Armenian Journal of Mathematics
Volume 11, Number 7, 2019, 1–14

On a Riemann boundary value problem for weighted spaces in the half-plane

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Abstract. The paper considers the Riemann boundary value problem in the half-plane in the class of functions that are $C(\rho)$-continuous with respect to the weight $\rho(x)$, when the weight function has infinite number of zeros. Necessary and sufficient conditions for solvability of the problem are established. If the problem is solvable, solutions are represented in an explicit form.

Key Words: Riemann boundary value problem, weighted space, factorization, homogeneous problem, Sokhotski-Plemelj formula
Mathematics Subject Classification 2010: 41A60, 68U10

1 Introduction. Formulation of the problem

Let $\Pi^{\pm} = \{x + iy : y \gtrless 0\}$ be the upper and lower half-planes respectively, and let A be the class of functions Φ analytic in $\Pi^+ \bigcup \Pi^-$, satisfying the condition

$$|\Phi(z)| \le C|z|^m, \qquad |Imz| \ge y_0 > 0,$$

where m is a natural number and C is a constant, possibly depending on y_0.

Let $\boldsymbol{\overline{C}(-\infty;+\infty)}$be the class of functions $f(x)$ continuous on the real axis for which the limits $f(+\infty), f(-\infty)$ exist and $f(+\infty) = f(-\infty) \neq 0$.

By $\boldsymbol{\overline{C}^{\delta}(-\infty;+\infty)}$we denote the class of functions $f \in \overline{C}(-\infty;+\infty)$ such that for any $C > 0$

$$|f(x_1) - f(x_2)| \le M \left|\frac{1}{x_1} - \frac{1}{x_2}\right|^{\delta}$$

for $x_1, x_2 > C > 0$. Further in this article we consider that $\delta \in (\frac{1}{2}, 1]$.

Denote by $\boldsymbol{\overline{C}(\rho)}$the class of functions f continuous on the real axis such that

$$f(x)\rho(x) \in \overline{C}(-\infty;+\infty) \quad \text{and} \quad \|f\|_{\overline{C}(\rho)} = \max_{x\in(-\infty,\infty)} |f(x)\rho(x)|,$$

where

$$\rho(x) = \prod_{k=1}^{\infty} \left| \frac{x - x_k}{x + i} \right|^{\alpha_k}, \qquad 0 < \alpha_k < 1, \tag{1}$$

$x_k \in (-\infty, +\infty)$ are real numbers and the sequences $\{\alpha_k\}_1^\infty, \{x_k\}_1^\infty$ satisfy some conditions.

The Riemann boundary value problem

$$\Phi^+(x) - a(x)\Phi^-(x) = f(x),$$

where $\Phi^\pm(z)$ are analytic functions in $\Pi^\pm$, has been investigated in weighted spaces $L^p(\rho)$, $p \in (1, \infty)$, $\rho(x) = \prod_{k=1}^N |x_k - x|^{\alpha_k}$ by many authors; let's note some of them: Khvedelidze B.V. [1]-[3], Simonenko I.B. [4], [5], Tovmasyan N.E. [8], [9], Soldatov A.P. [10], [11], Kazarian K., Soria F., Spitkovsky I. [12], Kazarian K.S. [13], [14]. Also see [15] - [18].

The case of the unit circle when the weight function has the form

$$\rho(t) = \prod_{k=1}^{\infty} |t_k - t|^{\alpha_k}, \qquad 0 < \alpha_k < 1, \ |t_k| = 1$$

and $\arg t_k \downarrow 0$ is investigated in $L^1(\rho)$ (see [19]) for

$$\lim_{r \to 1-0} \|\varphi^+(rt) - a(t)\varphi^-(r^{-1}t) - f(t)\|_{L^1(\rho)} = 0,$$

where functions $\varphi^\pm(z)$ are analytic in the domains $D^+ = \{z : |z| < 1\}$, $D^- = \{z : |z| > 1\}$ such that $\varphi^-(\infty) = 0$.

In the mentioned work some conditions on the numbers t_k, $k = 1, 2, ...$ and $\{\alpha_k\}_1^\infty$ are set. It is established that the Riemann homogeneous problem has infinite number of linearly independent solutions in $L^1(\rho)$, and the general solution is determined in explicit form.

In the work [20] the Riemann boundary value problem is investigated as follows: find an analytic function $\Phi \in A$ such that

$$\lim_{y \to +0} \|\Phi^+(x + iy) - a(x)\Phi^-(x - iy) - f(x)\|_{\overline{C}(\rho)} = 0,$$

where

$$\rho(x) = \prod_{k=1}^{N} \left| \frac{x - x_k}{x + i} \right|^{\alpha_k},$$

α_k are arbitrary real numbers and the sequence x_k satisfies the condition $x_k \in (-\infty, +\infty)$.

In this paper we investigate the Riemann boundary value problem for weighted spaces when the weight function has infinite number of zeros on the boundary, as follows:

Problem R *Let* $f \in \overline{C}(\rho)$. *Find an analytic function* $\Phi \in A$ *in* $\Pi^+ \cup \Pi^-$ *such that*

$$\lim_{y\to+0} \|\Phi^+(x+iy) - a(x)\Phi^-(x-iy) - f(x)\|_{\overline{C}(\rho)} = 0,$$

where $\rho(x)$ *is defined by* (1), $a(x) \in \overline{C}^{\delta}(-\infty;+\infty)$, $a(x) \neq 0$ *and*

$$|a(x) - a(\infty)| < C|x|^{-\delta} \qquad at\ |x| \geq C > 0.$$

It is proved that the homogeneous problem has one linearly independent solution when the index of the coefficient $a(x)$ greater than -1, otherwise it does not have solution. We determine certain conditions that guarantee the inhomogeneous problem R to have solutions.

2 Some auxiliary results

Let $\kappa = inda(t)$, $t \in (-\infty, +\infty)$,

$$\begin{aligned} S^+(z) &= exp\Big\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln a_1(t)dt}{t-z}\Big\}, \qquad z \in \Pi^+, \\ S^-(z) &= \Big(\frac{z+i}{z-i}\Big)^{\kappa} exp\Big\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln a_1(t)dt}{t-z}\Big\}, \qquad z \in \Pi^-, \end{aligned} \tag{2}$$

where

$$a_1(t) = \Big(\frac{t+i}{t-i}\Big)^{\kappa} a(t), \qquad inda_1(t) = 0.$$

Here we will consider two cases:

1. We assume that the sequence $\{x_k\}_1^\infty$ has a finite limit x_0 and

$$\sum_{k=1}^{\infty} \alpha_k < \infty.$$

Lemma 1 *Let the sequence* $\{x_k\}_1^\infty$ *satisfy the following conditions:*

$$\sum_{k=1}^{\infty} \alpha_k \ln|x_0 - x_k| > -\infty, \tag{3}$$

$$|x_k - x_j| > c|x_k - x_0|, \qquad j \neq k \tag{4}$$

for some fixed $c > 0$. *Then*

$$\inf \rho_m = \rho_0 > 0, \qquad m = 1, 2, ...,$$

where

$$\rho_m = \prod_{k\neq m}^{\infty} \Big|\frac{x_m - x_k}{x_m + i}\Big|^{\alpha_k}.$$

Proof. From condition (4) we have

$$\left|\frac{x_j - x_k}{x_j + i}\right|^{\alpha_k} > c^{\alpha_k}\left|\frac{x_0 - x_k}{x_j + i}\right|^{\alpha_k}$$

and

$$\prod_{k\neq j}^{\infty}\left|\frac{x_j - x_k}{x_j + i}\right|^{\alpha_k} > \prod_{k=1}^{\infty} c^{\alpha_k}\prod_{k\neq j}^{\infty}\left|\frac{x_0 - x_k}{x_j + i}\right|^{\alpha_k}.$$

According to the condition (3), there exists $\delta > 0$ such that $\inf \rho_m = \delta > 0$, $m = 1, 2, \ldots.$ $\square$

2. We assume that the limit of $\{x_k\}_1^\infty$ is infinite and

$$\sum_{k=1}^{\infty} \alpha_k \ln |x_k| < \infty.$$

Lemma 2 *Let the sequence $\{x_k\}_1^\infty$ satisfy the following condition:*

$$|x_k - x_j| > c|x_j|, \qquad j \neq k \tag{5}$$

for some fixed $c > 0$. Then $\inf \rho_m = \rho_0 > 0$*, $m = 1, 2, \ldots$*

Proof. Taking into account that $|x_k - x_j| > c|x_k|, j \neq k$, where $c > 0$ does not depend on k and j, we get

$$\prod_{k\neq j}^{\infty}\left|\frac{x_j - x_k}{x_j + i}\right|^{\alpha_k} > \prod_{k=1}^{\infty} c^{\alpha_k}\prod_{k\neq j}^{\infty}\left|\frac{x_j}{x_j + i}\right|^{\alpha_k} > C > 0.$$

Hence, $\inf \rho_m = \rho_0 > 0$, $m = 1, 2, \ldots$ $\square$

Let us denote

$$\delta_k(x) = \prod_{j\neq k}^{\infty}\left|\frac{x - x_j}{x + i}\right|^{\alpha_j}$$

and

$$\delta(x) = \delta_{k+1}(x) - \delta_k(x), \qquad x \in [x_k, x_{k+1}).$$

Lemma 3 *There exist $x'_k \in [x_k, x_{k+1})$, $k = 1, 2, \ldots$ such that $\delta(x'_k) = 0$.*

Proof. Consider

$$\delta_{k+1}(x) - \delta_k(x) = \prod_{j\neq k+1}^{\infty}\left|\frac{x_j - x}{x + i}\right|^{\alpha_j} - \prod_{j\neq k}^{\infty}\left|\frac{x_j - x}{x + i}\right|^{\alpha_j} =$$

$$= \prod_{j\neq k,k+1}^{\infty}\left|\frac{x_j - x}{x + i}\right|^{\alpha_j} \cdot \left|\left|\frac{x_k - x}{x + i}\right|^{\alpha_k} - \left|\frac{x_{k+1} - x}{x + i}\right|^{\alpha_{k+1}}\right|.$$

We can choose $c_1, c_2 > 0$ such that

$$\delta(x_{k+1} - c_1) = |-c_1|^{\alpha_{k+1}} - |x_{k+1} - x_k - c_1|^{\alpha_k} < 0$$

and

$$\delta(x_k + c_2) = |x_k - x_{k+1} + c_2|^{\alpha_{k+1}} - |c_2|^{\alpha_k} > 0.$$

Taking into account that $\delta(x)$ is continuous, we see that the equation $\delta(x) = 0$ has a solution. By pointing out those points with x'_k, we obtain the proof of the lemma. □

Let $X_1 = (-\infty, x'_1)$ and $X_k = [x'_{k-1}, x'_k)$, $k = 2, 3, ...$ It is clear that $X_k \cap X_{k+1} = \emptyset$, $k = 1, 2, 3, ...$

Lemma 4 *Let the sequence of points* $\{x_k\}_1^\infty$ *satisfy either condition* (3), (4) *or* (5). *Then there exists* $\delta > 0$ *such that for any* $k = 1, 2, ...$:

$$\inf_{x \in X_k} \delta_k(x) > \delta > 0.$$

Proof. Let $x \in (x'_{k-1}, x_k)$, then $|x_j - x| \geq |x_j - x_k|$ at $j \geq k+1$. If $j < k$, then we have $|x_j - x| > |x_j - x_{k-1}|$. Using Lemmas 1 and 2, we get

$$\prod_{j \geq k+1} \left| \frac{x_j - x}{x + i} \right|^{\alpha_j} > \prod_{j \geq k+1} \left| \frac{x_j - x_k}{x_k + i} \right|^{\alpha_j} > \delta > 0$$

and

$$\prod_{j<k} \left| \frac{x_j - x}{x + i} \right|^{\alpha_j} > \prod_{j<k} \left| \frac{x_j - x_{k-1}}{x_k + i} \right|^{\alpha_j} > \delta.$$

Hence, $\delta_k(x) \geq \delta^2$, $x \in (x'_{k-1}, x_k)$.

Let now $x \in (x_k, x'_k)$, then at $j < k$ we have $|x_j - x| > |x_j - x_k|$ and

$$\prod_{j=1}^{k-1} \left| \frac{x_j - x}{x + i} \right|^{\alpha_j} \geq \prod_{j=1}^{k-1} \left| \frac{x_j - x_k}{x_{k+1} + i} \right|^{\alpha_j} > \delta.$$

At $j \geq k+1$ we have $|x_j - x| > |x_j - x_{k+1}|$. We get

$$\prod_{j>k} \left| \frac{x_j - x}{x + i} \right|^{\alpha_j} > \prod_{j>k} \left| \frac{x_j - x_{k+1}}{x_{k+1} + i} \right|^{\alpha_j} > \delta.$$

Hence, $\delta_k > \delta^2$. □

Denote

$$\widetilde{\delta}(x) = \{\delta_k(x), x \in X_k\}, \qquad k = 1, 2, ...$$

From Lemmas 3 and 4 it follows that the function $\widetilde{\delta}(x)$ is continuous, and $\inf \widetilde{\delta}(x) > 0$, $x \in (-\infty, \infty)$.

We define the function $\Phi(z)$ as follows:

$$\Phi(z) = \frac{(z+i)S(z)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{(t+i)S^+(t)\widetilde{\delta}(t)} \frac{dt}{t - z}, \qquad z \in \Pi^{\pm}.$$

Lemma 5 *The estimate*

$$\|\Phi^+(x+iy) - a(x)\Phi^-(x-iy)\|_{\overline{C}(\rho)} \le C\|f\|_{\overline{C}(\rho)},$$

where the constant C is independent of y, is true. The limit relation

$$\lim_{y\to+0} \|\Phi^+(x+iy) - a(x)\Phi^-(x-iy) - f(x)\|_{\overline{C}(\rho)} = 0$$

also holds.

Proof. We have

$$\Phi^+(x+iy) - a(x)\Phi^-(x-iy) =$$

$$= \frac{(x+iy+i)S(x+iy)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{(t+i)S^+(t)\widetilde{\delta}(t)} \frac{dt}{t-x-iy} -$$

$$- \frac{a(x)(x-iy+i)S(x-iy)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{(t+i)S^+(t)\widetilde{\delta}(t)} \frac{dt}{t-x+iy} =$$

$$= I_1(f,x,y) + I_2(f,x,y),$$

where

$$I_1(f,x,y) = \frac{S(x+iy)(x+iy+i)}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{(t+i)S^+(t)\widetilde{\delta}(t)} \frac{ydt}{(t-x)^2+y^2},$$

$$I_2(f,x,y) = \frac{1}{2\pi i}\Big(S(x+iy)(x+iy+i) - a(x)S(x-iy)(x-iy+i)\Big)\cdot$$

$$\cdot \int_{-\infty}^{+\infty} \frac{f(t)}{(t+i)S^+(t)\widetilde{\delta}(t)} \frac{dt}{t-x+iy}.$$

Further,

$$I_1(f,x,y)\rho(x) = \frac{S(x+iy)\rho(x)}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)(x+iy+i)}{(t+i)S^+(t)\widetilde{\delta}(t)} \frac{ydt}{(t-x)^2+y^2}.$$

As $S^+(z)$ is bounded and $\widetilde{\delta}(x) > \delta > 0$ (see Lemma 4), we get

$$|I_1(f,x,y)\rho(x)| \le \widetilde{c_1}\rho(x) \int_{-\infty}^{+\infty} \Big|\frac{f(t)ydt}{(t-x)^2+y^2}\Big| +$$

$$+ \widetilde{c_1}\rho(x) \int_{-\infty}^{+\infty} \Big|\frac{f(t)\big((x+iy+i)-(t+i)\big)}{t+i} \frac{ydt}{(t-x)^2+y^2}\Big| =$$

$$= I_1^{(1)}(f,x,y) + I_1^{(2)}(f,x,y).$$

Thus,

$$I_1^{(1)}(f,x,y) \leq \widetilde{c}_1\rho(x) \max_{x\in(-\infty;+\infty)} |f(x)| \cdot \int_{-\infty}^{+\infty} \frac{y|dt|}{(t-x)^2+y^2} \leq c_1\|f\|_{\overline{C}(\rho)}.$$

As $|(x+iy+i)-(t+i)| = |t-x-iy|$, we get

$$I_1^{(2)}(f,x,y) \leq \widetilde{c}_2\rho(x) \max\nolimits_{x\in(-\infty;+\infty)} |f(x)| \cdot \int_{-\infty}^{+\infty} \frac{y|dt|}{|t+i||t-x+iy|} \leq$$

$$\leq \widetilde{c}_2\|f\|_{\overline{C}(\rho)} \cdot \left\{ \int_{-\infty}^{+\infty} \frac{y|dt|}{|t+i|^2} + \int_{-\infty}^{+\infty} \frac{y|dt|}{|t-x+iy|^2} \right\} \leq c_2\|f\|_{\overline{C}(\rho)}.$$

It may be noted that the constants c_1, c_2 do not depend on y. Therefore,

$$\|I_1(f,x,y)\|_{\overline{C}(\rho)} \leq M_1\|f\|_{\overline{C}(\rho)},$$

where M_1 is a constant independent of y.

Further we have

$$I_2(f,x,y)\rho(x) = \frac{\rho(x)}{2\pi i}\Big(S(x+iy)(x+iy+i) - a(x)S(x-iy)(x-iy+i)\Big)\cdot$$

$$\cdot \int_{-\infty}^{+\infty} \frac{f(t)}{(t+i)S^+(t)\widetilde{\delta}(t)} \frac{dt}{t-x+iy}.$$

As

$$S^+(x+iy)(x+iy+i) - a(x)S^-(x-iy)(x-iy+i) =$$

$$= (x+i)\big(S^+(x+iy) - a(x)S^-(x-iy)\big) +$$

$$+ iy\big(S^+(x+iy) + a(x)S^-(x-iy)\big),$$

and S^+, S^- are bounded, then using the fact (Lemma 1 in [21]) that for $y>0$ we have

$$|S^+(x+iy) - a(x)S^-(x-iy)| < A\frac{y^\delta}{|x+i|^{2\delta}}, \tag{6}$$

where $A>0$ is a constant, we get

$$I_2(f,x,y) \leq \widetilde{M}_2\rho(x) \max_{x\in(-\infty;+\infty)} |f(x)| \cdot \int_{-\infty}^{+\infty} \frac{|dt|}{|t+i||t-x+iy|} \leq M_2\|f\|_{\overline{C}(\rho)},$$

where M_2 is a constant independent of y.

Therefore,

$$\|\Phi^+(x+iy) - a(x)\Phi^-(x-iy)\|_{\overline{C}(\rho)} \leq M\|f\|_{\overline{C}(\rho)},$$

where $M = \max\{M_1, M_2\}$.

Now let us verify the second part of the lemma. Let $f_n(x) \in \overline{C}^{\delta}(\rho)$ be a sequence of finite functions such that

$$\lim_{n\to\infty} \|f_n(x) - f(x)\|_{\overline{C}(\rho)} = 0.$$

For an arbitrary n denote

$$\Phi_n(z) = \frac{(z+i)S(z)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f_n(t)}{S^+(t)(t+i)\widetilde{\delta}(t)} \frac{dt}{t-z}, \qquad z \in \Pi^{\pm}.$$

We will prove that

$$\lim_{y\to+0} \|\Phi_n^+(x+iy) - a(x)\Phi_n^-(x-iy) - f_n(x)\|_{\overline{C}(\rho)} = 0. \tag{7}$$

From the Sokhotski-Plemelj formula we have

$$\Phi_n^+(x+iy) - a(x)\Phi_n^-(x-iy) = f_n(x), \qquad x \in (-A, A). \tag{8}$$

If $|x| > A$, using the representation

$$\Phi_n^+(x+iy) - a(x)\Phi_n^-(x-iy) = J_1(x,y) + J_2(x,y),$$

where

$$J_1(x,y) = \frac{S^+(x+iy)(x+iy+i)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f_n(t)}{S^+(t)(t+i)\widetilde{\delta}(t)} \frac{ydt}{(t-x)^2+y^2},$$

$$J_2(x,y) = \frac{1}{2\pi i}\Big(S(x+iy)(x+iy+i) - a(x)S(x-iy)(x-iy+i)\Big)\cdot$$

$$\cdot \int_{-\infty}^{+\infty} \frac{f(t)}{(t+i)S^+(t)\widetilde{\delta}(t)} \frac{ydt}{t-x+iy},$$

we get

$$\max\nolimits_{|z|>A} \rho(x)|\Phi_n^+(x+iy) - a(x)\Phi_n^-(x-iy) - f_n(x)| \le$$

$$\le \max\nolimits_{|z|>A} \rho(x)|J_1(x,y)| + \max\nolimits_{|z|>A} \rho(x)|J_2(x,y)|.$$

Taking into account that $f_n(x) = 0, |x| > A$, for $J_1(x,y)$ we get the estimate

$$\rho(x)|J_1(x,y)| \le$$

$$\le C \max\nolimits_{|z|>A} \left\{ \rho(x) y|x+iy+i| \int_{-\infty}^{+\infty} \left| \frac{f_n(t)}{S^+(t)(t+i)} \right| \frac{ydt}{(t-x)^2+y^2} \right\} \le$$

$$\le C\|f_n\|_{\overline{C}} \max\nolimits_{|z|>A} \left\{ y|x+iy+i| \int_{-2A}^{2A} \frac{dt}{(t-x)^2} \right\} =$$

$$= C_1\|f_n\|_{\overline{C}} \max\nolimits_{|z|>A} \left\{ \frac{y|x+iy+i|}{|x^2-4A^2|} \right\},$$

where $C = const$. Hence, $J_1(x, y)$ tends to zero if $y \to +0$.

Using the scheme of the proof of the first part of this lemma, we get that $\rho(x)|J_2(x, y)|$ also tends to zero for $y \to +0$. Therefore, taking into account equality (8), we get (7).

Using the first estimate of this lemma and (7), we get

$$\|\Phi^+(x+iy) - a(x)\Phi^-(x-iy) - f(x)\|_{\overline{C}(\rho)} \leq$$

$$\leq \|\Phi_n^+(x+iy) - a(x)\Phi_n^-(x-iy) - f_n(x)\|_{\overline{C}(\rho)} + \|f_n(x) - f(x)\|_{\overline{C}(\rho)} +$$

$$+ \|[\Phi_n^+(x+iy) - \Phi^+(x+iy)] - a(x)[\Phi_n^-(x-iy) - \Phi^-(x-iy)]\|_{\overline{C}(\rho)} \leq$$

$$\leq \|\Phi_n^+(x+iy) - a(x)\Phi_n^-(x-iy) - f_n(x)\|_{\overline{C}(\rho)} + 2\|f_n(x) - f(x)\|_{\overline{C}(\rho)}.$$

Hence, we get

$$\lim_{y \to +0} \|\Phi^+(x+iy) - a(x)\Phi^-(x-iy) - f(x)\|_{\overline{C}(\rho)} = 0.$$

□

3 Main theorems

Theorem 1 *Consider the problem*

$$\lim_{y \to +0} \|\Phi^+(x+iy) - a(x)\Phi^-(x-iy)\|_{\overline{C}(\rho)} = 0. \tag{9}$$

a) In the case $\kappa \geq 0$, the general solution of the problem is $(Az + B)S(z)$, where A, B are constants and $S(z)$ is defined by (2).
b) When $\kappa = -1$, the general solution of the problem is $A(z + i)S(z)$.
c) Otherwise, the homogeneous problem (9) does not have solution.

Proof. Note, that the solutions of the problem (9) could be either

$$P(z)S(z), \qquad \frac{S(z)}{x_k - z}, \qquad k = 1, 2, ... \qquad \text{or} \qquad \frac{S(z)}{x_0 - z}, \tag{10}$$

where x_0 is an accumulation point of the points $\{x_k\}_1^\infty$ and $P(z)$ is a polynomial of degree m.

From (6) we have

$$|S^{+}(x+iy)(x+iy)^{k}-a(x)S^{-}(x-iy)(x-iy)^{k}|\leq$$

$$\leq|(x+iy)^{k}||S^{+}(x+iy)-a(x)S^{-}(x-iy)|+$$

$$+|(x+iy)^{k}||a(x)S^{-}(x-iy)|\cdot\left|1-\left(\frac{x-iy}{x+iy}\right)^{k}\right|\leq$$

$$\leq|(x+iy)^{k}|\cdot\left(|S^{+}(x+iy)-a(x)S^{-}(x-iy)|+\frac{A_{1}y}{|x|}\right)\leq$$

$$\leq y(A|x|^{k-1}+y^{\delta}|x|^{k-2\delta}).$$

Therefore, if $\kappa\geq 0$

$$|P(x+iy)S^{+}(x+iy)-a(x)P(x-iy)S^{-}(x-iy)|\leq y(A|x|^{m-1}+y^{\delta}|x|^{m-2\delta}).$$

Hence, m must be 0 or 1 in order for $P(z)S(z)$ to be a solution of the problem (9). So the solution is represented in the form $(Az+B)S(z)$.

If $\kappa=-1$, then $S^{-}(z)$ has pole of order 1 at the point of $z=-i$. Therefore, $P(z)$ is represented in the form $A(z+i)S(z)$.

In the case $\kappa<-1$, $S^{-}(z)$ has pole of order $|\kappa|$ at the point of $z=-i$. Since $m\leq 1$, we get that $P(z)S(z)$ is not a solution of the problem (9).

Let us show that the last two of (10) do not satisfy the condition of the problem (9). Indeed, consider

$$\Phi_{1}(z)=\frac{S(z)}{x_{k}-z}.$$

Observe that

$$\Phi_{1}^{+}(x+iy)-a(x)\Phi_{1}^{-}(x-iy)=\frac{S(x+iy)}{x_{k}-x-iy}-a(x)\frac{S(x-iy)}{x_{k}-x+iy}=$$

$$=I_{1}(x,y)+I_{2}(x,y),$$

where

$$I_{1}(x,y)=\frac{2iyS(x+iy)}{(x_{k}-x)^{2}+y^{2}},\qquad I_{2}(x,y)=\frac{S(x+iy)-a(x)S(x-iy)}{x_{k}-x+iy}.$$

In the case $x=x_{k}$, we get

$$|I_{1}(x,y)\cdot\rho(x)|=C\frac{\rho(x)}{y}.$$

Hence,

$$\lim_{y\to+0}\|I_1(x,y)\|_{\overline{C}(\rho)}=\infty.$$

Now using the inequality (6), we get

$$|I_2(x,y)\cdot\rho(x)|\le\widetilde{c}_2\frac{y^\delta}{|x+i|^{2\delta}}\frac{|x-x_k|^{\alpha_k}}{|x_k-x+iy|}\le$$

$$\le c_2\frac{y^\delta}{|x+i|^{2\delta}}\frac{|x-x_k|^{\alpha_k}}{|x_k-x|^{\frac12}y^{\frac12}}=c_2\frac{y^{\delta-\frac12}}{|x-x_k|^{\frac12-\alpha_k}}\longrightarrow 0\qquad\text{as } y\to+0.$$

Hence,

$$\lim_{y\to+0}\|I_2(x,y)\|_{\overline{C}(\rho)}=0.$$

Therefore, $\Phi_1(z)$ is not a solution of the problem (9).

Further, considering

$$\Phi_0(z)=\frac{S(z)}{x_0-z},$$

we get

$$\Phi_0^+(x+iy)-a(x)\Phi_0^-(x-iy)=J_1(x,y)+J_2(x,y),$$

where

$$J_1(x,y)=\frac{2iyS(x+iy)}{(x_0-x)^2+y^2},\qquad J_2(x,y)=\frac{S(x+iy)-a(x)S(x-iy)}{x_0-x+iy}.$$

Taking into account that $x_k\to x_0$ as $k\to\infty$, similarly we have

$$\lim_{y\to+0}\|J_1(x,y)\|_{\overline{C}(\rho)}=\infty,\qquad\lim_{y\to+0}\|J_2(x,y)\|_{\overline{C}(\rho)}=0.$$

Therefore, $\Phi_0(z)$ is also not a solution of the homogeneous problem (9). □

Theorem 2 *Let $f\in\overline{C}(\rho)$, and the sequence of points $\{x_k\}_1^\infty$ satisfy either condition (3), (4) or (5). Then the following assertions hold:*
a) If $\kappa\ge0$, then the general solution of the inhomogeneous Problem R may be represented as

$$\Phi(z)=\frac{(z+i)S(z)}{2\pi i}\int_{-\infty}^{+\infty}\frac{f(t)}{(t+i)S^+(t)\widetilde{\delta}(t)}\frac{dt}{t-z}+(Az+B)S(z),\qquad(11)$$

$z\in\Pi^\pm$, where A and B are constants.
b) If $\kappa=-1$, then the general solution of the inhomogeneous Problem R may be represented as

$$\Phi(z)=\frac{(z+i)S(z)}{2\pi i}\int_{-\infty}^{+\infty}\frac{f(t)}{(t+i)S^+(t)\widetilde{\delta}(t)}\frac{dt}{t-z}+A(z+i)S(z),\qquad z\in\Pi^\pm.$$

c) If $\kappa < -1$, then the inhomogeneous Problem R is solvable if and only if the function f satisfies the conditions

$$\int_{-\infty}^{+\infty} \frac{f(t)}{\widetilde{\delta}(t)S^{+}(t)} \frac{dt}{(t+i)^{j}} = 0, \qquad j = 1, 2, ..., -\kappa - 1.$$

In this case, the general solution can be represented by (11)*, where $A = B = 0$.*

Proof. The proofs of the points a) and b) follow from Lemma 4 and Theorem 1.

c) Let $\kappa < -1$ and denote

$$\Phi^{+}(x+iy) - a(x)\Phi^{-}(x-iy) = f_{y}(x). \tag{12}$$

Multiplying (12) by $(x+i)^{-1}$ and taking into account that $a(x) = \dfrac{S^{+}(x)}{S^{-}(x)}$, we get

$$\frac{\Phi^{+}(x+iy)}{S^{+}(x)(x+i)} - \frac{\Phi^{-}(x-iy)}{S^{-}(x)(x+i)} = \frac{f_{y}(x)}{S^{+}(x)(x+i)}.$$

Denoting

$$\Phi_{y}^{+}(z) = \frac{\Phi^{+}(z+iy)}{S^{+}(z)(z+i)}, \; z \in \Pi^{+}, \qquad \Phi_{y}^{-}(z) = \frac{\Phi^{-}(z-iy)}{S^{-}(z)(z+i)}, \; z \in \Pi^{-},$$

we get

$$\Phi_{y}^{+}(x) - \Phi_{y}^{-}(x) = \frac{f_{y}(x)}{S^{+}(x)(x+i)}.$$

In the case $\kappa < -1$, the function $\Phi_{y}^{-}(z)$ has zero of order $|\kappa - 1|$ at the point of $z = -i$. Consequently, $f(x)$ satisfies the conditions

$$\int_{-\infty}^{+\infty} \frac{f(t)}{\widetilde{\delta}(t)S^{+}(t)} \frac{dt}{(t+i)^{j}} = 0, \qquad j = 1, 2, ..., -\kappa - 1.$$

□

Research was conducted with the support of SCS MES RA in the scope of a joint scientific project NYSU-SFU-16/1, financed based on the results of international competition "SCS MES RA-YSU-SFU RF-2016".

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Please, cite to this paper as published in
Armen. J. Math., V. **11**, N. 7(2019), pp. 1–14