variety V of algebras is said to satisfy a certain hyperidentity, if this hyperidentity is satisfied in any algebra of V. In this case the hyperidentity is called a hyperidentity of variety V.

For example, if $Q(\cdot)$ is a distributive quasigroup, then an algebra $Q(\cdot, \backslash, /)$ satisfies the hyperidentities of distributivity:

$$X(x, Y(y, z)) = Y(X(x, y), X(x, z)), \tag{d_1}$$

$$X(Y(x, y), z) = Y(X(x, z), X(y, z)). \tag{d_2}$$

Moreover, if $Q(A)$ is a distributive quasigroup, then an algebra

$$Q\left(A, A^{-1}, {}^{-1}A, {}^{-1}(A^{-1}), ({}^{-1}A)^{-1}, A^{*}\right)$$

satisfies these hyperidentities.

The identity (set of identities) F is called solid if a quasigroup $Q(\cdot)$ satisfies F it follows,that an algebra $Q(\cdot, \backslash, /)$ satisfies a hyperidentity (set of hyperidentities) corresponding to F.

Open Problem: Characterize the solid identities (set of identities).

We consider the category of algebras with bihomomorphisms $(\varphi, \tilde{\psi})$ as morphisms, where:

$$\varphi\left[A(x_1, \ldots, x_n)\right] = (\tilde{\psi}A)\left(\varphi x_1, \ldots, \varphi x_n\right).$$

This category we denote by $\widetilde{Alg}$. Product in this category is called a superproduct of algebras. The "identity relations" of this category are hyperidentities.

Bilattices, algebras with two separate lattice structures, have found their main applications in the algebraization of systems of inference in artificial intelligence and logic programming. The two lattice orderings of a bilattice are viewed as respectively representing the relative degree of truth and knowledge of possible events. Bounded Bilattices were first introduced by M.Ginsberg and M.Fitting in 1988-90 ([**1**], [**2**]).

We will discuss the concepts of interlaced, modular, distributive, Boolean (DeMorgan) and Ginsberg bilattices with their characterizations by hyperidentities and superproducts.

2.. Some preliminary results on hyperidentities

For the proofs of the following result see [**3**] (also see [**4**], [**5**]).

Theorem A. *The variety of lattices satisfies the following hyperidentities:*

$$X(x,x)=x, \tag{1}$$

$$X(x,y)=X(y,x), \tag{2}$$

$$X(x,X(y,z))=X(X(x,y),z), \tag{3}$$

$$X(Y(X(x,y),z),Y(y,z))=Y(X(x,y),z). \tag{4}$$

Conversely, every hyperidentity of the variety of lattices is a consequence of hyperidentities (1), (2), (3), (4).

Theorem B. *The variety of modular lattices satisfies the following hyperidentities:* (1), (2), (3), (4) *and*

$$X(Y(x,X(y,z)),Y(y,z))=Y(X(x,Y(y,z)),X(y,z)). \tag{5}$$

Conversely, every hyperidentity of the variety of modular lattices is a consequence of hyperidentities (1), (2), (3), (4), (5).

Theorem C. *The variety of distributive lattices satisfies the following hyperidentities:* (1), (2), (3) *and* (d_1). *Conversely, every hyperidentity of the variety of distributive lattices is a consequence of hyperidentities* (1), (2), (3), (d_1).

THEOREM D. *The variety of Boolean algebras satisfies the following hyperidentities:* (1), (2), (3), (d_1) *and*

$$F(F(x)) = x, \tag{6}$$

$$X(F(x), y) = X(F(X(x, y)), y), \tag{7}$$

$$F(X(F(X(x, y)), F(X(x, F(y))))) = x. \tag{8}$$

Conversely, every hyperidentity of the variety of Boolean algebras is a consequence of hyperidentities (1), (2), (3), (d_1), (6), (7), (8).

Open Problem: Characterize the hyperidentities of variety of (associative) rings. For example, the following hyperidentity is satisfied in every ring:

$$X(X(Y(x, x), Y(x, x)), Y(X(x, x), X(x, x)))$$
$$= X(Y(X(x, x), X(x, x)), X(Y(x, x), Y(x, x))).$$

Historical Notes: Hyperidentities of associativity and distributivity was first considered in binary algebras with quasigroup operations in [**6**] and [**7**]. The general concept of hyperidentity in algebras is studied in monographs [**8**], [**9**].

Let $\mathcal{J}(\mathfrak{A})$ be the term (or polynomial) algebra of algebra $\mathfrak{A}$. We call the hyperidentities of algebra $\mathcal{J}(\mathfrak{A})$ term hyperidentities of $\mathfrak{A}$. Term hyperidentities were first considered in [**10**] and [**11**], and studied in monographs [**12**], [**13**].

Many related papers can be found in these references.

3.. Bilattices

An algebra $(B, \wedge, \vee, \cdot, +)$ with four binary operations is called a *bilattice* if both the reducts $B_1 = (B, \wedge, \vee)$ and $B_2 = (B, \cdot, +)$ are lattices. A bilattice is *bounded if both its reducts* B_1 and B_2 are bounded lattice. A bilattice is called *interlaced* if each of the four basic operations preserves both the ordering relations: $\leqslant_1$ in the lattice B_1 and $\leqslant_2$ in the lattice B_2. A bilattice is called *distributive* if it satisfies distributive hyperidentity (d_1). An interlaced bilattice is called *modular* if it satisfies the hyperidentity (5). Define a bilattice B with one unary operation $'$ to be *Boolean* (see [**14**]) if it is distributive and satisfies the hyperidentities

$$x'' = x,$$

$$X(x, X(y, y')') = x.$$

THEOREM E. *A bilattice is interlaced iff it satisfies the hyperidentities satisfied in the variety of lattices.A bilattice is modular iff it satisfies the hyperidentities satisfied in the variety of modular lattices.A bilattice is distributive iff it satisfies the hyperidentities satisfied in the variety of distributive lattices.A bilattice is Boolean iff it satisfies the hyperidentities satisfied in the variety of Boolean algebras.*

Let $(L_1, \wedge_1, \vee_1)$ and $(L_2, \wedge_2, \vee_2)$ be lattices. Let $B = L_1 \bowtie L_2$ be the *superproduct* of the lattices L_1 and L_2, i.e.

$$B = (L_1 \times L_2, \wedge, \vee, \cdot, +)$$

with basic operations defined by

$$\wedge = (\wedge_1, \vee_2),\ \ \vee = (\vee_1, \wedge_2),$$

$$\cdot = (\wedge_1, \wedge_2),\ \ + = (\vee_1, \vee_2).$$

THEOREM F. *An algebra $(B, \wedge, \vee, \cdot, +)$ is an interlaced bilattice iff it is isomorphic to the superproduct of two lattices.An algebra $(B, \wedge, \vee, \cdot, +)$ is a modular bilattice iff it is isomorphic to the superproduct of two modular lattices.An algebra $(B, \wedge, \vee, \cdot, +)$ is a distributive bilattice iff it is isomorphic to the superproduct of two distributive lattices.An algebra $(B, \wedge, \vee, \cdot, +)$ is a Boolean bilattice iff it is isomorphic to the superproduct of two Boolean algebras.*

The paper [**15**] contains a representation theorem for bounded interlaced bilattices with the unary negation introduced by Ginsberg and

Fitting . We now provide a general version of the representation theorem considering arbitrary (unbounded) interlaced bilattices with the unary negation.

The Ginsberg negation $'$ is defined on a bilattice $(B, \wedge, \vee, \cdot, +)$ as a unary operation satisfying the following identities

$$x'' = x, (x \wedge y)' = x' \vee y', (x \vee y)' = x' \wedge y',$$
$$(x + y)' = x' + y', (x \cdot y)' = x' \cdot y'$$

A bilattice with Ginsberg negation is called a *Ginsberg bilattice.*An interlaced (modular, distributive) bilattice with Ginsberg negation is called a Ginsberg interlaced (modular, distributive) bilattice.

A Ginsberg bilattice $B(L)$ is a superproduct $L \bowtie L$ with Ginsberg negation

$$(x, y)' = (y, x).$$

THEOREM G. *An algebra $(B, \wedge, \vee, \cdot, +,')$ is a Ginsberg interlaced bilattice iff it is isomorphic to the $B(L)$ for some lattice L.An algebra $(B, \wedge, \vee, \cdot, +,')$ is a Ginsberg modular bilattice iff it is isomorphic to the $B(L)$ for some modular lattice L.An algebra $(B, \wedge, \vee, \cdot, +,')$ is a Ginsberg distributive bilattice iff it is isomorphic to the $B(L)$ for some distributive lattice L.*

These results are extended to the DeMorgan bilattices. An algebra $(B, \wedge, \vee, \cdot, +,')$ with four binary and one unary operations is called a *DeMorgan bilattice* if both the reducts $B_1 = (B, \wedge, \vee,')$ and $B_2 = (B, \cdot, +,')$ are DeMorgan algebras and $(B, \wedge, \vee, \cdot, +)$ is a distributive bilattice.

Open Problem: Characterize the Ginsberg bilattices.

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