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Finite subgroups of the free groups of the infinitely based varieties of S. I. Adian

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Abstract. It is proved that all finite subgroups of the free groups of the infinitely based varieties of S. I. Adian are cyclic. The set of all non-isomorphic free groups of rank m in this varieties is of continuum cardinality for every finite rank $m > 1$.

Key Words: infinitely based variety, torsion-free groups, free Burnside group
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Introduction

First examples of infinite independent systems of group identities (in 2 generators) were constructed by S. I. Adian in [1] (see also [2]). It was a constructive solution of the posed by B. H. Neumann in 1937 famous problem whether there exists an infinite system of group identities not equivalent to any finite system. This result was included in the monograph [3] (see. [3, Chapter VII]), where it is proved that for any odd number $n > 1003$ the family of identities in 2 generators

$$(x^{pn}y^{pn}x^{-pn}y^{-pn})^n = 1, \tag{1}$$

where p runs through all prime numbers, is independent, that is, none of the identities is the consequence of the others. This implies that for every odd $n \geq 1003$ there is a continuum set of distinct varieties $\mathcal{A}_n(\Pi)$ corresponding to distinct sets of primes Π. At the same time, there exists a continuum set of non-isomorphic groups $\Gamma(m, n, \Pi)$ in the variety $\mathcal{A}_n(\Pi)$ for any finite $m > 1$, where $\Gamma(m, n, \Pi)$ is the relatively free group of fixed rank m in $\mathcal{A}_n(\Pi)$.

It is easy to understand that all varieties defined by identities of the form (1) contain the Burnside variety $\mathcal{B}_n$ of all groups satisfying the identity $x^n = 1$. Recall that the *free Burnside group* $B(m, n)$ is the free group of rank m in the variety $\mathcal{B}_n$ (for the definitions of the notions "relatively free group", "rank", "variety", etc see [7]).

Further studies of the groups $\Gamma(m, n, \Pi)$ can be found in [4], where it is proved that the centralizer of any non-trivial element for each relatively free group $\Gamma(m, n, \Pi)$ is cyclic. It is shown that all specified groups have trivial centre, any of their abelian subgroups is cyclic and any non-trivial normal subgroup is infinite. For the free groups $\Gamma(m, n, \Pi)$ in the varieties $\mathcal{A}_n(\Pi)$, the answer to the question about describing automorphisms of the endomorphism semigroup $End(\Gamma)$ raised by B. Plotkin in 2000 is also obtained. In particular, it is proved that for any of these groups $\Gamma(m, n, \Pi)$ the automorphism group of endomorphism semigroup $End(\Gamma(m, n, \Pi))$ is canonically embedded in the groups $Aut(\Gamma(m, n, \Pi))$.

We will prove the following theorem.

Theorem 1 *Any finite subgroup of every free group* $\Gamma(m, n, \Pi)$ *of the infinitely based varieties* $\mathcal{A}_n(\Pi)$ *is cyclic for any rank* $m \geq 1$.

Emphasize that the analogous statement for the free Burnside groups $B(m, n)$ of any odd period $n \geq 665$ and any rank was proved by S. I. Adian in [3] (see Theorem 1.8, Chapter VII). On the other hand, the proof of this result for absolutely free groups is an easy exercise.

1 Definitions and preliminary lemmas

We will carry out the proof of the main result using the scheme proposed in chapter VII of the monograph [3]. As in [3], first we will construct central extensions of the groups $\Gamma(m, n, \Pi)$. We will complete the proof by using the method of the proof of Theorem 1.8 from [3, Chapter VII] and Baer's theorem on the finiteness of the commutant of a group with finite quotient group by the centre.

To study the further properties of the groups $\Gamma = \Gamma(m, n, \Pi)$, we recall the construction of auxiliary groups $\Gamma(m, n, \Pi, \alpha)$ that are built by the induction on rank α (similarly to the definition of the groups $B(m, n, \alpha)$ in IV.2.2 [3], see also [4, Paragraph 2]).

For every $\alpha > 0$ we denote by $\mathcal{E}_\alpha$ the set of all marked elementary periods A of rank α such that the following conditions hold:

(a) for ervery elementary word E of rank α there is one and only one word A such that

$$A \in \mathcal{E}_\alpha \text{ and } (\,\mathrm{Rel}(E, A^n) \text{ or } \mathrm{Rel}(E, A^{-n})\,),$$

(b) if $A \in \mathcal{E}_\alpha$, then the membership relation $PA^nQ \in \overline{\mathcal{M}}_{\alpha-1}$ holds for some P and Q.

Further, we denote

$$\mathcal{E} = \bigcup_{i=1}^{\infty} \mathcal{E}_i. \tag{2}$$

Let $\Gamma(m,n,\Pi,0)$ be a free group with $m > 1$ generators. For every $\alpha > 0$ we denote by $\Gamma(m,n,\Pi,\alpha)$ the group with the same generators and the system of defining relations $A^n = 1$, where $A \in \bigcup_{i=1}^{n} \mathcal{E}_i$. We note that the group $\Gamma(m,n,\Pi)$ coincides with the group with the same generators and the system of defining relations $A^n = 1$, where $A \in \mathcal{E} = \bigcup_{i=1}^{\infty} \mathcal{E}_i$.

The following lemma is proved in [3].

Lemma 1 *(see Lemma 2.7, Chapter VII, [3]) For evry word C which is not equal to 1 one can find in the group $\Gamma(m,n,\Pi)$ the words T and E such that $C = TE^rT^{-1}$ in $\Gamma(m,n,\Pi)$ for some integer r, where either $E \in \mathcal{E}$ or E is an unmarked elementary period of some rank γ and the word E^q occurs in some word of the class $\overline{\mathcal{M}}_{\gamma-1}$.*

The following two lemmas are proved in [4].

Lemma 2 *If E is a marked elementary period of some rank $\gamma \geq 1$, then E has order n in the group $\Gamma(m,n,\Pi)$.*

Lemma 3 *If E is an unmarked elementary period of some rank γ, then E has infinite order in $\Gamma(m,n,\Pi)$.*

The set $\mathcal{E}$ is countable (see Theorem 2.13 in Chapter VI [3]), that is, its elements can be enumerated by natural numbers. We fix some numbering, and let $\mathcal{E} = \{A_j | j \in \mathbb{N}\}$ ($\mathbb{N}$ is the set of natural numbers).

We also fix an at most countable arbitrary abelian group $\mathcal{D}$ given by generators and defining relations in the following form:

$$\mathcal{D} = \langle d_1, d_2, \ldots, d_i, \ldots \mid r = 1, \, r \in \mathcal{R} \rangle, \tag{3}$$

where $\mathcal{R}$ is an arbitrary set of words in the group alphabet $d_1, d_2, \ldots, d_i, \ldots$

By $A_{\mathcal{D}}(m,n,\Pi)$ we denote the group given by a system of generators of two types:

$$a_1, a_2, ..., a_m \tag{4}$$

and

$$d_1, d_2, ..., d_i, ...$$

and by a system of defining relations of the following three types:

$$r = 1 \text{ for all } r \in \mathcal{R},$$

$$a_i d_j = d_j a_i,$$

$$A_j^n = d_j \tag{5}$$

for all $A_j \in \mathcal{E}$ (see (2)), $i = 1, 2, ..., m$ and $j \in \mathbb{N}$.

From the defining relation (5), it follows that the groups $A_{\mathcal{D}}(m,n,\Pi)$ are m-generated groups with the generators (4). The following proposition holds for the groups $A_{\mathcal{D}}(m,n,\Pi)$.

PROPOSITION **1** *For every $m > 1$, odd $n \geq 1003$ and for any abelian group $\mathcal{D}$ having the presentation* (3) *the following conditions hold:*

1. the centre of $A_{\mathcal{D}}(m, n, \Pi)$ coincides with $\mathcal{D}$,

2. the quotient group of $A_{\mathcal{D}}(m, n, \Pi)$ by the subgroup $\mathcal{D}$ is isomorphic to the free group $\Gamma(m, n, \Pi)$.

Proposition 1 is proved just like items 3 and 4 of Theorem 1 from [4].

As a group $\mathcal{D}$, we choose infinite cyclic group

$$\mathcal{D} = \langle d_1, d_2, \ldots, d_i, \ldots \mid d_j d_k^{-1} = 1,\ j, k \in \mathbb{N}\rangle.$$

Then, according to Proposition 1, the centre of the obtained group $A_{\mathcal{Z}}(m, n, \Pi)$ coincides with the infinite cyclic group with generator d.

Lemma 4 *The groups $A_{\mathcal{Z}}(m, n, \Pi)$ are torsion-free.*

Proof. By virtue of the item 2 of Proposition 1, each non-trivial element x of the group $A_{\mathcal{Z}}(m, n, \Pi)$ can be represented as $x = Xd^j$, where X is a word in the generators of the group $\Gamma(m, n, \Pi)$ and d is the generator of its centre. If $X \neq 1$ in $\Gamma(m, n, \Pi)$, then by Lemma 1 there are words T and E such that $X = TE^rT^{-1}$ in the group $\Gamma(m, n, \Pi)$ for some integer r, and either $E \in \mathcal{E}$ or E is an unmarked elementary period of some rank γ and word E^q occurs in some word of class $\overline{\mathcal{M}}_{\gamma-1}$. By virtue of the item 1 of Proposition 1, the centre of the $A_{\mathcal{D}}(m, n, \Pi)$ is an infinite cyclic group generated by d. Therefore, with $X = 1$ the statement of theorem is obvious and it remains to consider the case when $X \neq 1$ in $\Gamma(m, n, \Pi)$.

Let E be an unmarked elementary period of some rank γ, and let the word E^q occurs in some word of class $\overline{\mathcal{M}}_{\gamma-1}$. Then by Lemma 3 element E and, therefore, X has an infinite order in the quotient group $\Gamma(m, n, \Pi)$. Hence its preimage x in $A_{\mathcal{D}}(m, n, \Pi)$ also has infinite order. If $E \in \mathcal{E}$, then using Lemma 2 we will literally repeat proof of Theorem 1.6 from [3] and will get that x has an infinite order. Lemma is proved.

□

2 The Proof of Theorem 1

Let the finite subgroup G of the group $\Gamma(m, n, \Pi)$ be generated by elements $g_1, g_2, ..., g_k$. Consider the subgroup G_1 of the group $A_{\mathcal{Z}}(m, n, \Pi)$ generated by $g_1, g_2, ..., g_k, d$. According to Proposition 1, the element d belongs to the centre of the group G_1. Hence, the quotient group of G_1 is finite by its centre. But according to the well-known theorem of Baer (see [6]) from the finiteness of quotient group by the centre the finiteness of the commutator subgroup follows. Therefore, the commutator subgroup of the group G_1 is finite. Since by Lemma 4 the group $A_{\mathcal{Z}}(m, n, \Pi)$ is torsion-free, the only finite subgroup in it is the trivial subgroup. Hence the commutator subgroup of G_1 is trivial, which means that the group G_1 is abelian. Therefore, the image G in $\Gamma(m, n, \Pi)$ of the group G_1 is also an abelian group. By virtue of Corollary 1 of paper [4] every abelian subgroup of the group $\Gamma(m, n, \Pi)$ is cyclic. Thus, G is cyclic. Theorem is proved.

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