Armenian Journal of Mathematics
Volume 12, Number 4, 2020, 1–12

New refinements of the Jensen-Mercer inequality associated to positive n-tuples

M. Adil Khan[1] and J. Pečarić

Abstract. In this manuscript, we propose new refinements for the Jensen-Mercer as well as variant of the Jensen-Mercer inequalities associated to certain positive tuples. We give some related integral version and present applications for different means. At the end, further generalizations are given which are associated to m finite sequences.[2]

Key Words: Convex functions, Jensen-Mercer inequality, means
Mathematics Subject Classification 2010: 26D15

Introduction

The celebrated Jensen inequality states that if $[a, b]$ is an interval in $\mathbb{R}$, $y_j \in [a, b]$, $\zeta_j \in \mathbb{R}^+$ $(j = 1, 2, \ldots, n)$, and $\psi : I \to \mathbb{R}$ is a convex function, then

$$\psi\left(\frac{\sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma}{\sum_{\gamma=1}^{n} \zeta_\gamma}\right) \leq \frac{\sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma)}{\sum_{\gamma=1}^{n} \zeta_\gamma}. \tag{1}$$

Jensen's inequality is one of the fundamental inequalities in Mathematics, and it underlies many vital statistical concepts and proofs. Some important applications involve derivation of the AM-GM mean inequality, estimations for Shannon and Zipf-Mandelbrot entropies, the convergence property of the expectation maximization algorithm, positivity of Kullback-Leibler divergence, etc. [5, 4, 12, 6, 7]. Also this inequality has been applied to solve many problems in different fields of science and technology e.g engineering, physics, financial economics, computer science, etc.

There are several classical important inequalities which may be deduced from (1), for example Hölder, Levinson's, Ky Fan, Young's inequalities, etc.

[1]Corresponding author
[2]The publication was supported by the Ministry of Education and Science of the Russian Federation (the Agreement number No. 02.a03.21.0008.

Due to the great importance of this inequality, several researchers have focused on this inequality and derived its many improvements, refinements and extensions. The Jensen inequality also has been given for some other generalized convex functions such as s-convex, preinvex, h-convex, η-convex functions, etc. [10, 22, 15, 17, 13, 8, 9, 21, 23].

In 2003 Mercer proved the following variant of Jensen's inequality [16].

Theorem 1 *Let $\psi : [a,b] \to \mathbb{R}$ be a convex function and let $y_\gamma \in [a,b]$, $\zeta_\gamma \in \mathbb{R}^+$, $\gamma = 1, 2, ..., n$, $\bar{\zeta} = \sum_{\gamma=1}^{n} \zeta_\gamma$. Then*

$$\psi\left(a + b - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma\right) \leq \psi(a) + \psi(b) - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma). \tag{2}$$

The following variant of Jensen-Steffensen's inequality has been given in [1].

Theorem 2 *Let $\psi : [a,b] \to \mathbb{R}$ be a convex function and let $y_\gamma \in [a,b]$, $\zeta_\gamma \in \mathbb{R}$, $\zeta_\gamma \neq 0$, $\gamma = 1, 2, \ldots, n$, $\bar{\zeta} = \sum_{\gamma=1}^{n} \zeta_\gamma$. If $y_1 \leq y_2 \leq \cdots \leq y_n$ or $y_1 \geq y_2 \geq \cdots \geq y_n$ and*

$$\sum_{\gamma=1}^{n} \zeta_\gamma > 0, \qquad 0 \leq \sum_{\gamma=1}^{k} \zeta_\gamma \leq \sum_{\gamma=1}^{n} \zeta_\gamma, \quad k = 1, 2, \ldots, n, \tag{3}$$

then

$$\psi\left(a + b - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma\right) \leq \psi(a) + \psi(b) - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma). \tag{4}$$

For some of the results concerning to the Jensen-Mercer inequality, we recommend [2, 18, 20, 11, 19, 3, 14].

The concept of convexity has a great impact on our everyday lives, and there are various applications of this concept in business, industry, medicine, art, etc. The applications of the convexity in equilibrium of non-cooperative games and the problems of optimum allocation of resources are significant. Jensen's inequality is one of the most important results which holds for convex (concave) functions. The classical Jensen's inequality, the complete form of Jensen's inequality, and the generalized Jensen's inequality for convex functions are important results in theoretical and applied Mathematics. In 2003 Mercer proved an interesting variant of Jensen's inequality which has been studied by many mathematicians in recent years. The purpose of this paper is to study the Jensen-Mercer inequality and to present new refinements of the Jensen-Mercer inequality in discrete as well as integral case. Also, several applications have been given for different means.

1 Refinements

Theorem 3 *Let $\psi : [a,b] \to \mathbb{R}$ be a convex function and let $y_\gamma \in [a,b]$, $\zeta_\gamma, \eta_\gamma, \theta_\gamma \in \mathbb{R}^+$, $\eta_\gamma + \theta_\gamma = 1$ for each $\gamma \in \{1,2,\ldots,n\}$, and let $\bar{\zeta} = \sum_{\gamma=1}^{n} \zeta_\gamma$. Then*

$$\begin{aligned} &\psi\left(a+b-\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma\right) \\ &\quad\le \sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma\psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}\right) \\ &\qquad + \sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma\psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right) \\ &\quad\le \psi(a)+\psi(b)-\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma\psi(y_\gamma). \end{aligned} \tag{5}$$

Proof. Since $\eta_\gamma + \theta_\gamma = 1$ for $\gamma \in \{1,2,\ldots,n\}$, we can write

$$\begin{aligned} \psi\left(a+b-\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma\right) &= \psi\left(\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma(a+b-y_\gamma)\right) \\ &= \psi\left(\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)+\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)\right) \\ &= \psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}{\bar{\zeta}}\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}\right. \\ &\qquad \left.+\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}{\bar{\zeta}}\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right) \\ &\le \frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}{\bar{\zeta}}\psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}\right) \\ &\qquad +\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}{\bar{\zeta}}\psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right) \\ &\le \frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}{\bar{\zeta}}\left(\psi(a)+\psi(b)-\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma\psi(y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}\right) \\ &\qquad +\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}{\bar{\zeta}}\left(\psi(a)+\psi(b)-\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma\psi(y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right) \\ &= \psi(a)+\psi(b)-\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma\psi(y_\gamma). \end{aligned}$$

The first inequality holds due to the definition of convexity, and the second inequality holds due to Theorem 1. □

In the following theorem we present integral version of the above theorem.

Theorem 4 *Let $\psi : [a,b] \to \mathbb{R}$ be a convex function defined on the interval $[a,b]$. Let $p,u,v,g : [\alpha,\beta] \to \mathbb{R}$ be integrable functions such that $g(\omega) \in [a,b]$, $u(\omega), v(\omega), p(\omega) \in \mathbb{R}^+$ for all $\omega \in [\alpha,\beta]$, $v(\omega)+u(\omega)=1$, and let $P=\int_\alpha^\beta p(\omega)d\omega$. Then*

$$\begin{aligned}
&\psi\left(a+b-\frac{1}{P}\int_\alpha^\beta p(\omega)g(\omega)d\omega\right)\\
&\qquad\le \frac{1}{P}\int_\alpha^\beta u(\omega)p(\omega)d\omega\psi\left(\frac{\int_\alpha^\beta p(\omega)u(\omega)(a+b-g(\omega))d\omega}{\int_\alpha^\beta p(\omega)u(\omega)d\omega}\right)\\
&\qquad+\frac{1}{P}\int_\alpha^\beta p(\omega)v(\omega)d\omega\psi\left(\frac{\int_\alpha^\beta p(\omega)v(\omega)(a+b-g(\omega))d\omega}{\int_\alpha^\beta p(\omega)v(\omega)d\omega}\right)\\
&\le \psi(a)+\psi(b)-\frac{1}{P}\int_\alpha^\beta p(\omega)\psi(g(\omega))d\omega. \qquad (6)
\end{aligned}$$

If the function ψ is concave, then the reverse inequalities hold in (6).

In the following theorem we present a refinement of variant of Jensen-Steffensen's inequality:

Theorem 5 *Let $\psi : [a,b] \to \mathbb{R}$ be a convex function. Let $y_\gamma \in [a,b]$, $\zeta_\gamma, \eta_\gamma, \theta_\gamma \in \mathbb{R}$, $\zeta_\gamma\eta_\gamma, \zeta_\gamma\theta_\gamma \neq 0$ and $\eta_\gamma+\theta_\gamma=1$ for all $\gamma \in \{1,2,\ldots,n\}$, and let $\bar{\zeta}=\sum_{\gamma=1}^n \zeta_\gamma$. If $y_1 \le y_2 \le \cdots \le y_n$ or $y_1 \ge y_2 \ge \cdots \ge y_n$ and*

$$\sum_{\gamma=1}^n \zeta_\gamma\eta_\gamma > 0, \qquad 0 \le \sum_{\gamma=1}^k \zeta_\gamma\eta_\gamma \le \sum_{\gamma=1}^n \zeta_\gamma\eta_\gamma, \ \ k=1,2,\ldots,n, \qquad (7)$$

$$\sum_{\gamma=1}^n \zeta_\gamma\theta_\gamma > 0, \qquad 0 \le \sum_{\gamma=1}^k \zeta_\gamma\theta_\gamma \le \sum_{\gamma=1}^n \zeta_\gamma\theta_\gamma, \ \ k=1,2,\ldots,n, \qquad (8)$$

then

$$\begin{aligned}
\psi\left(a+b-\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^n \zeta_\gamma y_\gamma\right) &\le \frac{1}{\bar{\zeta}}\sum_{\gamma=1}^n \eta_\gamma\zeta_\gamma\psi\left(a+b-\frac{\sum_{\gamma=1}^n \zeta_\gamma\eta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma\eta_\gamma}\right)\\
&\quad+\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^n \zeta_\gamma\theta_\gamma\psi\left(\frac{\sum_{\gamma=1}^n \zeta_\gamma\theta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^n \zeta_\gamma\theta_\gamma}\right)\\
&\le \psi(a)+\psi(b)-\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^n \zeta_\gamma\psi(y_\gamma). \qquad (9)
\end{aligned}$$

If the function ψ is concave, then the reverse inequalities hold in (9).

Proof. Since $\eta_\gamma + \theta_\gamma = 1$ for all $\gamma \in \{1, 2, \dots, n\}$, we have

$$
\begin{aligned}
&\psi\left(a+b-\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma\right) = \psi\left(\frac{1}{\bar{\zeta}}\textstyle\sum_{\gamma=1}^{n}\zeta_\gamma(a+b-y_\gamma)\right)\\
&= \psi\left(\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)+\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)\right)\\
&= \psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}{\bar{\zeta}}\cdot\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}\right.\\
&\qquad\left.+\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}{\bar{\zeta}}\cdot\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right).
\end{aligned}
\tag{10}
$$

Note, that $\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma > 0$ and $\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma > 0$, and therefore $\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma + \sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma > 0$, which gives $\bar{\zeta} > 0$.

Further, by applying convexity of ψ on the right side of (10) and Theorem 2, one can obtain

$$
\begin{aligned}
\psi\left(\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma(a+b-y_\gamma)\right) &\leq \frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}{\bar{\zeta}}\psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}\right)\\
&\quad+\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}{\bar{\zeta}}\psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right)\\
&\leq \frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma\left(\psi(a)+\psi(b)-\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma\psi(y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}\right)\\
&\quad+ \frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma\left(\psi(a)+\psi(b)-\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma\psi(y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right)\\
&= \psi(a)+\psi(b)-\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma\psi(y_\gamma).
\end{aligned}
$$

□

Remark 1 *If we add (7) and* (8)*, then the variant of Jensen-Steffensen inequality conditions (3) will be obtained.*

2 Applications to means

Let $y_\gamma \in [a, b]$, $\gamma = 1, 2, .., n$, and let $\zeta_1, ..., \zeta_n > 0$ with $\bar{\zeta} = \sum_{\gamma=1}^{n}\zeta_\gamma$. Let $A_n(\mathbf{y};\boldsymbol{\zeta})$, $\tilde{A}_n(\mathbf{y};\boldsymbol{\zeta})$, $G_n(\mathbf{y};\boldsymbol{\zeta})$, $\tilde{G}_n(\mathbf{y};\boldsymbol{\zeta})$, $H_n(\mathbf{y};\boldsymbol{\zeta})$, $\tilde{H}_n(\mathbf{y};\boldsymbol{\zeta})$, and $M_n^{[r]}(\mathbf{y};\boldsymbol{\zeta})$,

$\tilde{M}_n^{[r]}(\mathbf{y};\boldsymbol{\zeta})$ denote the weighted arithmetic, geometric, harmonic, and power means defined as:

$$A_n(\mathbf{y};\boldsymbol{\zeta}) := \frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma, \quad \tilde{A}_n := a+b-\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma = a+b-A_n(\mathbf{y};\boldsymbol{\zeta}),$$

$$G_n(\mathbf{y};\boldsymbol{\zeta}) := \left(\prod_{\gamma=1}^{n} y_\gamma^{\zeta_\gamma}\right)^{\frac{1}{\bar{\zeta}}}, \quad \tilde{G}_n(\mathbf{y};\boldsymbol{\zeta}) := \frac{ab}{\left(\prod_{\gamma=1}^{n} y_\gamma^{\zeta_\gamma}\right)^{\frac{1}{\bar{\zeta}}}} = \frac{ab}{G_n(\mathbf{y};\boldsymbol{\zeta})},$$

$$H_n(\mathbf{y};\boldsymbol{\zeta}) := \frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma^{-1}, \quad \tilde{H}_n(\mathbf{y};\boldsymbol{\zeta}) := \left(a^{-1}+b^{-1}-H_n^{-1}(\mathbf{y};\boldsymbol{\zeta})\right)^{-1},$$

$$M_n^{[s]}(\mathbf{y};\boldsymbol{\zeta}) := \begin{cases} \left(\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma^{s}\right)^{\frac{1}{s}}, & s\neq 0,\\ G_n(\mathbf{y};\boldsymbol{\zeta}), & s=0,\end{cases}$$

$$\tilde{M}_n^{[s]}(\mathbf{y};\boldsymbol{\zeta}) := \begin{cases} \left(a^s+b^s-\left(M_n^{[s]}(\mathbf{y};\boldsymbol{\zeta})\right)^{s}\right)^{\frac{1}{s}}, & s\neq 0,\\ \tilde{G}_n(\mathbf{y};\boldsymbol{\zeta}), & s=0.\end{cases}$$

Also, assume that $\eta_\gamma,\theta_\gamma \in \mathbb{R}^+$ are such that $\eta_\gamma+\theta_\gamma = 1$ for each $\gamma \in \{1,2,\ldots,n\}$.

Under the above assumptions we give the following corollaries.

Corollary 1 *The following inequalities are valid*

$$\tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta}) \leq \sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma\tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\eta}) + \sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma\tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\theta}) \leq \tilde{A}_n(\boldsymbol{y};\boldsymbol{\zeta}). \tag{11}$$

Proof. Using the function $\psi(x) = \exp(x)$ and replacing a, b, and y_γ by $\ln a, \ln b$, and $\ln y_\gamma$ in (5), we get (11). □

Corollary 2 *By taking $a \to \frac{1}{a}, b \to \frac{1}{b}, y_\gamma \to \frac{1}{y_\gamma}$, in (11) we have*

$$\frac{1}{\tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta})} \leq \frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}{\tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\eta})} + \frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}{\tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\theta})} \leq \frac{1}{\tilde{H}_n(\boldsymbol{y};\boldsymbol{\zeta})}. \tag{12}$$

Corollary 3 *The following inequalities hold:*

$$\tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta}) \leq \left(\tilde{A}_n(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\eta})\right)^{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}\left(\tilde{A}_n(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\theta})\right)^{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma} \leq \tilde{A}_n(\boldsymbol{y};\boldsymbol{\zeta}). \tag{13}$$

Proof. Using the function $f(x) = -\ln x$ in (5) and then applying the exponential function, we get (13). □

Corollary 4 *For $s \neq 0$ and $s \leq 1$, we have*

$$\tilde{M}_n^{[s]}(\boldsymbol{y};\boldsymbol{\zeta}) \leq \sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma\tilde{M}_n^{[s]}(\boldsymbol{y};\boldsymbol{\zeta.\eta})+\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma\tilde{M}_n^{[s]}(\boldsymbol{y};\boldsymbol{\zeta.\theta}) \leq \tilde{A}_n(\boldsymbol{y};\boldsymbol{\zeta}). \quad (14)$$

Proof. Use the function $\psi(x) = x^{\frac{1}{s}}$ and replace a, b, and x_i by a^s, b^s, and x_γ^s respectively in (5) to get (14). □

Corollary 5 *For $t, s \in \mathbb{R}$ with $0 < t \leq s$, we have*

$$\begin{aligned}\left(\tilde{M}_n^{[t]}(\boldsymbol{y};\boldsymbol{\zeta})\right)^s &\leq \sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma\left(\tilde{M}_n^{[t]}(\boldsymbol{y};\boldsymbol{\zeta.\eta})\right)^s+\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma\left(\tilde{M}_n^{[t]}(\boldsymbol{y};\boldsymbol{\zeta.\theta})\right)^s \\ &\leq \tilde{M}_n^{[s]}(\boldsymbol{y};\boldsymbol{\zeta}).\end{aligned} \quad (15)$$

Proof. Use the function $\psi(x) = x^{\frac{s}{t}}$ and replace a, b, and x_γ by a^t, b^t, and x_γ^t respectively in (5) to get (15). □

Now, we generalize the given applications for the *quasi-arithmetic mean.*

Let $\phi : [a, b] \to \mathbb{R}$ be a strictly monotonic and continuous function. Then for a given n-tuple $\mathbf{y} = (y_1, ..., y_n) \in [a, b]^n$ and positive n-tuple $\boldsymbol{\zeta} = (\zeta_1, ..., \zeta_n)$ with $\bar{\zeta} = \sum_{\gamma=1}^{n}\zeta_\gamma$, the value

$$M_\phi^{[n]}(\mathbf{y};\boldsymbol{\zeta}) = \phi^{-1}\left(\sum_{\gamma=1}^{n}\zeta_\gamma\phi(y_\gamma)\right)$$

is well defined and is called *quasi-arithmetic mean* of $\mathbf{y}$ with wight $\boldsymbol{\zeta}$. If we define

$$\tilde{M}_\phi^{[n]}(\mathbf{y};\boldsymbol{\zeta}) = \phi^{-1}\left(\phi(a)+\phi(b)-\sum_{\gamma=1}^{n}\zeta_\gamma\phi(y_\gamma)\right),$$

then we have the following result.

Corollary 6 *The following inequalities hold:*

$$\begin{aligned}\psi\left(\tilde{M}_\phi^{[n]}(\boldsymbol{y};\boldsymbol{\zeta})\right) &\leq \sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma\psi\left(\tilde{M}_\phi^{[n]}(\boldsymbol{y};\boldsymbol{\zeta.\eta})\right)+\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma\psi\left(\tilde{M}_\phi^{[n]}(\boldsymbol{y};\boldsymbol{\zeta.\theta})\right) \\ &\leq \psi\left(\tilde{M}_\psi^{[n]}(\boldsymbol{y};\boldsymbol{\zeta})\right)\end{aligned} \quad (16)$$

provided that $\psi \circ \phi^{-1}$ is convex and ψ is strictly increasing.

Proof. Replace $\psi(x)$ by $\psi \circ \phi^{-1}(x)$ and a, b, y_γ by $\phi(a), \phi(b), \phi(y_\gamma)$ respectively in (5), then apply ψ^{-1} to get (16). □

3 Further generalizations

In this section, we present a further refinement of the Jensen-Mercer as well as variant of the Jensen-Mercer inequalities concerning to m sequences whose sum is equal to unity.

Theorem 6 *Let $\psi : [a,b] \to \mathbb{R}$ be a convex function defined on the interval $[a,b]$. Let $y_\gamma \in [a,b]$, $\zeta_\gamma, \theta_\gamma^l \in \mathbb{R}^+$, $\gamma = 1,2,\ldots,n$, $l = 1,2,\ldots,m$, be such that $\sum_{l=1}^m \theta_\gamma^l = 1$ for each $\gamma \in \{1,2,\ldots,n\}$, and let $\bar{\zeta} = \sum_{\gamma=1}^n \zeta_\gamma$. Assume that L_1 and L_2 are non-empty disjoint subsets of $\{1,2,...,m\}$ such that $L_1 \cup L_2 = \{1,2,...,m\}$. Then*

$$\begin{aligned}\psi\left(a+b\right. &- \left.\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma\right)\\ &\leq \frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma\psi\left(\frac{\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma}\right)\\ &\quad+\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma\psi\left(\frac{\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma}\right)\\ &\leq \psi(a)+\psi(b)-\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma\psi(y_\gamma).\end{aligned} \tag{17}$$

If the function ψ is concave, then the reverse inequalities hold in (17).

Proof. Since $\sum_{l=1}^m \theta_\gamma^l = 1$ for each $\gamma \in \{1,2,\ldots,n\}$, we can write

$$a+b-\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma = \frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)+\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma).$$

Therefore, we have

$$\begin{aligned}&\psi\left(a+b-\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma\right)\\ &\quad=\psi\left(\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)+\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)\right)\\ &\quad=\psi\left(\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma\frac{\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma}\right.\\ &\qquad\left.+\frac{1}{\bar{\zeta}}\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma\frac{\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma}\right)\end{aligned}$$

$$\leq \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^{n} \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma \psi \left(\frac{\sum_{\gamma=1}^{n} \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma (a+b-y_\gamma)}{\sum_{\gamma=1}^{n} \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma} \right)$$
$$+ \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^{n} \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma \psi \left(\frac{\sum_{\gamma=1}^{n} \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma (a+b-y_\gamma)}{\sum_{\gamma=1}^{n} \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma} \right)$$
$$\leq \psi(a) + \psi(b) - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma). \tag{18}$$

The first inequality takes place due to the definition of convexity, and the second one holds by virtue of Jensen-Mercer's inequality. □

Remark 2 *We can give applications of Theorem 6 for means as given in Section 2.*

The following theorem is the integral analogue of Theorem 6.

Theorem 7 *Let $\psi : \mathtt{G} \to \mathbb{R}$ be a convex function defined on the interval $\mathtt{G}$. Let $p, g, u_l \in L[a,b]$ be such that $g(\omega) \in \mathtt{G}$, $p(\omega), u_l(\omega) \in \mathbb{R}^+$ for all $\omega \in [a,b]$ $(l = 1, 2, ..., n)$, and let $\sum\limits_{l=1}^{n} u_l(\omega) = 1$, $P = \int_a^b p(\omega) d\omega$. Assume that L_1 and L_2 are non-empty disjoint subsets of $\{1, 2, ..., n\}$ such that $L_1 \cup L_2 = \{1, 2, ..., n\}$. Then*

$$\frac{1}{P} \int_a^b p(\omega) \psi(g(\omega)) d\omega$$
$$\geq \frac{1}{P} \int_a^b \sum_{l \in L_1} u_l(\omega) p(\omega) d\omega \psi \left(\frac{\int_a^b \sum_{l \in L_1} u_l(\omega) p(\omega) g(\omega) d\omega}{\int_a^b \sum_{l \in L_1} u_l(\omega) p(\omega) d\omega} \right)$$
$$+ \frac{1}{P} \int_a^b \sum_{l \in L_2} u_l(\omega) p(\omega) d\omega \psi \left(\frac{\int_a^b \sum_{l \in L_2} u_l(\omega) p(\omega) g(\omega) d\omega}{\int_a^b \sum_{l \in L_2} u_l(\omega) p(\omega) d\omega} \right)$$
$$\geq \psi \left(\frac{1}{P} \int_a^b p(\omega) g(\omega) d\omega \right). \tag{19}$$

If the function ψ is concave, then the reverse inequalities hold in (19).

In the following theorem, we present a further refinement of the variant of Jensen-Steffensen's inequality associated to m certain sequences.

Theorem 8 *Let $\psi : [a,b] \to \mathbb{R}$ be a convex function defined on the interval $[a,b]$. Let $y_\gamma \in I$, let $\zeta_\gamma, \theta_\gamma^l \in \mathbb{R}$, $\gamma = 1, 2, \ldots, n$, $l = 1, 2, \ldots, m$, be such that $\zeta_\gamma \eta_\gamma, \zeta_\gamma \theta_\gamma^l \neq 0$ and $\sum_{l=1}^{m} \theta_\gamma^l = 1$ for each $\gamma \in \{1, 2, \ldots, n\}$, and let $\bar{\zeta} = \sum_{\gamma=1}^{n} \zeta_\gamma$. Assume that L_1 and L_2 are non-empty disjoint subsets of*

$\{1,2,...,m\}$ such that $L_1 \cup L_2 = \{1,2,...,m\}$. If $y_1 \leq y_2 \leq \cdots \leq y_n$ or $y_1 \geq y_2 \geq \cdots \geq y_n$ and for each $l \in \{1,2,\ldots,m\}$

$$\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma^l > 0, \qquad 0 \leq \sum_{\gamma=1}^{k} \zeta_\gamma \theta_\gamma^l \leq \sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma^l, \ \ k = 1,2,\ldots,n, \tag{20}$$

then

$$\begin{aligned} \psi\Bigg(a+b \ &- \ \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma\Bigg) \\ &\leq \ \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^{n} \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma \psi\left(\frac{\sum_{\gamma=1}^{n} \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma (a+b-y_\gamma)}{\sum_{\gamma=1}^{n} \sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma}\right) \\ &\quad + \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^{n} \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma \psi\left(\frac{\sum_{\gamma=1}^{n} \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma (a+b-y_\gamma)}{\sum_{\gamma=1}^{n} \sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma}\right) \\ &\leq \ \psi(a) + \psi(b) - \frac{1}{\bar{\zeta}} \sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma). \end{aligned} \tag{21}$$

If the function ψ is concave, then the reverse inequalities hold in (21).

Proof. Since $\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma^l > 0$ for each $l \in \{1,2,\ldots,m\}$, one has $\sum_{l=1}^{m} \sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma^l > 0$. Also, $\sum_{l=1}^{m} \theta_\gamma^l = 1$. Hence, we can conclude that $\bar{\zeta} > 0$.

Further, proceeding in the same way as in the proof of Theorem 6 but using variant of Jensen-Steffensen's inequality instead of Jensen-Steffensem's inequality, we obtain (21). □

References

[1] S. Abramovich, M. Klaričić Bakula, M. Matić, and J. Pečarić, *A variant of Jensen-Steffensen's inequality and quasi-arithmetic means,* J. Math. Anal. Appl., **307**(2005), pp. 370–386.

[2] M. Adil Khan, A. R. Khan, and J. Pečarić, *On the refinements of Jensen-Mercer's inequality*, Revue D'Analyse Numerique Et De Theorie De L'Approximation, **41**(2012), no. 1, pp. 62–81.

[3] M. Adil Khan, G. Ali Khan, M. Jamil, K. A. Khan, and A. Kilicman, *New refinements of Jensen-Mercer inequality,* Journal of Computational and Theoretical Nanoscience, **12**(2015), pp. 4408–4414.

[4] M. Adil Khan, Đ. Pečarić, and J. Pečarić, *Bounds for Shannon and Zipf-mandelbrot entropies,* Math. Methods Appl. Sci., **40**(2017), no. 18, pp. 7316–7322.

[5] M. Adil Khan, Đ. Pečarić, and J. Pečarić, *On Zipf-Mandelbrot entropy*, J. Comput. Appl. Math., **346**(2019), pp. 192–204.

[6] M. Adil Khan, M. Hanif, Z. A, Khan, K. Ahmad, and Yu-Ming Chu, *Association of Jensen inequality for s-convex function*, J. Inequal. Appl., (2019), Article ID 162, 14 pages.

[7] M. Adil Khan, Z. M. Al-sahwi, and Yu Ming Chu, *New estimations for Shannon and Zipf-Mandelbrot entropies*, Entropy, **20**(2018), no. 608, pp. 1–10.

[8] M. Adil Khan, S. Zaheer Ullah, and Y. Chu, *The concept of coordinate strongly convex functions and related inequalities*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A, Matematicas, **113**(2019), pp. 2235–2251.

[9] M. Adil Khan, J. Khan, and J. Pečarić, *Generalization of Jensen's and Jensen-Steffensen's inequalities by generalized majorization theorem*, J. Math. Inequal., **11**(2017), no. 4, pp. 1049–1074.

[10] M. K. Bakula and K. Nikodem, *Converse Jensen inequality for strongly convex set-valued maps*, J. Math. Inequal., **12**(2018), no. 2, pp. 545–550.

[11] J. Barić and A. Matković, *Bounds for the normalized Jensen-Mercer functional*, J. Math. Inequal., **3**(2009), no. 4, pp. 529–541.

[12] A. P. Dempster, N. M. Laird, and D. B. Rubin, *Maximum Likelihood from Incomplete Data via the EM Algorithm*, Journal of the Royal Statistical Society, Series B, (1977), pp. 1–38.

[13] J. Khan, M. Adil Khan, and J. Pečarić, *On Jensen's type inequalities via generalized majorization Inequalities*, Filomat, **32**(2018), no. 16, pp. 5719–5733.

[14] K. A. Khan, M. Adil Khan, and U. Sadaf, *New refinement of Jensen-Mercer's operator inequality and applications to means*, Punjab Univ. J. Math., **49**(2017), no. 2, pp. 127–151.

[15] J. Mićić Hot and Y. Seo, *An interpolation of Jensen's inequality and its converses with applications to quasi-arithmetic mean inequalities*, J. Math. Inequal., **12**(2018), no. 2, pp. 303–313.

[16] A. McD. Mercer, *A variant of Jensen's inequality*, JIPAM, **4** (2003), Article 73.

[17] R. Mikić, Đ. Pečarić, and J. Pečarić, *Inequalities of the Jensen and Edmundson-Lah-Ribarić type for 3-convex functions with applications*, J. Math. Inequal., **12**(2018), no. 3, pp. 677–692.

[18] A. Matković and J. Pečarić, *Refinements of the Jensen-Mercer inequality for index set functions with applications*, Revue d'analyse numérique et de théorie de l'approximation, **35**(2006), no. 1, pp. 71–82.

[19] A. Matković, *Generalization of the Jensen–Mercer inequality by Taylor's polynomial*, Math. Inequal. Appl. **19**(2016), no. 4, pp. 1387–1398.

[20] A. Matković and J. Pečarić, *Note on generalization of the Jensen–Mercer inequality by Taylor's polynomial*, Math. Inequal. Appl., **22**(2019), no. 4, pp. 1379–1384.

[21] H. R. Moradi, M. E. Omidvar, M. Adil Khan, and K. Nikodem, *Around Jensen's inequality for strongly convex functions*, Aequationes Mathematicae, **92**(2018), no. 1, pp. 25–37.

[22] M. Sababheh, *Improved Jensen's inequality*, Math. Inequal. Appl., **20**(2017), no. 2, pp. 389–403.

[23] Ying-Qing Song, M. Adil Khan, S. Zaheer Ullah, and Y. Ming-Chu, *Integral inequalities for strongly convex functions*, Journal of function Spaces, (2018), Article ID 6595921, pp. 1–8.

Muhammad Adil Khan
Department of Mathematics,
University of Peshawar,
Peshawar 25000, Pakistan.
adilswati@gmail.com, madilkhan@uop.edu.pk

Josip Pečarić
RUDN University,
Miklukho-Maklaya str. 6,
117198 Moscow, Russia.
pecaric@element.hr

Please, cite to this paper as published in
Armen. J. Math., V. **12**, N. 4(2020), pp. 1–12