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On the distribution of primitive roots that are (k, r)-integers

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Abstract. Let k and r be fixed integers with $1 < r < k$. A positive integer is called r-free if it is not divisible by the r^{th} power of any prime. A positive integer n is called a (k, r)-integer if n is written in the form $a^k b$ where b is an r-free integer. Let p be an odd prime and let $x > 1$ be a real number. In this paper an asymptotic formula for the number of (k, r)-integers which are primitive roots modulo p and do not exceed x is obtained.

Key Words: (k, r)-integer, primitive root
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1 Introduction and result

The problem of counting primitive roots in a given set is a topic in analytic number theory. Let $prim(x)$ be the number of positive primitive roots modulo a fixed prime p that are $\leq x$. In [8] Shapiro showed that

$$prim(x) = \frac{\phi(p-1)}{p-1}\left(x + O(p^{1/2}(\log p)^{1/2}2^{\omega(p-1)})\right),$$

where the O is uniform in x and p, $\phi(n)$ is Euler's function, and $\omega(n)$ denotes the number of distinct prime divisors of n. Moreover, Shapiro proved that

$$prim(x, k, l) \quad = \quad \frac{\phi(p-1)}{p-1}\left(\frac{x}{k} + O(p^{1/2}(\log p)^{1/2}2^{\omega(p-1)})\right),$$

where $prim(x, k, l)$ is the number of positive primitive roots modulo a fixed prime p, that are $\leq x$ and $\equiv l \pmod{k}$. Shapiro showed that the number of positive square-free primitive roots modulo a fixed prime p that are $\leq x$ equals

$$\frac{\phi(p-1)}{p-1}\left(\prod_{p}\left(1-\frac{1}{p^2}\right)x + O(x^{1/2}p^{1/4}(\log p)^{1/2}2^{\omega(p-1)})\right), \tag{1}$$

and the number of positive square-full primitive roots modulo a fixed prime p that are $\leq x$ is

$$\frac{\phi(p-1)}{p-1}\left(cx^{1/2}+O(x^{1/3}p^{1/6}(\log p)^{1/3}2^{\omega(p-1)})\right), \tag{2}$$

where

$$c=2\left(1-\frac{1}{p}\right)\sum_{(q|p)=-1}\frac{\mu^2(q)}{q^{3/2}},$$

$(q|p)$ being *Legendre's symbol.* Later, Liu and Zhang [3] improved (1) and showed that the number of positive square-free primitive roots modulo a fixed prime p that are $\leq x$ equals

$$\frac{p\phi(p-1)}{(p^2-1)\zeta(2)}x+O(x^{1/2+\epsilon}p^{9/44+\epsilon}). \tag{3}$$

Since the Euler product of $\zeta(2)=\prod_p(1-p^{-2})^{-1}$, the main term of (1) and (3) are identical. Recently, Munsch and Trudgian [5] improved (2) and showed that the number of positive square-full primitive roots modulo a fixed prime p that are $\leq x$ equals

$$\frac{\phi(p-1)}{p-1}\left(\left(\frac{p^2}{p^2+p+1}\right)\frac{C_p x^{1/2}}{\zeta(3)}+O(x^{1/3}(\log x)p^{1/9}(\log p)^{1/6}2^{\omega(p-1)})\right), \tag{4}$$

where $C_p \gg p^{-\frac{1}{8\sqrt{e}}}$. Very recently, the first author used the same method as in this paper to improve (4) and proved in [11] that for a given odd prime $p\leq x^{1/5}$ the number of positive square-full primitive roots modulo p that are $\leq x$ equals

$$\frac{\phi(p-1)}{p}\left\{\left(\frac{L(3/2,\chi_0)-L(3/2,\chi_1)}{L(3,\chi_0)}\right)x^{1/2}+\right.$$

$$\left.+\left(\frac{L(2/3,\chi_0)-L(2/3,\chi_2^2)}{L(2,\chi_0)}\right)x^{1/3}\right\}+O\left(\phi(p-1)3^{\omega_{1,3}(p-1)}p^{1/2+\epsilon}x^{1/6}\right);$$

here χ_0, $\chi_1\neq\chi_0$, and $\chi_2\neq\chi_0$ denote, respectively, the principal, quadratic and cubic characters modulo p. The terms with the cubic characters $\chi_2\neq\chi_0$ occur if $3|p-1$. The symbol $\omega_{1,3}(n)$ denotes the number of distinct primes $q\equiv 1 \pmod 3$ which are divisors of n. For a complex number $s=\sigma+it$, let $L(s,\chi)$ denote the Dirichlet L-function defined by $L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}$, $\sigma>1$.

It is natural to ask for generalized r-free primitive roots. (A positive integer n is called r-full if for all primes $p|n$, we have $p^r|n$). In this paper we study the distribution of the number of positive primitive roots modulo a fixed odd prime p that are (k, r)-integers. Subbarao and Suryanarayana remarked in [9] that in the case k tends to ∞ an (∞, r)-integer is the same as an r-free integer. One might consider the (k, r)-integers as generalized r-free integers. This is a motivation of this paper.

A positive integer n is called semi r-free if in the canonical factorization of n no exponent is equal to r. The (k, r)-integers also include the semi r-free integers when $k = r + 1$. Thus, our paper also includes those semi r-free integers that are primitive roots. The method used here is the same as in the proof of Theorem 2.1 in [10].

Let $prim_{(k,r)}(x)$ be the number of positive primitive roots modulo a fixed odd prime p that are (k, r)-integer and do not exceed x. Our main result is

Theorem 1 *For a given odd prime p and real $x \geq p^{k+1}$, we have*

$$prim_{(k,r)}(x) = \frac{\phi(p-1)}{p} \frac{L(k, \chi_0)}{L(r, \chi_0)} x +$$

$$+ O\Big(x^{1/r} \phi(p-1) 2^{\omega(p-1)} p^{1/2+\epsilon} \times \exp\left(-B \log^{3/5} x (\log\log x)^{-1/5}\right)\Big),$$

where B is a positive constant depending on r and k.

Theorem 1 contains the results for square-free integers (1) and (3) as an $(\infty, 2)$-integer. This follows from the fact that $L(k, \chi_0)$ tends to 1 as k tends to ∞ and the identity $L(2, \chi_0) = \zeta(2)(1 - p^{-2})$.

Throughout this paper ϵ denotes a fixed positive constant, not necessarily the same in all occurrences. As usual, let $\mu(n)$, $\phi(n)$, and $\omega(n)$ denote the Möbius function, the Euler-phi function, and the number of prime factors of n, respectively. Let $\psi(x) = x - \lfloor x \rfloor - 1/2$. For $r = 1, 2, ...$ the exponent pair is

$$(k_r, l_r) = \left(\frac{1}{2} - \frac{r+1}{2(2\Lambda - 1)}, \frac{1}{2} + \frac{1}{2(2\Lambda - 1)}\right), \qquad \Lambda = 2^r.$$

2 Prerequisites

In this section we state and prove lemmas which are needed in our proof.

Lemma 1 *[See Lemma 8.5.1 in [8]] For a given odd prime p, the characteristic function of the primitive root modulo p is*

$$\frac{\phi(p-1)}{p-1} \sum_{d|p-1} \frac{\mu(d)}{\phi(d)} \sum_{\chi \in G_d} \chi(n) = \begin{cases} 1, & \text{if } n \text{ is a primitive root mod } p, \\ 0, & \text{otherwise,} \end{cases}$$

where G_d denotes the set of characters of the character group modulo p that are of order d.

Lemma 2 *[See Lemma 2.6 in [9]] If $q_{k,r}$ denotes the characteristic function of the set of (k,r)-integers, then*

$$q_{k,r}(n) = \sum_{a^k b^r c = n} \mu(b).$$

Lemma 3 *[See Lemma 2 in [6]] Let ω und κ be two real numbers satisfying $\omega > 0$ snd $0 < \kappa \neq 1$. Then*

$$\sum_{n \le \omega} n^{-\kappa} = \zeta(\kappa) - \frac{1}{\kappa - 1}\omega^{1-\kappa} - \psi(\omega)\omega^{-\kappa} + O(\omega^{-\kappa-1}).$$

Lemma 4 *[See Lemma 8 in [6]] Let α, β, γ be three positive real numbers, and let (k,l) be an exponent pair with $k > 0$. Then for $x \ge 2$*

$$\sum_{n \le x^\alpha} \psi\left(\frac{x^\beta}{n^\gamma}\right) = O(x^{\alpha - \frac{1}{2}(\beta - \alpha\gamma)}) + \begin{cases} O\big(x^{\frac{\alpha l + (\beta - \alpha\gamma)k}{k+1}}\big) & \text{for } l > \gamma k, \\ O\big(x^{\frac{\beta k}{k+1}} \log x\big) & \text{for } l = \gamma k, \\ O\big(x^{\frac{\beta k}{1 + (1+\gamma)k - l}}\big) & \text{for } l < \gamma k. \end{cases}$$

Lemma 5 *[See Lemma 13 in [7]] If $f(n)$ is an arithmetic function, then*

$$\sum_{n \le \omega, (n,q)=1} f(n) = \sum_{d|q} \mu(d) \sum_{m \le \frac{\omega}{d}} f(md).$$

Lemma 6 *[See Lemma 14 in [7]] For $\alpha > 0, \alpha \neq 1$, and $0 < \beta \le 1$, we have*

$$\sum_{\substack{n \le X \\ n \equiv l \pmod q}} n^{-\alpha} = q^{-\alpha}\zeta\left(\alpha, \frac{l}{q}\right) + \frac{1}{1-\alpha} \cdot \frac{X^{1-\alpha}}{q} - \psi\left(\frac{X - l}{q}\right) X^{-\alpha} + O(qX^{-\alpha-1}),$$

where

$$\zeta(\alpha, \beta) = \sum_{n=0}^{\infty} (n + \beta)^{-\alpha}.$$

Lemma 7 *[See Lemma 17 in [7]] Let x, η, α, ω be real numbers, let j and q be positive numbers, where $x \ge 1, \alpha > 0, \eta \ge 1, 1 \le j \le q$, let (k,l) be an exponent pair with $k > 0$, and let*

$$R(x, \eta, \alpha; q, j; \omega) = \sum_{\substack{n \le \eta: \\ n \equiv j \pmod q}} \psi\left(\frac{x}{n^\alpha} + \omega\right),$$

if ω is independent of n. Then

$$R(x, \eta, \alpha; q, j; \omega) =$$

$$= O(1) + O(x^{-1/2}\eta^{1+\frac{\alpha}{2}}q^{-1}) + \begin{cases} O\big(x^{\frac{k}{k+1}}\eta^{\frac{l-\alpha k}{k+1}}q^{\frac{-l}{k+1}}\big) & \text{for } l > \alpha k, \\ O\big(x^{\frac{k}{k+1}} \log \eta q^{\frac{-\alpha k}{k+1}}\big) & \text{for } l = \alpha k, \\ O\big((xq^{-\alpha})^{\frac{k}{1+(1+\alpha)k - l}}\big) & \text{for } l < \alpha k, \end{cases}$$

where the O-constants depend on α only.

Throughout this paper we apply Lemmas 4 and 7 with the exponent pair $(2/7, 4/7)$.

Lemma 8 *Let χ be a Dirichlet character modulo p, let χ_0 denote the principal character, let $L(s,\chi)$ be the associated Dirichlet L-function, let ϵ be a fixed positive number, and let Γ_k be the set of all non principal characters modulo p order d where $d|k$. Then for $p \leq x^{1/(k+1)}$ we have*

$$\frac{p}{p-1} \sum_{\substack{m \leq x \\ m \text{ is } (k,r)\text{-integer}}} \chi_0(m) = \frac{L(k,\chi_0)}{L(r,\chi_0)} x +$$

$$+ O\left(x^{1/r} p^{1/2+\epsilon} \exp\left(-B \log^{3/5} x (\log\log x)^{-1/5}\right)\right);$$

if there exist characters $\chi_1 \in \Gamma_k$, then

$$\frac{p}{p-1} \sum_{\substack{m \leq x \\ m \text{ is } (k,r)\text{-integer}}} \chi_1(m) =$$

$$= O\left(x^{1/r} p^{3/2+\epsilon} \exp\left(-B \log^{3/5} x (\log\log x)^{-1/5}\right)\right);$$

if there exist characters $\chi_2 \neq \chi_0$ and $\chi_2 \notin \Gamma_k$, then

$$\frac{p}{p-1} \sum_{\substack{m \leq x \\ m \text{ is } (k,r)\text{-integer}}} \chi_2(m) = O\left(x^{1/r} p^{3/2} \exp\left(-B \log^{3/5} x (\log\log x)^{-1/5}\right)\right),$$

where B is a positive constant depending on r and k.

Proof. Let χ be any Dirichlet character modulo p. For $x \geq p^{k+1}$, let

$$T_\chi(x;k,r) := \sum_{\substack{n \leq x \\ n \text{ is } (k,r)\text{-integer}}} \chi(n).$$

In view of Lemma 2, we have

$$T_\chi(x;k,r) := \sum_{a^k b^r c \leq x} \mu(b) \chi(a^k b^r c).$$

Let $z = x^{1/r}$ and $0 < \rho(x) < 1$ (we will choose $\rho(x)$ later). We write

$$T_\chi(x;k,r) =$$

$$= \sum_{\substack{a^k b^r c \leq x \\ b \leq \rho z}} \mu(b)\chi(a^k b^r c) + \sum_{\substack{a^k b^r c \leq x \\ a^k c \leq 1/\rho^r}} \mu(b)\chi(a^k b^r c) - \sum_{\substack{b \leq \rho z \\ a^k c \leq 1/\rho^r}} \mu(b)\chi(a^k b^r c).$$

We bound the second and the last terms as

$$\sum_{\substack{a^kb^rc\le x\\ a^kc\le 1/\rho^r}}\mu(b)\chi(a^kb^rc)=\sum_{a^kc\le 1/\rho^r}\chi(a^kc)\sum_{b\le\sqrt[r]{(x/a^kc)}}\mu(b)\chi(b^r)\ll$$

$$\ll\sum_{a^kc\le 1/\rho^r}\left|\sum_{b\le\sqrt[r]{(x/a^kc)}}\mu(b)\chi(b^r)\right|\ll\zeta(k)\rho^{1-r}x^{1/r}$$

and

$$\sum_{\substack{b\le\rho z\\ a^kc\le 1/\rho^r}}\mu(b)\chi(a^kb^rc)=\sum_{a^kc\le 1/\rho^r}\chi(a^kc)\sum_{b\le\rho z}\mu(b)\chi(b^r)\ll$$

$$\ll\rho z\rho^{-r}\zeta(k)\ll\zeta(k)\rho^{1-r}z.$$

Thus,

$$T_\chi(x;k,r)=\sum_{\substack{a^kb^rc\le x\\ b\le\rho z}}\mu(b)\chi(a^kb^rc)+O(\zeta(k)\rho^{1-r}x^{1/r}).$$

We reduce the first term to the form

$$\sum_{\substack{a^kb^rc\le x\\ b\le\rho z}}\mu(b)\chi(a^kb^rc)=\sum_{b\le\rho z}\mu(b)\chi^r(b)\sum_{a^kc\le\frac{x}{b^r}}\chi(a^kc).$$

Denote by

$$S_\chi(y;k)=\sum_{a^kc\le y}\chi(a^kc).$$

Then

$$\sum_{\substack{a^kb^rc\le x\\ b\le\rho z}}\mu(b)\chi(a^kb^rc)=\sum_{b\le\rho z}\mu(b)\chi^r(b)S_\chi\left(\frac{x}{b^r};k\right).$$

We first investigate $S_\chi(y;k)$. Write

$$S_\chi(y;k)=\sum_{a\le y^{1/(k+1)}}\chi^k(a)\sum_{c\le y/a^k}\chi(c)+\sum_{c\le y^{1/(k+1)}}\chi(c)\sum_{a\le(y/c)^{1/k}}\chi(a)-$$

$$-\sum_{c\le y^{1/(k+1)}}\chi(c)\sum_{a\le y^{1/(k+1)}}\chi^k(a).$$

If $\chi=\chi_0$, we have

$$S_{\chi_0}(y;k)=\sum_{\substack{a\le y^{1/(k+1)}\\ (a,p)=1}}\sum_{\substack{c\le y/a^k\\ (c,p)=1}}1+\sum_{\substack{c\le y^{1/(k+1)}\\ (c,p)=1}}\sum_{\substack{a\le(y/c)^{1/k}\\ (a,p)=1}}1-\sum_{\substack{c\le y^{1/(k+1)}\\ (c,p)=1}}1\sum_{\substack{a\le y^{1/(k+1)}\\ (a,p)=1}}1.$$

In view of Lemma 5, we have

$$S_{\chi_0}(y;k) = \sum_{\substack{a\le y^{1/(k+1)}\\(a,p)=1}} \sum_{d|p}\mu(d)\left\lfloor \frac{y}{da^k}\right\rfloor + \sum_{\substack{c\le y^{1/(k+1)}\\(c,p)=1}} \sum_{d|p}\mu(d)\left\lfloor \frac{y^{1/k}}{dc^{1/k}}\right\rfloor - $$
$$- \left(\sum_{d|p}\mu(d)\left\lfloor \frac{y^{1/(k+1)}}{d}\right\rfloor\right)^2.$$

From Lemma 3 it follows

$$S_{\chi_0}(y;k) = \frac{\phi(p)}{p}L(k,\chi_0)x + \frac{\phi(p)}{p}L\left(\frac{1}{k},\chi_0\right)x^{1/k} -$$

$$-\sum_{d|p}\mu(d)\sum_{t|p}\mu(t)\sum_{a\le \frac{y^{1/(k+1)}}{t}}\left(\psi\left(\frac{y}{dt^k a^k}\right)+\psi\left(\frac{y^{1/k}}{dt^{1/k}a^{1/k}}\right)\right)+O(p^{1+\epsilon}).$$

In view of Lemma 4,

$$\sum_{d|p}\mu(d)\sum_{t|p}\mu(t)\sum_{a\le \frac{y^{1/(k+1)}}{t}}\psi(\frac{y}{dt^k a^k}) \ll$$
$$\ll \sum_{d|p}\sum_{t|p}\left(1+y^{1/(2(k+1))}d^{1/2}t^{-1}+\frac{y^{2/(5+2k)}}{d^{2/(5+2k)}t^{2k/(5+2k)}}\right) =$$
$$= O\left(p^{\epsilon}+y^{1/(2(k+1))}p^{1/2+\epsilon}+y^{2/(5+2k)}p^{\epsilon}\right) = O\left(y^{2/(5+2k)}p^{1/2+\epsilon}\right).$$

Similarly,

$$\sum_{d|p}\mu(d)\sum_{t|p}\mu(t)\sum_{a\le \frac{y^{1/(k+1)}}{t}}\psi\left(\frac{y^{1/k}}{dt^{1/k}a^{1/k}}\right) = O\left(y^{2/(5+2k)}p^{1/2+\epsilon}\right).$$

Thus,

$$S_{\chi_0}(y;k) = \frac{\phi(p)}{p}L(k,\chi_0)y + \frac{\phi(p)}{p}L(\frac{1}{k},\chi_0)y^{1/k} + O\left(y^{2/(5+2k)}p^{1/2+\epsilon}+p^{1+\epsilon}\right).$$

If $\chi\in\Gamma_k$, we have

$$S_\chi(y;k) =$$

$$= \sum_{\substack{a\le y^{1/(k+1)}\\(a,p)=1}}\sum_{c\le y/a^k}\chi(c) + \sum_{c\le y^{1/(k+1)}}\chi(c)\sum_{\substack{a\le (y/c)^{1/k}\\(a,p)=1}}1 - \sum_{c\le y^{1/(k+1)}}\chi(c)\sum_{\substack{a\le y^{1/(k+1)}\\(a,p)=1}}1.$$

In view of

$$\sum_{m\le z}\chi(m)=\sum_{h\le p}\chi(h)\left\lfloor \frac{z}{p}-\frac{h}{p}+1\right\rfloor, \qquad \text{for } \chi\ne\chi_0,$$

and Lemma 5, we obtain

$$\begin{aligned} S_\chi(y;k) &= \sum_{h\le p}\chi(h)\sum_{\substack{a\le y^{1/(k+1)}\\ (a,p)=1}}\left\lfloor \frac{y}{a^kp}-\frac{h}{p}+1\right\rfloor + \\ &+ \sum_{h\le p}\chi(h)\sum_{d|p}\mu(d)\sum_{\substack{a\le y^{1/(k+1)}\\ a\equiv h \pmod p}}\left\lfloor \frac{y^{1/k}}{a^{1/k}d}\right\rfloor - \\ &- \sum_{h\le p}\chi(h)\left\lfloor \frac{y^{1/(k+1)}}{p}-\frac{h}{p}+1\right\rfloor\sum_{d|p}\mu(d)\left\lfloor \frac{y^{1/k}}{d}\right\rfloor . \end{aligned}$$

By Lemma 6, further we can write

$$\begin{aligned} S_\chi(y;k) &= \frac{\phi(p)}{p}L\left(\frac{1}{k},\chi\right)x^{1/k}-\sum_{h\le p}\chi(h)\sum_{d|p}\mu(d)\sum_{a\le \frac{y^{1/(k+1)}}{d}}\psi\left(\frac{y}{pt^ka^k}-\frac{h}{p}\right)- \\ &- \sum_{h\le p}\chi(h)\sum_{d|p}\mu(d)\sum_{\substack{a\le y^{1/(k+1)}\\ a\equiv h \pmod p}}\psi\left(\frac{y^{1/k}}{da^{1/k}}\right)+O(p^2). \end{aligned}$$

In view of Lemma 7, we have

$$\begin{aligned} &\sum_{h\le p}\chi(h)\sum_{d|p}\mu(d)\sum_{a\le \frac{y^{1/(k+1)}}{d}}\psi\left(\frac{y}{pt^ka^k}-\frac{h}{p}\right)\ll \\ \ll\ & O(1)\sum_{h\le p}\sum_{d|p}\left(1+y^{1/(2(k+1))}p^{1/2}d^{-1}+y^{2/(5+2k)}p^{-2/(5+2k)}d^{-2k/(5+2k)}\right)= \\ =\ & O\left(p^{1+\epsilon}+y^{1/(2(k+1))}p^{3/2+\epsilon}+y^{2/(5+2k)}p^{(3+2k)/(5+2k)+\epsilon}\right)= \\ =\ & O\left(y^{2/(5+2k)}p^{3/2+\epsilon}\right) \end{aligned}$$

and

$$\begin{aligned} &\sum_{h\le p}\chi(h)\sum_{d|p}\mu(d)\sum_{\substack{a\le y^{1/(k+1)}\\ a\equiv h \pmod p}}\psi\left(\frac{y^{1/k}}{da^{1/k}}\right) \\ \ll\ & O(1)\sum_{h\le p}\sum_{d|p}\left(1+y^{1/(2(k+1))}d^{1/2}p^{-1}+y^{2/(5+2k)}d^{-2/(5+2k)}p^{-2k/(5+2k)}\right) \\ =\ & O\left(p^{1+\epsilon}+y^{1/(2(k+1))}p^{1/2+\epsilon}+y^{2/(5+2k)}p^{5/(5+2k)+\epsilon}\right)=O\left(y^{2/(5+2k)}p^{1/2+\epsilon}\right). \end{aligned}$$

Thus, for any $\chi \in \Gamma_k$,

$$S_{1,\chi}(y;k) = \frac{\phi(p)}{p} L\left(\frac{1}{k}, \chi\right) y^{1/k} + O\left(y^{2/(5+2k)} p^{3/2+\epsilon}\right) + O(p^2).$$

If $\chi \neq \chi_0$ and $\chi \notin \Gamma_k$, we have

$$\begin{aligned} S_{1,\chi}(y;k) &= \sum_{h \le p} \chi(h) \sum_{a \le y^{1/(k+1)}} \chi^k(a) \left\lfloor \frac{y}{a^k p} - \frac{h}{p} + 1 \right\rfloor + \\ &+ \sum_{h \le p} \chi^k(h) \sum_{a \le y^{1/(k+1)}} \chi(a) \left\lfloor \frac{y^{1/k}}{a^{1/k} p} - \frac{h}{p} + 1 \right\rfloor - \\ &- \sum_{h \le p} \chi(h) \sum_{j \le p} \chi^k(j) \left\lfloor \frac{y^{1/(k+1)}}{p} - \frac{j}{p} + 1 \right\rfloor \left\lfloor \frac{y^{1/(k+1)}}{p} - \frac{h}{p} + 1 \right\rfloor = \\ &= -\sum_{h \le p} \chi(h) \sum_{j \le p} \chi^k(j) \sum_{\substack{a \le y^{1/(k+1)} \\ a \equiv j \pmod p}} \psi\left(\frac{y}{pa^k} - \frac{h}{p}\right) - \\ &- \sum_{h \le p} \chi^k(h) \sum_{j \le p} \chi(j) \sum_{\substack{a \le y^{1/(k+1)} \\ a \equiv j \pmod p}} \psi\left(\frac{y^{1/k}}{pa^{1/k}} - \frac{h}{p}\right) + O(p^2). \end{aligned}$$

It follows from Lemma 7, similarly to the proof on the previous case, that

$$\sum_{h \le p} \chi(h) \sum_{j \le p} \chi^k(j) \sum_{\substack{a \le y^{1/(k+1)} \\ a \equiv j \pmod p}} \psi\left(\frac{y}{pa^k} - \frac{h}{p}\right) \ll$$

$$\ll \sum_{h \le p} \sum_{j \le p} \left(1 + y^{1/(2(k+1))} p^{-1/2} + y^{2/(5+2k)} p^{-2(k+1)/(5+2k)}\right) =$$

$$= O\left(p^2 + y^{1/(2(k+1))} p^{3/2} + y^{2/(5+2k)} p^{(8+2k)/(5+2k)+\epsilon}\right) = O\left(y^{2/(5+2k)} p^{3/2}\right)$$

and

$$\sum_{h \le p} \chi^k(h) \sum_{j \le p} \chi(j) \sum_{\substack{a \le y^{1/(k+1)} \\ a \equiv j \pmod p}} \psi\left(\frac{y^{1/k}}{pa^{1/k}} - \frac{h}{p}\right) \ll$$

$$\ll \sum_{h \le p} \sum_{j \le p} \left(1 + y^{1/(2(k+1))} p^{-1/2} + y^{2/(5+2k)} p^{-2(k+1)/(5+2k)}\right) = O\left(y^{2/(5+2k)} p^{3/2}\right).$$

Thus,

$$\begin{aligned}
&\sum_{b\le\rho z}\mu(b)\chi_0^r(b)S_{\chi_0}\left(\frac{x}{b^r};k\right)=\\
=&\sum_{b\le\rho z}\mu(b)\chi_0^r(b)\left(\frac{\phi(p)}{p}L(k,\chi_0)\frac{x}{b^r}+\frac{\phi(p)}{p}L\left(\frac{1}{k},\chi_0\right)\left(\frac{x}{b^r}\right)^{1/k}\right)+\\
+&\sum_{b\le\rho z}\mu(b)\chi_0^r(b)\left(O\left(\left(\frac{x}{b^r}\right)^{2/(5+2k)}p^{1/2+\epsilon}\right)\right)=\\
=&\frac{\phi(p)}{p}\cdot\frac{L(k,\chi_0)}{L(r,\chi_0)}x+O\left(\rho^{1-r}x^{1/r}+\rho^{1-r/k}x^{1/r}+\rho^{1-2r/(5+2k)}x^{1/r}p^{1/2+\epsilon}\right)=\\
=&\frac{\phi(p)}{p}\cdot\frac{L(k,\chi_0)}{L(r,\chi_0)}x+O\left(\rho^{1-2r/(5+2k)}x^{1/r}p^{1/2+\epsilon}\right);
\end{aligned}$$

for $\chi\in\Gamma_k$,

$$\begin{aligned}
\sum_{b\le\rho z}\mu(b)\chi^r(b)S_{\chi}\left(\frac{x}{b^r};k\right)&=\sum_{b\le\rho z}\mu(b)\chi^r(b)\frac{\phi(p)}{p}L\left(\frac{1}{k},\chi\right)\left(\frac{x}{b^r}\right)^{1/k}+\\
&+\sum_{b\le\rho z}\mu(b)\chi^r(b)O\left(\left(\frac{x}{b^r}\right)^{2/(5+2k)}p^{3/2+\epsilon}\right)=\\
&=O\left(\rho^{1-2r/(5+2k)}x^{1/r}p^{3/2+\epsilon}\right);
\end{aligned}$$

for $\chi\neq\chi_0$ and $\chi\notin\Gamma_k$,

$$\begin{aligned}
\sum_{b\le\rho z}\mu(b)\chi^r(b)S_{\chi}\left(\frac{x}{b^r};k\right)&\ll\sum_{b\le\rho z}\left(\left(\frac{x}{b^r}\right)^{2/(5+2k)}p^{3/2}\right)\ll\\
&=O\left(\rho^{1-2r/(5+2k)}x^{1/r}p^{3/2}\right).
\end{aligned}$$

As in [9], we choose $\rho=\exp\left(\frac{-A}{r}\log^{3/5}(x^{1/2r})(\log\log(x^{1/2r}))^{-1/5}\right)$. This is based on the Vinogradov-Korobov zero-free region for the Riemann zeta-function. Thus, we have $B=\frac{r^{-8/5}(5+2k-2r)}{5+2k}A$, and the proof of Lemma is complete. $\square$

3 Proof of Theorem 1

In view of Lemma 1, for a given odd prime p, we have

$$\begin{aligned}
&prim_{k,r}(x)=\\
&=\frac{\phi(p-1)}{p-1}\left\{\sum_{\substack{m\le x\\ m\text{ is }(k,r)\text{ integer}}}\chi_0(m)+\sum_{\substack{d\mid p-1\\ d>1}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in G_d}\sum_{\substack{m\le x\\ m\text{ is }(k,r)\text{ integer}}}\chi(m)\right\}.
\end{aligned}$$

We bound the last sum by using the last two cases in Lemma 8. Thus,

$$\left| \sum_{\substack{d|p-1 \\ d>1}} \frac{\mu(d)}{\phi(d)} \sum_{\chi \in G_d} \sum_{\substack{m \leq x \\ m \text{ is } (k,r) \text{ integer}}} \chi(m) \right| \ll$$
$$\ll 2^{\omega(p-1)} p^{3/2+\epsilon} x^{1/r} \exp\left(-B \log^{3/5} x (\log\log x)^{-1/5}\right).$$

We use the first case in Lemma 8 to obtain the main term as

$$\frac{p-1}{p} \cdot \frac{L(k, \chi_0)}{L(r, \chi_0)} x,$$

and the error term of the first case is dominated by

$$2^{\omega(p-1)} p^{3/2+\epsilon} x^{1/r} \exp\left(-B \log^{3/5} x (\log\log x)^{-1/5}\right),$$

which establishes the formula.

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References

[1] D. A. Burgess, *On character sums and primitive roots*, Proc. London Math. Soc., **12** (1962), no. 3, pp. 179–192.

[2] S. D. Cohen, and T. Trudgian, *On the least square-free primitive root modulo p*, J. Number Theory, **170** (2017), pp. 10–16.

[3] H. Liu, and W. Zhang, *On the squarefree and squarefull numbers*, J. Math. Kyoto Univ, **45** (2005), no. 2, pp. 247–255.

[4] M. Munsch, *Character sums over squarefree and squarefull numbers*, Arch. Math, **102** (2014), no. 6, pp. 555–563.

[5] M. Munsch and T. Trudgian, *Square-full primitive roots*, Intern. J. Number Theory, **14** (2018), no. 4, pp. 1013–1021.

[6] H. E. Richert, *Über die Anzahl Abelscher Gruppen gegebener Ordnung. I*, Math. Z, **56** (1952), no. 1, pp. 21–32.

[7] H. E. Richert, *Über die Anzahl Abelscher Gruppen gegebener Ordnung. II*, Math. Z, **58** (1953), no. 1, pp. 71–84.

[8] H. N. Shapiro, *Introduction to the Theory of Numbers*, Wiley, New York, (1983).

[9] M. V. Subbarao and D. *Suryanarayana, On the order of the error function of the (k, r)-integers*, J. Number Theory, **6.2** (1974), pp. 112–123.

[10] T. Srichan, *Square-full and cube-full numbers in arithmetic progressions*, Siauliai Math. Semin, **8** (2013), no. 16, pp. 223–248.

[11] T. Srichan, *On the distribution of square-full and cube-full primitive roots*, Period. Math. Hungar (to appear).

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